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Exponential Ratio Type Estimators of Population Mean under Non-Response

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ABSTRACT

This paper proposes some exponential ratio type estimators of population mean under the situations when certain observations for some sampling units are missing. These missing observations may be for either auxiliary variable or study variable. The biases and mean square errors of the proposed estimators have been derived, up to the first order of approximation. The proposed estimators are compared theoretically with that of the existing ratio type estimators defined by [1]. It has been found that the proposed exponential ratio type estimators perform better than the mean per unit estimator even for the low positive correlation between study variable and auxiliary variable. Moreover, we obtained the conditions for which our proposed estimators are better than the corresponding ratio type estimators of [1]. To verify the theoretical results obtained, a simulation study is carried out finally.

KEYWORDS

Auxiliary Variable; Bias; Mean Square Error; Non-Response; Simple Random Sampling without Replacement; Study Variable

Mathematical Subject Classification: 62 D 05

1. Introduction

In survey sampling situations, auxiliary information is generally used to improve the precision or accuracy of the estimator of unknown population parameter of interest under the assumption that all the observations in the sample are available. But in many survey sampling situations, this assumption is not true. This is the case of incomplete information which may arise due to some non-response in the given sample. There are various practical reasons for this incomplete information due to non-response like,

1) unwillingness of the respondent to answer some particular questions,

2) accidental loss of information caused by unknown factors,

3) failure on the part of investigator to collect correct information, etc.

Such type of incomplete information is very common in the studies related to medical research, market research surveys, opinion polls, socio economic investigations, etc.

In survey sampling, when information about all the sampling units is available then it is conventional to estimate unknown population mean of study variable using ratio estimator provided that there is a positive correlation between study variable and auxiliary variable (see [2]). But, when information about all the units is not available then the traditional complete data estimating procedures could not be used straight forwardly to analyze the data. [3] discussed the situation that certain incomplete information may occur on either study variable

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or auxiliary variable or both of these variables. In the early investigation with non-response situations of survey sampling, [1,4-6] considered the problem of estimation of population mean. Further investigation is carried out from various angles recently by many authors viz. [7-24], etc.

We have seen that [1] considered various ratio type estimators of population mean, under different situations, when some of the observations on either the study variable or auxiliary variable are missing. In this paper, we have assumed the same situations of non-response as considered by them. In Section 2, we have proposed the corresponding exponential ratio type estimators of population mean for these situations. In Section 3, we have obtained the expressions for the biases and mean square errors (MSE) of the proposed estimators. In Section 4, the comparisons (with respect to biases and mean square errors) have been made for the proposed estimators with the corresponding existing estimators of [1]. Finally, in Section 5, a simulation study has been performed to support the theoretical results obtained earlier in this paper.

2. Notations and the Proposed Estimators

Let y and x denote the positive valued study variable and positive valued auxiliary variable respectively. Assume that there is a positive correlation between y and x. Let (Y_i, X_i); $i = 1, 2, 3, \cdots, N$ denote the values of bivariate (y, x) on the i^{th} unit of population of size N. Consider a sample, say "s", of size n is drawn with simple random sampling without replacement (SRSWOR) from this population. Now, the problem of interest is to estimate the unknown population mean $\bar{Y}$ of study variable y when the population mean $\bar{X}$ of auxiliary variable x is assumed to be known.

It is assumed that $(n-p-q)$ observations of (y, x), namely $(y_1, x_1), (y_2, x_2), \cdots, (y_{n-p-q}, x_{n-p-q})$ measured on selected units in the sample are completely available. In addition to these available observations, let $x_1^*, x_2^*, \cdots, x_p^*$ denote the available observations of x variable on other p units in the sample but the corresponding observations of y variable are missing on these p sample units. Similarly, we have a set of other q available observations of y variable, namely $y_1^{**}, y_2^{**}, \cdots, y_q^{**}$ in the sample but the associated values of x variable are missing on these q sample units. Thus we have the following sub samples of the sample "s":

Sample "s" of size n

For $(n-p-q)$ sampling units, the observations $(y_i, x_i)'s$ of (y, x) are known which constitutes a sub sample "s_1"

For the other p sampling units, the observations $x_i^{*'s}$ of x are known but the corresponding observations of y are not known which constitutes a sub sample "s_2"

For the remaining q sampling units the observations $y_i^{**'s}$ of y are known but the corresponding observations of x are not known which constitutes a sub sample "s_3"

We note that

$$s = s_1 \cup s_2 \cup s_3 \text{ and } s_i \cap s_j = \phi(\text{an empty set}) \text{ for } i, j = 1, 2, 3 \text{ and } i \neq j.$$

Here, the quantities p and q denote the numbers of distinct sampling units in the sample "s" with incomplete observations on bivariate (y, x) and these must be random in nature.

Let

$$\left.\begin{aligned}
&\bar{x} = \frac{1}{n-p-q}\sum_{i\in s_1} x_i && \left(\text{sample mean of } x \text{ based on subsample } s_1\right)\\
&\bar{y} = \frac{1}{n-p-q}\sum_{i\in s_1} y_i && \left(\text{sample mean of } y \text{ based on subsample } s_1\right)\\
&\bar{x}^* = \frac{1}{p}\sum_{i\in s_2} x_i^* && \left(\text{sample mean of } x \text{ based on subsample } s_2\right)\\
&\bar{y}^{**} = \frac{1}{q}\sum_{i\in s_3} y_i^{**} && \left(\text{sample mean of } y \text{ based on subsample } s_3\right)\\
&\bar{y}_A = \frac{(n-p-q)\bar{y} + q\bar{y}^{**}}{(n-p)} && \left(\text{sample mean of } y \text{ based on subsample } s_1 \cup s_3\right)\\
&\bar{x}_A = \frac{(n-p-q)\bar{x} + p\bar{x}^{*}}{(n-q)} && \left(\text{sample mean of } x \text{ based on subsample } s_1 \cup s_2\right)
\end{aligned}\right] \tag{1}$$

[1] defined the following four ratio type estimators of $\bar{Y}$:

$$\bar{y}_{r_1} = \frac{\bar{y}}{\bar{x}}\bar{X} \qquad \left(\text{based on subsample } s_1\right) \tag{2}$$

$$\bar{y}_{r_2} = \frac{\bar{y}}{\bar{x}_A}\bar{X} = \frac{(n-q)\bar{y}}{(n-p-q)\bar{x} + p\bar{x}^*}\bar{X} \qquad \left(\text{based on subsample } s_1 \cup s_2\right) \tag{3}$$

$$\bar{y}_{r_3} = \frac{\bar{y}_A}{\bar{x}}\bar{X} = \frac{(n-p-q)\bar{y} + q\bar{y}^{**}}{(n-p)\bar{x}}\bar{X} \qquad \left(\text{based on subsample } s_1 \cup s_3\right) \tag{4}$$

$$\bar{y}_{r_4} = \frac{\bar{y}_A}{\bar{x}_A}\bar{X} = \frac{(n-q)}{(n-p)}\left[\frac{(n-p-q)\bar{y} + q\bar{y}^{**}}{(n-p-q)\bar{x} + p\bar{x}^*}\right]\bar{X} \qquad \left(\text{based on whole sample } s\right) \tag{5}$$

In the ordinary circumstances, when there is no non-response then [25] introduced an exponential ratio type estimator of $\bar{Y}$ which is better than mean per unit estimator of $\bar{Y}$ even for the low positive correlation between y and x. On the other hand, the ordinary ratio estimator of $\bar{Y}$ (due to [2]) is better than mean per unit estimator for high positive correlation between y and x and under certain conditions. On taking this advantage of exponential ratio type estimators and then considering the concept of ratio type estimators defined by [1], we have got a motivation to propose the following exponential ratio type estimators of $\bar{Y}$:

$$\bar{y}_{\text{Re}_1} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \qquad \left(\text{based on subsample } s_1\right) \tag{6}$$

$$\bar{y}_{\text{Re}_2} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}_A}{\bar{X}+\bar{x}_A}\right) = \bar{y}\exp\left(\frac{\bar{X}-\left\{\dfrac{(n-p-q)\bar{x}+p\bar{x}^*}{(n-q)}\right\}}{\bar{X}+\left\{\dfrac{(n-p-q)\bar{x}+p\bar{x}^*}{(n-q)}\right\}}\right), \qquad \left(\text{based on subsample } s_1 \cup s_2\right) \tag{7}$$

$$\bar{y}_{\text{Re}_3} = \bar{y}_A\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) = \left[\frac{(n-p-q)\bar{y}+q\bar{y}^{**}}{(n-p)}\right]\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right], \qquad \left(\text{based on subsample } s_1 \cup s_3\right) \tag{8}$$

$$\bar{y}_{\text{Re}_4} = \bar{y}_A\exp\left(\frac{\bar{X}-\bar{x}_A}{\bar{X}+\bar{x}_A}\right) = \left[\frac{(n-p-q)\bar{y}+q\bar{y}^{**}}{(n-p)}\right]\times\exp\left[\frac{\bar{X}-\left\{\dfrac{(n-p-q)\bar{x}+p\bar{x}^*}{(n-q)}\right\}}{\bar{X}+\left\{\dfrac{(n-p-q)\bar{x}+p\bar{x}^*}{(n-q)}\right\}}\right], \ \left(\text{based on whole sample } s\right) \tag{9}$$

These four proposed estimators will now be compared with the corresponding four estimators of [1] with respect to their biases and mean square errors.

3. Biases and Mean Square Errors of Proposed Estimators

To obtain the biases and mean square errors of the proposed estimators, we proceed as follows.

Let

$$U=\frac{\bar{x}}{\bar{X}}-1,\ V=\frac{\bar{y}}{\bar{Y}}-1,\ U^{*}=\frac{\bar{x}^{*}}{\bar{X}}-1,\ V^{**}=\frac{\bar{y}^{**}}{\bar{Y}}-1 \tag{10}$$

Now we state the following lemma.

Lemma 3.1: Under SRSWOR, we have the following expectations:

$$\left.\begin{aligned}&E(U)=E(V)=E(U^{*})=E(V^{**})=0,\\&E(U^2)=f_{p+q}C_x^2,\ E(V^2)=f_{p+q}C_y^2,\ E(UV)=f_{p+q}\rho C_yC_x,\\&E(U^{*2})=\left(\frac{1}{p}-\frac{1}{N}\right)C_x^2,\ E(V^{**2})=\left(\frac{1}{q}-\frac{1}{N}\right)C_y^2,\\&E(UU^{*})=E(U^{*}V)=E(VV^{**})=E(UV^{**})=0\end{aligned}\right] \tag{11}$$

where

$$f_{p+q}=E_1\left(\frac{1}{n-p-q}\right)-\frac{1}{N},\ C_x^2=\frac{1}{(N-1)\bar{X}^2}\sum_{i=1}^{N}(X_i-\bar{X})^2,\ C_y^2=\frac{1}{(N-1)\bar{Y}^2}\sum_{i=1}^{N}(Y_i-\bar{Y})^2$$

$$\rho=\operatorname{corr}(y,x)=\frac{1}{(N-1)\bar{X}\bar{Y}C_xC_y}\sum_{i=1}^{N}(Y_i-\bar{Y})(X_i-\bar{X}),\text{ and}$$

E_1 denotes the unconditional expectation based on the whole sample *s*.

Proof: See Appendix 1.

Remark 3.1: We note that $f_p\le f_{p+q}$ and $f_q\le f_{p+q}$ always hold good, where as $f_p \lesseqgtr f_q$, hold good according as $p \lesseqgtr q$.

From (10), we can rewrite the following:

$$\bar{x}=(1+U)\bar{X},\ \bar{y}=(1+V)\bar{Y},\ \bar{x}^{*}=(1+U^{*})\bar{X},\ \bar{y}^{**}=(1+V^{**})\bar{Y} \tag{12}$$

Now we state the following lemma.

Lemma 3.2: On using (12) and retaining the terms up to second degree of $U's, V's, U^{*}'s$ and $V^{**}'s$ only, we can write the following:

$$\bar{y}_{\mathrm{Re}_1}-\bar{Y}=\bar{Y}\left(-\frac{U}{2}+\frac{3}{8}U^2+V-\frac{VU}{2}\right) \tag{13}$$

$$\begin{aligned}\bar{y}_{\mathrm{Re}_2}-\bar{Y}=\bar{Y}\Bigg[&-\frac{(n-p-q)}{2(n-q)}U+\frac{3(n-p-q)^2}{8(n-q)^2}U^2+V+\frac{2p(n-p-q)}{4(n-q)^2}UU^{*}-\frac{p}{2(n-q)}U^{*}\\&+\frac{3p^2}{8(n-q)^2}U^{*2}-\frac{(n-p-q)}{2(n-q)}UV-\frac{p}{2(n-q)}U^{*}V\Bigg]\end{aligned} \tag{14}$$

$$\bar{y}_{\mathrm{Re}_3}-\bar{Y}=\bar{Y}\left[-\frac{U}{2}+\frac{3}{8}U^2+\frac{(n-p-q)}{(n-p)}V-\frac{(n-p-q)}{2(n-p)}UV+\frac{q}{n-p}V^{**}+\frac{q}{2(n-p)}UV^{**}\right] \tag{15}$$

$$\bar{y}_{\mathrm{Re}_4}-\bar{Y}=\bar{Y}\Bigg[-\frac{(n-p-q)}{2(n-q)}U+\frac{3(n-p-q)^2}{8(n-q)^2}U^2+\frac{3p(n-p-q)}{4(n-q)^2}UU^*-\frac{p}{2(n-q)}U^*$$
$$+\frac{3p^2}{8(n-q)^2}U^{*2}+\frac{n-p-q}{n-p}V-\frac{(n-p-q)^2}{2(n-p)(n-q)}UV-\frac{p(n-p-q)}{2(n-p)(n-q)}U^*V \tag{16}$$
$$+\frac{q}{n-p}V^{**}-\frac{q(n-p-q)}{2(n-p)(n-q)}UV^{**}-\frac{pq}{2(n-p)(n-q)}U^*V^{**}\Bigg]$$

Proof: See **Appendix 1**.

Theorem 3.1: The expressions of biases and mean square errors of the four proposed exponential ratio type estimators of $\bar{Y}$, up to first order of approximation, are

$$Bias\left(\bar{y}_{\mathrm{Re}_1}\right)=\bar{Y}f_{p+q}\left(\frac{3}{8}C_x^2-\frac{1}{2}\rho C_y C_x\right) \tag{17}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_1}\right)=\bar{Y}^2 f_{p+q}\left(\frac{C_x^2}{4}+C_y^2-\rho C_y C_x\right) \tag{18}$$

$$Bias\left(\bar{y}_{\mathrm{Re}_2}\right)=\bar{Y}f_q\left(\frac{3C_x^2}{8}-\frac{1}{2}\rho C_y C_x\right) \tag{19}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_2}\right)=\bar{Y}^2\left(\frac{1}{4}C_x^2 f_q+C_y^2 f_{p+q}-\rho C_y C_x f_q\right) \tag{20}$$

$$Bias\left(\bar{y}_{\mathrm{Re}_3}\right)=\bar{Y}\left(\frac{3}{8}C_x^2 f_{p+q}-\frac{1}{2}\rho C_y C_x f_p\right) \tag{21}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_3}\right)=\bar{Y}^2\left(\frac{1}{4}C_x^2 f_{p+q}+C_y^2 f_p-\rho C_y C_x f_p\right) \tag{22}$$

$$Bias\left(\bar{y}_{\mathrm{Re}_4}\right)=\bar{Y}\left(\frac{3}{8}C_x^2 f_q-\frac{1}{2}\rho C_y C_x f_p\right) \tag{23}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right)=\bar{Y}^2\left(\frac{C_x^2}{4}f_q+f_p C_y^2-\rho C_y C_x f_p\right) \tag{24}$$

where

$$f_{p+q}=E_1\left(\frac{1}{n-p-q}\right)-\frac{1}{N},\quad f_p=E_1\left(\frac{1}{n-p}\right)-\frac{1}{N},\quad f_q=E_1\left(\frac{1}{n-q}\right)-\frac{1}{N}$$

Proof: See **Appendix 2**.

Remarks 3.2: The expressions for biases and mean square errors of the proposed estimators (as obtained in Theorem 3.1) involve some unknown population parameters like $\bar{Y}, C_y, C_x$ and ρ. To find the estimates of these biases and mean square errors, the general practice in survey sampling is to replace the unknown population parameters with their respective consistent estimators based on the same sample.

To test the superiority of our proposed estimators over the existing estimators of [1], we compare the biases and the mean square errors of these estimators.

4. Comparison of Proposed Estimators with Existing Estimators

To compare the biases and mean square errors of the proposed estimators with the estimators defined by [1], we require the expressions of their biases and mean square errors, up to first order of approximation and these are given in **Table 1**.

Remark 4.1: Note that $MSE\left(\bar{y}_A\right)=f_p\bar{Y}^2C_y^2$ and $Bias\left(\bar{y}_A\right)=0$.

Remark 4.2: While comparing the biases and mean square errors, we have taken p and q as fixed quantities in the given sample, so we must have

$$E_1\left(\frac{1}{n-p}\right)=\frac{1}{n-p},\ \ E_1\left(\frac{1}{n-q}\right)=\frac{1}{n-q},\ \ E_1\left(\frac{1}{n-p-q}\right)=\frac{1}{n-p-q}.$$

Remark 4.3: [26] has shown that the values of parameter $K=\rho\frac{C_y}{C_x}$ remain stable in any repetitive survey. So while comparing biases and mean square errors of various estimators, we shall try to find the conditions on the values of K under which one estimator is superior to the other estimator. In the present situations, we also note that the value of K always lies in the interval $0<K<\infty$.

Theorem 4.1: Up to the terms of order n^{-1}, we have

$$\left|Bias\left(\bar{y}_{\mathrm{Re}_1}\right)\right|<\left|Bias\left(\bar{y}_{r_1}\right)\right| \quad \text{if } K\in\left(0,\frac{88}{96}\right)\cup\left(\frac{5}{4},\infty\right) \tag{25}$$

$$\left|Bias\left(\bar{y}_{\mathrm{Re}_2}\right)\right|<\left|Bias\left(\bar{y}_{r_2}\right)\right| \quad \text{if } K\in\left(0,\frac{88}{96}\right)\cup\left(\frac{5}{4},\infty\right) \tag{26}$$

$$\left|Bias\left(\bar{y}_{\mathrm{Re}_3}\right)\right|<\left|Bias\left(\bar{y}_{r_3}\right)\right| \quad \text{if } K\in\left(0,\frac{88 f_{p+q}}{96 f_p}\right)\cup\left(\frac{5 f_{p+q}}{4 f_p},\infty\right) \tag{27}$$

$$\left|Bias\left(\bar{y}_{\mathrm{Re}_4}\right)\right|<\left|Bias\left(\bar{y}_{r_4}\right)\right| \quad \text{if } K\in\left(0,\frac{88}{96}\frac{f_q}{f_p}\right)\cup\left(\frac{5}{4}\frac{f_q}{f_p},\infty\right) \tag{28}$$

Proof: See Appendix 3.

Theorem 4.2: Up to the term of order n^{-1}, we have

$$MSE\left(\bar{y}_{\mathrm{Re}_1}\right)<MSE\left(\bar{y}_{r_1}\right) \quad \text{if } K<\frac{3}{4} \tag{29}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_2}\right)<MSE\left(\bar{y}_{r_2}\right) \quad \text{if } K<\frac{3}{4} \tag{30}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_3}\right)<MSE\left(\bar{y}_{r_3}\right) \quad \text{if } K<\frac{3}{4}\frac{f_{p+q}}{f_p} \tag{31}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right)<MSE\left(\bar{y}_{r_4}\right) \quad \text{if } K<\frac{3}{4}\frac{f_q}{f_p} \tag{32}$$

Proof: See Appendix 3.

From the above two theorems, we see that our proposed estimators are superior to the existing estimators under some very simple conditions.

Theorem 4.3: Up to the term of order n^{-1}, we have

$$MSE\left(\bar{y}_{r_1}\right)<MSE\left(\bar{y}_A\right), \quad \text{if } K>\frac{1}{2}\left[1+\frac{C_y^2}{C_x^2}\left(1-\frac{f_p}{f_{p+q}}\right)\right] \tag{33}$$

Table 1. Biases and mean square errors of the existing estimators, up to first order of approximation.

Estimator	$Bias(.)$	$MSE(.)$
$\bar{y}_{r_1}$	$\bar{Y}f_{p+q}\left(C_x^2-\rho C_x C_y\right)$	$\bar{Y}^2 f_{p+q}\left(C_x^2+C_y^2-2\rho C_y C_x\right)$
$\bar{y}_{r_2}$	$\bar{Y}\left(f_q C_x^2-\rho C_y C_x f_q\right)$	$\bar{Y}^2\left(f_{p+q}C_y^2+f_q C_x^2-2\rho C_y C_x f_q\right)$
$\bar{y}_{r_3}$	$\bar{Y}\left(f_{p+q}C_x^2-\rho C_y C_x f_p\right)$	$\bar{Y}^2\left(C_x^2 f_{p+q}+C_y^2 f_p-2\rho C_y C_x f_p\right)$
$\bar{y}_{r_4}$	$\bar{Y}\left(f_q C_x^2-\rho C_y C_x f_p\right)$	$\bar{Y}^2\left(f_p C_y^2+f_q C_x^2-2\rho C_y C_x f_p\right)$

$$MSE\left(\bar{y}_{r_2}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{2}\left[1+\frac{C_y^2}{C_x^2}\left(\frac{f_{p+q}-f_p}{f_q}\right)\right] \tag{34}$$

$$MSE\left(\bar{y}_{r_3}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{2}\frac{f_{p+q}}{f_p} \tag{35}$$

$$MSE\left(\bar{y}_{r_4}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{2}\frac{f_q}{f_p} \tag{36}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_1}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{4}+\frac{C_y^2}{C_x^2}\left(1-\frac{f_p}{f_{p+q}}\right) \tag{37}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_2}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{4}+\frac{C_y^2}{C_x^2}\left(\frac{f_{p+q}-f_p}{f_q}\right) \tag{38}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_3}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{4}\frac{f_{p+q}}{f_p} \tag{39}$$

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right) < MSE\left(\bar{y}_A\right), \quad \text{if } K > \frac{1}{4}\frac{f_q}{f_p} \tag{40}$$

Proof: This theorem can be proved in the similar way as Theorem 4.2.

Corollary 4.1: On combining Theorems 4.2 and 4.3, we have the following results:

1) The mean square error of the proposed estimator $\bar{y}_{\mathrm{Re}_1}$ is less than that of both $\bar{y}_{r_1}$ and $\bar{y}_A$ if $K \in \left(\frac{1}{4}+\frac{C_y^2}{C_x^2}\left(1-\frac{f_p}{f_{p+q}}\right), \frac{3}{4}\right)$, provided that $\frac{C_y^2}{C_x^2}\left(1-\frac{f_p}{f_{p+q}}\right) < \frac{1}{2}$.

2) The mean square error of the proposed estimator $\bar{y}_{\mathrm{Re}_2}$ is less than that of both $\bar{y}_{r_2}$ and $\bar{y}_A$ if $K \in \left(\frac{1}{4}+\frac{C_y^2}{C_x^2}\left(\frac{f_{p+q}-f_p}{f_q}\right), \frac{3}{4}\right)$, provided that $\frac{C_y^2}{C_x^2}\left(\frac{f_{p+q}-f_p}{f_q}\right) < \frac{1}{2}$.

3) The mean square error of the proposed estimator $\bar{y}_{\mathrm{Re}_3}$ is less than that of both $\bar{y}_{r_3}$ and $\bar{y}_A$ if $K \in \left(\frac{1}{4}\frac{f_{p+q}}{f_p}, \frac{3}{4}\frac{f_{p+q}}{f_p}\right)$.

4) The mean square error of the proposed estimator $\bar{y}_{\mathrm{Re}_4}$ is less than that of both $\bar{y}_{r_4}$ and $\bar{y}_A$ if $K \in \left(\frac{1}{4}\frac{f_q}{f_p}, \frac{3}{4}\frac{f_q}{f_p}\right)$.

Remark 4.4: From the results of Theorem 4.3, we can conclude that four proposed exponential ratio type estimators are really better than mean per unit estimator for even the lower positive values of K (or equivalently even for lower positive correlation between y and x).

5. A Simulation Study

To support the facts proved in earlier sections of this paper, we perform a simulation study here. For this purpose, we have taken an empirical population of size 34 from [27] [page No. 177]. In this population, variable "y" is area under wheat in 1973 and variable "x" is area under wheat in 1971. For this population, we have following requisite parameters:

$$\bar{X} = 856.41,\ \bar{Y} = 208.88,\ C_x = 0.86,\ C_y = 0.72,\ \rho = 0.45,\ K = 0.38$$

We have simulated sample (with SRSWOR) 50 times from the above fixed population using R software (version 2.14.0).

While performing a simulation study, we use the following steps in sequence:

1) A sample, say "s", of size n is drawn from the fixed population using simple random sampling without replacement (SRSWOR).

2) Take the fixed values of missingness rates, that is, p and q.

3) Randomly, we deleted q observations from the set of observations of study variable and p observations from the set of observations of auxiliary variable.

4) Identify the subsamples, namely s_1, s_2 and s_3.

5) The values of the estimators $\bar{y}_{r_1}, \bar{y}_{r_2}, \bar{y}_{r_3}, \bar{y}_{r_4}, \bar{y}_{\mathrm{Re}_1}, \bar{y}_{\mathrm{Re}_2}, \bar{y}_{\mathrm{Re}_3}, \bar{y}_{\mathrm{Re}_4}$ and $\bar{y}_A$ are calculated for each triplet (n, p, q).

6) Calculate the variances (or approximate mean square errors) of these estimators by using their 50 values that are obtained from 50 different simulated samples drawn from the given fixed population.

We have taken the different values of triplet (n, p, q) as shown in **Table 2**.

In **Tables 3** and **4**, we have mentioned the variances of values of various estimators, considered in this paper, obtained for the simulated 50 different samples drawn from the given population on taking various values of sample sizes and values of missingness rates.

Table 2. Considered sample sizes and missingness rates.

		n = 12 with		
p = 3, q = 3	p = 3, q = 4	p = 3, q = 5	p = 4, q = 3	p = 5, q = 3
		n = 17 with		
p = 3, q = 3	p = 3, q = 4	p = 3, q = 5	p = 4, q = 3	p = 5, q = 3

Table 3. Variances of various estimators for sample size "n" = 12.

Missingness rate	Variances of various estimators for some fixed values of p and q								
	$\bar{y}_{r_1}$	$\bar{y}_{\mathrm{Re}_1}$	$\bar{y}_{r_2}$	$\bar{y}_{\mathrm{Re}_2}$	$\bar{y}_{r_3}$	$\bar{y}_{\mathrm{Re}_3}$	$\bar{y}_{r_4}$	$\bar{y}_{\mathrm{Re}_4}$	$\bar{y}_A$
p = 3, q = 3	3341.9487	2268.5374	3287.6161	2583.8184	4683.8065	2072.6869	3628.4179	2121.2399	1944.3183
p = 3, q = 4	3695.3966	2647.3249	3568.2378	2983.9004	6007.6042	2511.0964	4579.5597	2472.3673	1948.7899
p = 3, q = 5	3927.4125	3730.6319	5083.6828	5031.9734	11471.7880	2587.7638	4397.0302	2106.5239	1751.5138
p = 4, q = 3	3800.1203	2708.5279	3760.8063	3071.0838	5486.9688	2439.7666	4021.7138	2451.9012	2226.7277
p = 5, q = 3	3943.2391	3626.4965	5626.9421	5207.1947	11593.1195	3521.8459	4970.5069	3493.3012	3317.3329

Table 4. Variances of various estimators for sample size "n" = 17.

Missingness rate	Variances of various estimators for some fixed values of p and q								
	$\bar{y}_{r_1}$	$\bar{y}_{\mathrm{Re}_1}$	$\bar{y}_{r_2}$	$\bar{y}_{\mathrm{Re}_2}$	$\bar{y}_{r_3}$	$\bar{y}_{\mathrm{Re}_3}$	$\bar{y}_{r_4}$	$\bar{y}_{\mathrm{Re}_4}$	$\bar{y}_A$
p = 3, q = 3	1668.5370	1227.9175	1374.6055	1078.7541	2154.1255	1251.3020	1728.3800	1071.2678	1018.2884
p = 3, q = 4	1763.7731	1216.2798	1528.6801	1128.6377	2404.9459	1321.0321	1801.9670	1097.1925	1018.2884
p = 3, q = 5	1942.1475	1372.8471	1693.9537	1271.8256	2318.1452	1224.2231	1842.7394	1036.7207	1010.5828
p = 4, q = 3	1910.8782	1324.8379	1584.9994	1212.9682	2370.6976	1368.2326	1980.4694	1239.9672	1102.7907
p = 5, q = 3	2004.6414	1481.4818	1841.2174	1476.9390	2394.4015	1469.5580	2253.3091	1474.1011	1308.9791

Discussion, Conclusions and Interpretations

For the given fixed population, $K = 0.38$, which satisfies all the conditions, obtained in the results (29) to (32), for all values of triplet (n, p, q), considered in **Tables 3** and **4**. Therefore, the variances of all the proposed exponential ratio type estimators are less than that of all the corresponding existing ratio type estimators of [1]. It can be verified through **Tables 3** and **4**.

Again for the given fixed population, $K = 0.38$, which does not satisfy all the conditions, obtained in the results (33) to (40), for all values of triplet (n, p, q), considered in **Tables 3** and **4**. Therefore, the variances of all the eight ratio type estimators of $\bar{Y}$ [four proposed estimators and four [1]'s estimators] considered in this paper are not less than that of the mean per unit estimator $\bar{y}_A$. It can be verified again through **Tables 3** and **4**.

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Appendix 1

Proof of Lemma 3.1: Taking

$$E(U)=E\left(\frac{\bar{x}}{\bar{X}}-1\right)=\frac{1}{\bar{X}}E(\bar{x})-1=\frac{1}{\bar{X}}\left[E_1\left(E_2\left(\bar{x}/p,q\right)\right)\right]-1$$

where E_2 denotes the conditional expectation based on the sub sample (either s_1 or s_2 or s_3) under the condition that both p and q are fixed for the given sample s.

$$\text{Now}\quad E(U)=\frac{1}{\bar{X}}\left[E_1\left(\bar{x}_s\right)\right]-1=\frac{1}{\bar{X}}\left(\bar{X}\right)-1=0$$

where $\bar{x}_s=E_2\left(\bar{x}/p,q\right)$ is the sample mean of *x* based on whole complete sample "*s*".

Hence the required result.

Again taking

$$E\left(U^2\right)=E_1\left(E_2\left(U^2/p,q\right)\right)=E_1\left[\left(\frac{1}{n-p-q}-\frac{1}{N}\right)C_x^2\right]=C_x^2 f_{p+q},$$

$$\because E_2\left(U^2/p,q\right)=\left(\frac{1}{n-p-q}-\frac{1}{N}\right)C_x^2$$

Hence the required result again.

Similarly, we can prove the other results of Lemma 3.1.

Proof of Lemma 3.2:

Due to complex form of estimator $\bar{y}_{\mathrm{Re}_4}$, we are recalling estimator as defined in (9).

$$\bar{y}_{\mathrm{Re}_4}=\left[\frac{(n-p-q)\bar{y}+q\bar{y}^{**}}{(n-p)}\right]\exp\left[\frac{\bar{X}-\left\{\frac{(n-p-q)\bar{x}+p\bar{x}^{*}}{(n-q)}\right\}}{\bar{X}+\left\{\frac{(n-p-q)\bar{x}+p\bar{x}^{*}}{(n-q)}\right\}}\right]$$

$$=\left[\frac{(n-p-q)\left(V\bar{Y}+\bar{Y}\right)+q\left(V^{**}\bar{Y}+\bar{Y}\right)}{n-p}\right]\exp\left[\frac{\bar{X}-\left\{\frac{(n-p-q)\left(U\bar{X}+\bar{X}\right)+p\left(U^{*}\bar{X}+\bar{X}\right)}{(n-q)}\right\}}{\bar{X}+\left\{\frac{(n-p-q)\left(U\bar{X}+\bar{X}\right)+p\left(U^{*}\bar{X}+\bar{X}\right)}{(n-q)}\right\}}\right]$$

$$\left(\text{using }(12)\right)$$

On retaining the terms only up to second degrees of $U's, V's, U^{*}'s$ and $V^{**}'s$, we have

$$\bar{y}_{\mathrm{Re}_4}=\bar{Y}\left[\frac{(n-p)+(n-p-q)V+qV^{**}}{n-p}\right]\left[1-\frac{(n-p-q)}{(2n-2q)}U+\frac{3}{8}\frac{(n-p-q)^2}{(n-q)^2}U^2\right.$$

$$\left.+\frac{3p}{4}\frac{(n-p-q)}{(n-q)^2}UU^{*}-\frac{pU^{*}}{2(n-q)}+\frac{3}{8}\frac{p^2U^{*2}}{(n-q)^2}\right]$$

$$\bar{y}_{\mathrm{Re}_4}=\bar{Y}\left[1-\frac{(n-p-q)}{2(n-q)}U+\frac{3}{8}\frac{(n-p-q)^2}{(n-q)^2}U^2+\frac{3p}{4}\frac{(n-p-q)}{(n-q)^2}UU^{*}-\frac{pU^{*}}{2(n-q)}+\frac{3}{8}\frac{p^2U^{*2}}{(n-q)^2}+\frac{(n-p-q)}{(n-p)}V\right.$$

$$\left.-\frac{(n-p-q)^2}{2(n-p)(n-q)}UV-\frac{p(n-p-q)}{2(n-p)(n-q)}U^{*}V+\frac{q}{(n-p)}V^{**}-\frac{q(n-p-q)}{2(n-p)(n-q)}UV^{**}-\frac{pq}{2(n-p)(n-q)}U^{*}V^{**}\right]$$

$$\Rightarrow \bar{y}_{\mathrm{Re}_4} - \bar{Y} = \bar{Y}\left[-\frac{(n-p-q)}{2(n-q)}U + \frac{3(n-p-q)^2}{8(n-q)^2}U^2 + \frac{3p(n-p-q)}{4(n-q)^2}UU^* - \frac{p}{2(n-q)}U^* \right.$$
$$+\frac{3p^2}{8(n-q)^2}U^{*2} + \frac{n-p-q}{n-p}V - \frac{(n-p-q)^2}{2(n-p)(n-q)}UV - \frac{p(n-p-q)}{2(n-p)(n-q)}U^*V$$
$$\left. + \frac{q}{n-p}V^{**} - \frac{q(n-p-q)}{2(n-p)(n-q)}UV^{**} - \frac{pq}{2(n-p)(n-q)}U^*V^{**} \right]$$

which is the required result.

Similarly, we can prove the other results of Lemma 3.2.

Appendix 2

<u>Proof of Theorem 3.1:</u> Again due to complex nature of estimator $\bar{y}_{\mathrm{Re}_4}$, we prove the results (23) and (24) only. So, from (16), we have

$$\Rightarrow \bar{y}_{\mathrm{Re}_4} - \bar{Y} = \bar{Y}\left[-\frac{(n-p-q)}{2(n-q)}U + \frac{3(n-p-q)^2}{8(n-q)^2}U^2 + \frac{3p(n-p-q)}{4(n-q)^2}UU^* - \frac{p}{2(n-q)}U^* \right.$$
$$+\frac{3p^2}{8(n-q)^2}U^{*2} + \frac{n-p-q}{n-p}V - \frac{(n-p-q)^2}{2(n-p)(n-q)}UV - \frac{p(n-p-q)}{2(n-p)(n-q)}U^*V$$
$$\left. + \frac{q}{n-p}V^{**} - \frac{q(n-p-q)}{2(n-p)(n-q)}UV^{**} - \frac{pq}{2(n-p)(n-q)}U^*V^{**} \right]$$

The bias of $\bar{y}_{\mathrm{Re}_4}$, up to the terms of order n^{-1}, is given by

$$Bias\left(\bar{y}_{\mathrm{Re}_4}\right) = E\left(\bar{y}_{\mathrm{Re}_4}\right) - \bar{Y} = E\left(\bar{y}_{\mathrm{Re}_4} - \bar{Y}\right)$$
$$= \bar{Y}E\left[-\frac{(n-p-q)}{2(n-q)}U + \frac{3(n-p-q)^2}{8(n-q)^2}U^2 + \frac{3p(n-p-q)}{4(n-q)^2}UU^* - \frac{p}{2(n-q)}U^* \right.$$
$$+\frac{3p^2}{8(n-q)^2}U^{*2} + \frac{n-p-q}{n-p}V - \frac{(n-p-q)^2}{2(n-p)(n-q)}UV - \frac{p(n-p-q)}{2(n-p)(n-q)}U^*V$$
$$\left. + \frac{q}{n-p}V^{**} - \frac{q(n-p-q)}{2(n-p)(n-q)}UV^{**} - \frac{pq}{2(n-p)(n-q)}U^*V^{**} \right]$$
$$= \bar{Y}\left[-\frac{(n-p-q)}{2(n-q)}E(U) + \frac{3(n-p-q)^2}{8(n-q)^2}E\left(U^2\right) + \frac{3p(n-p-q)}{4(n-q)^2}E\left(UU^*\right) \right.$$
$$-\frac{p}{2(n-q)}E\left(U^*\right) + \frac{3p^2}{8(n-q)^2}E\left(U^{*2}\right) + \frac{n-p-q}{n-p}E(V) - \frac{(n-p-q)^2}{2(n-p)(n-q)}E(UV)$$
$$-\frac{p(n-p-q)}{2(n-p)(n-q)}E\left(U^*V\right) + \frac{q}{n-p}E\left(V^{**}\right) - \frac{q(n-p-q)}{2(n-p)(n-q)}E\left(UV^{**}\right)$$
$$\left. -\frac{pq}{2(n-p)(n-q)}E\left(U^*V^{**}\right) \right]$$

On using the values of expectations, as obtained in Lemma 3.1, and then simplifying, we have

$$Bias\left(\bar{y}_{\mathrm{Re}_4}\right) = \bar{Y}\left(\frac{3}{8}C_x^2 f_q - \frac{1}{2}\rho C_y C_x f_p \right)$$

Hence the result (23) is proved.

Again by definition, the mean square error of $\bar{y}_{\mathrm{Re}_4}$ is given by

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right) = E\left(\bar{y}_{\mathrm{Re}_4} - \bar{Y}\right)^2$$

The mean square error of $\bar{y}_{\mathrm{Re}_4}$, up to the terms of order n^{-1}, is

$$MSE\ \left(\bar{y}_{\mathrm{Re}_4}\right) = \bar{Y}^2 E\left[-\frac{(n-p-q)}{2(n-q)}U + \frac{3(n-p-q)^2}{8(n-q)^2}U^2 + \frac{3p(n-p-q)}{4(n-q)^2}UU^* - \frac{p}{2(n-q)}U^*\right.$$
$$+\frac{3p^2}{8(n-q)^2}U^{*2} + \frac{n-p-q}{n-p}V - \frac{(n-p-q)^2}{2(n-p)(n-q)}UV - \frac{p(n-p-q)}{2(n-p)(n-q)}U^*V$$
$$\left.+\frac{q}{n-p}V^{**} - \frac{q(n-p-q)}{2(n-p)(n-q)}UV^{**} - \frac{pq}{2(n-p)(n-q)}U^*V^{**}\right]^2$$

On retaining the terms only up to second degrees of $U's, V's, U^{*\prime}s$ and $V^{**\prime}s$, we have

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right) = \bar{Y}^2 E\left[\frac{(n-p-q)^2}{4(n-q)^2}U^2 + \frac{p^2}{4(n-q)^2}U^{*2} + \frac{(n-p-q)^2}{(n-p)^2}V^2 + \frac{q^2}{(n-p)^2}V^{**2}\right.$$
$$+\frac{(n-p-q)p}{2(n-q)^2}UU^* - \frac{(n-p-q)^2}{(n-p)(n-q)}UV - \frac{(n-p-q)q}{(n-p)(n-q)}UV^{**}$$
$$\left.-\frac{(n-p-q)p}{(n-p)(n-q)}U^*V - \frac{p}{(n-p)(n-q)}U^*V^{**} + \frac{2(n-p-q)q}{(n-p)^2}VV^{**}\right]$$
$$= \bar{Y}^2\left[\frac{(n-p-q)^2}{4(n-q)^2}E\left(U^2\right) + \frac{p^2}{4(n-q)^2}E\left(U^{*2}\right) + \frac{(n-p-q)^2}{(n-p)^2}E(V)^2 + \frac{q^2}{(n-p)^2}E\left(V^{**2}\right)\right.$$
$$+\frac{(n-p-q)p}{2(n-q)^2}E\left(UU^*\right) - \frac{(n-p-q)^2}{(n-p)(n-q)}E(UV) - \frac{(n-p-q)q}{(n-p)(n-q)}E\left(UV^{**}\right)$$
$$\left.-\frac{(n-p-q)p}{(n-p)(n-q)}E\left(U^*V\right) - \frac{p}{(n-p)(n-q)}E\left(U^*V^{**}\right) + \frac{2(n-p-q)q}{(n-p)^2}E\left(VV^{**}\right)\right]$$

On using the values of expectations, as obtained in Lemma 3.1 and then simplifying, we have

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right) = \bar{Y}^2\left(\frac{C_x^2}{4}f_q + f_p C_y^2 - \rho C_y C_x f_p\right)$$

Hence the result (24) is proved.
In the similar way, we can prove the other results (17) to (22).

Appendix 3

Proof of Theorem 4.1:
Here we give the proof of the result (28) only. The proofs of other results of this Theorem are similar.
Taking

$$\left|Bias\left(\bar{y}_{\mathrm{Re}_4}\right)\right| < \left|Bias\left(\bar{y}_{r_4}\right)\right|$$

$$\text{if } \left|\bar{Y}\left(\frac{3}{8}C_x^2 f_q - \frac{1}{2}\rho C_y C_x f_p\right)\right| < \left|\bar{Y}\left(f_q C_x^2 - \rho C_y C_x f_p\right)\right|$$

(on using (23) and **Table 1**)

$$\text{or if } C_x^4\left[-\frac{55}{64}f_q^2-\frac{3}{4}\frac{\rho^2C_y^2}{C_x^2}f_p^2+\frac{13}{8}\frac{C_y\rho}{C_x}f_pf_q\right]<0$$

(on squaring both sides of the inequality and then simplifying)

$$\text{or if } 48f_p^2K^2-104f_pf_qK+55f_q^2>0 \quad \left(\text{where } K=\rho\frac{C_y}{C_x}\right)$$

$$\text{or if } K^2-K\left(\frac{104}{48}\frac{f_q}{f_p}\right)+\frac{55}{48}\frac{f_q^2}{f_p^2}>0$$

$$\text{or if } \left(K-K_1\right)\left(K-K_2\right)>0, \tag{A3.1}$$

where

$$K_1=\frac{5}{4}\frac{f_q}{f_p} \text{ and } K_2=\frac{88}{96}\frac{f_q}{f_p}.$$

Noting that $K_1>K_2$.

For $\left(K-K_1\right)\left(K-K_2\right)>0$, we have the following two cases:

Either case 1: When $\left(K-K_1\right)>0$ and $\left(K-K_2\right)>0$ then it implies that $K>K_1$ and $K>K_2$. Thus we must have $K>K_1$.

Or case 2: When $\left(K-K_1\right)<0$ and $\left(K-K_2\right)<0$ then it implies that $K<K_1$ and $K<K_2$.

Thus we must have $K<K_2$.

On combining Case 1 and Case 2, we conclude that for $\left(K-K_1\right)\left(K-K_2\right)>0$ we must have either $K>K_1$ or $K<K_2$, that is, $K\in\left(0,K_2\right)\cup\left(K_1,\infty\right)$. (A3.2)

Thus from the results (A3.1) and (A3.2), we have got that

$$\left|Bias\left(\bar{y}_{\mathrm{Re}_4}\right)\right|<\left|Bias\left(\bar{y}_{r_4}\right)\right| \text{ if } K\in\left(0,\frac{88}{96}\frac{f_q}{f_p}\right)\cup\left(\frac{5}{4}\frac{f_q}{f_p},\infty\right)$$

Hence the result (28) is proved.

Proof of Theorem 4.2:

We give here the proof of the result (32) only. The proofs of other results of this Theorem are similar.

Taking

$$MSE\left(\bar{y}_{\mathrm{Re}_4}\right)<MSE\left(\bar{y}_{r_4}\right)$$

$$\text{if } \frac{C_x^2}{4}f_q+f_pC_y^2-\rho C_yC_xf_p<f_pC_y^2+f_qC_x^2-2\rho C_yC_xf_p$$

(on using (24) and **Table 1**)

$$\text{or if } \frac{C_x^2}{4}f_q-\rho C_yC_xf_p<f_qC_x^2-2\rho C_yC_xf_p$$

$$\text{or if } K<\frac{3}{4}\frac{f_q}{f_p}.$$

Hence the proof of result (32) is complete.