

Multiple Solutions for a Class of Semilinear Elliptic Equations with Nonlinear Boundary Conditions

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ABSTRACT

In this paper, using Local Linking Theorem, we obtain the existence of multiple solutions for a class of semilinear elliptic equations with nonlinear boundary conditions, in which the nonlinearities are compared with higher Neumann eigenvalue and the first Steklov eigenvalue.

KEYWORDS

Multiple Solutions; Nonlinear Boundary Conditions; Local Linking Theorem

1. Introduction

In this paper, we investigate the multiple solutions for semilinear elliptic equation with nonlinear boundary conditions

$$\begin{cases} -\Delta u + c(x)u = f(x, u), & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g(x, u), & x \in \partial\Omega \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu} = \nu \cdot \nabla$ is the outward normal derivative on $\partial\Omega$, and the function $c(x)$ satisfies

$$C) \quad c(x) \in L^\infty(\Omega), \text{ and } c(x) \geq 0 \text{ a.e. on } \Omega \text{ with } \int_{\Omega} c(x) dx > 0.$$

Problems of the above type have been discussed extensively. In 1902, Steklov (see [1]) studied the eigenvalue problem

$$\begin{cases} -\Delta u = 0, & x \in \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u, & x \in \partial\Omega \end{cases} \quad (1.2)$$

Auchmuty (see [2]) considered the eigenvalue problem

$$\begin{cases} -\Delta u(x) + c(x)u = 0, & x \in \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u(x), & x \in \partial\Omega \end{cases} \quad (1.3)$$

where $c(x)$ satisfies the condition C) and proved that the eigenfunctions provide a complete orthonormal

bases of certain closed subspace of $H^1(\Omega)$. Using sub and super-solutions method, Amann (see [3]), Mawhin and Schmitt (see [4]) obtained some existence results for the problem (1.1). However, since it is based on comparison techniques, the sub and super-method does not apply when the nonlinearities are compared with higher eigenvalues.

In this paper, using the Local Linking Theorem, we obtain multiple solutions for the problem (1.1), which the nonlinearities are compared with higher Neumann eigenvalue and the first Steklov eigenvalue.

2. Preliminaries and Main Results

Let $L^p(\Omega)$ denote the Lebesgue space with the norm $\|u\|_p = \left(\int_{\Omega} |u|^p dx\right)^{\frac{1}{p}}$, and $L^p(\partial\Omega, d\sigma)$ with the norm

$\|u\|_{p, \partial\Omega} = \left(\int_{\partial\Omega} |u|^p d\sigma\right)^{\frac{1}{p}}$. Obviously, the space $L^2(\Omega)$ and the space $L^2(\partial\Omega, d\sigma)$ inner product are denoted by

$$(u, v) = \int_{\Omega} uv dx, (u, v)_{\partial} = \int_{\partial\Omega} uv d\sigma.$$

$H^1(\Omega)$ is a Hilbert space under the standard inner product

$$[u, v]_1 = \int_{\Omega} (u(x)v(x) + \nabla u(x) \nabla v(x)) dx, \tag{2.1}$$

with the associated norm $\|u\|_1$. As the function $c(x)$ satisfies the condition C), we define the weighted $H^1 - c$ -inner product by

$$[u, v]_c = \int_{\Omega} (c(x)u(x)v(x) + \nabla u(x) \nabla v(x)) dx, \tag{2.2}$$

and the associated norm $\|u\|_c$. By Corollary 3.3 in [2], we obtain that the norm $\|u\|_c$ is equivalent to the standard norm $\|u\|_1$. As the function $c(x)$ satisfies the condition C), by Equation (2.2), we can split

$$H^1(\Omega) = H_0^1(\Omega) \oplus_c [H_0^1(\Omega)]^{\perp},$$

as a direct orthogonal sum.

Now, we state the Local Linking theorem introduced by [5].

Lemma 2.1 Let X is a reflexive Banach space, $X = Y \oplus V$ with $\dim V < +\infty, \Phi \in C^1(X, \mathbb{R})$ satisfies the (PS) condition, if

1) there exists a constant $\rho > 0$ such that

$$\Phi(x) \leq 0, \forall x \in V \text{ with } \|x\|_V \leq \rho,$$

$$\Phi(x) \geq 0, \forall x \in Y \text{ with } \|x\|_Y \leq \rho,$$

2) Φ is bounded below and $\inf_X \Phi < 0$, then the functional Φ has at least two nontrivial critical points.

Proof. See Theorem 4 in [5].

For the problem (1.3), Auchmuty (see [2]) obtained that

$$\mu_1 \oint_{\partial\Omega} (\Gamma u)^2 d\sigma \leq \int_{\Omega} (|\nabla u|^2 + c(x)u^2) dx,$$

holds for all $u \in H^1(\Omega)$, where μ_1 is the first Steklov eigenvalue for the problem (1.3). In [6], for the Neuman eigenvalue problem

$$\begin{cases} -\Delta u(x) + c(x)u = \lambda u(x), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

they obtain that the above problem has a sequence of real eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \rightarrow \infty, \text{ as } j \rightarrow \infty, \tag{2.3}$$

with finite dimensional eigenspaces.

Assume that, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions satisfying

H1) There exist $c_1 > 0$ and $2 < q_1 < 2^* = \frac{2N}{N-2}$ such that

$$|f(x, t)| \leq c_1 (1 + |t|^{q_1-1}),$$

for all $t \in \mathbb{R}$ and for a.e. $x \in \Omega$;

H2) There exist $c_2 > 0$ and $2 < q_2 < 2_* = \frac{2(N-1)}{N-2}$ such that

$$|g(x, t)| \leq c_2 (1 + |t|^{q_2-1}),$$

for all $t \in \mathbb{R}$ and for a.e. $x \in \partial\Omega$;

H3) There exist $\lambda, \mu > 0$ such that

$$\liminf_{|u| \rightarrow \infty} \frac{g(x, u)}{u} \leq \limsup_{|u| \rightarrow \infty} \frac{g(x, u)}{u} \leq \mu < \mu_1$$

and

$$\liminf_{|u| \rightarrow \infty} \frac{f(x, u)}{u} \leq \limsup_{|u| \rightarrow \infty} \frac{f(x, u)}{u} \leq \lambda < \lambda_1,$$

uniformly for $x \in \bar{\Omega}$ with

$$\lambda_1 \mu + \mu_1 \lambda < \mu_1 \lambda_1.$$

H4) There exist a integer $k \geq 1, \delta > 0$ and four constants $\alpha, \beta, a, b \in \mathbb{R}$ such that

$$\frac{1}{2} a t^2 \leq G(x, t) \leq \frac{1}{2} b t^2 \text{ and } \frac{1}{2} \alpha t^2 \leq F(x, t) \leq \frac{1}{2} \beta t^2$$

uniformly for $x \in \bar{\Omega}$, $0 < |t| < \delta$, where $F(x, t) = \int_0^t f(x, s) ds$, $G(x, t) = \int_0^t g(x, s) ds$,

$$\lambda_k < \alpha \leq \beta < \lambda_{k+1}, \lambda_k \mu_1 < \lambda_k a + \mu_1 \alpha \text{ and } \lambda_{k+1} b + \mu_1 \beta < \lambda_{k+1} \mu_1$$

Theorem 2.2 Suppose $c(x)$ satisfies C), and H1)-H4) hold, Then the problem (1.1) has at least two distinct nontrivial solutions.

3. The Proofs of Theorem 2.2

Now, we define the functional $\Phi : H^1(\Omega) \rightarrow \mathbb{R}$

$$\Phi(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + c(x)u^2) dx - \int_{\Omega} F(x, u) dx - \int_{\partial\Omega} G(x, u) d\sigma = \frac{1}{2} \|u\|_c^2 - J(u), \tag{3.1}$$

where $J(u) = \int_{\Omega} F(x, u) dx + \int_{\partial\Omega} G(x, u) d\sigma$.

Since the function $f(x, u)$ satisfies H1), $g(x, u)$ satisfies H2), by the Sobolev embedding of $H^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$, the continuity of the trace operator from $H^1(\Omega)$ into $L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$ and the Holder inequality, we obtain that the functional $\Phi(u)$ is well defined. Moreover, by Lemma 2.1, and Lemma 4.2 in [7], we obtain that $\Phi \in C^1(H^1(\Omega), \mathbb{R})$, and

$$\Phi'(u)v = \int_{\Omega} (\nabla u \nabla v + c(x)uv) dx - \int_{\Omega} f(x,u)v dx - \int_{\partial\Omega} g(x,u)v d\sigma, \forall v \in H^1(\Omega). \tag{3.2}$$

Furthermore, the functional $J(u)$ is weakly continuous, and $J'(u)$ is compact. Let $u = v$ in (3.2) and a simple computation, we obtain that the critical point of the functional $\Phi(u)$ is the weak solution of the problem (1.1).

Lemma 3.1 (see [7]) Assume that the function $c(x)$ satisfies the condition C), H1) and H2) hold. If $\{u_m\}$ is a (PS) sequence for the functional Φ , and $\{u_m\}$ is bounded in $H^1(\Omega)$, then $\{u_m\}$ has a strongly convergence subsequence. *i.e.* $\Phi(u)$ satisfies the (PS) condition.

Lemma 3.2 Assume that $c(x)$ satisfies the condition C), and H1)-H3) hold, the functional $\Phi(u)$ is coercive on $H^1(\Omega)$.

Proof By H3), we obtain that there exist some constants $0 < \varepsilon < \lambda$, $0 < \varepsilon < \mu$, and $M_1 > 0$, $C > 0$ such that

$$F(x,t) \leq \frac{1}{2}(\lambda - \varepsilon)t^2 + c, \forall |t| \geq M_1, G(x,t) \leq \frac{1}{2}(\mu - \varepsilon)t^2 + c, \forall |t| \geq M_1. \tag{3.3}$$

From H1), H2) and Equation (3.3), we obtain that there exist $f_0 \in L^1(\Omega)$, $g_0 \in L^1(\partial\Omega)$ such that

$$F(x,u) \leq \frac{1}{2}(\lambda - \varepsilon)u^2 + f_0, G(x,u) \leq \frac{1}{2}(\mu - \varepsilon)u^2 + g_0, \tag{3.4}$$

Hence, we obtain that

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + c(x)u^2) dx - \int_{\Omega} F(x,u) dx - \int_{\partial\Omega} G(x,u) d\sigma \\ &\geq \frac{1}{2} \|u\|_c^2 - \frac{1}{2}(\lambda - \varepsilon) \int_{\Omega} u^2 dx - \frac{1}{2}(\mu - \varepsilon) \int_{\partial\Omega} u^2 d\sigma - \int_{\Omega} f_0 dx - \int_{\partial\Omega} g_0 d\sigma \end{aligned} \tag{3.5}$$

Assume that $\|u\|_c \rightarrow \infty$, then using the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$, we obtain either $\|u\|_{\partial} \rightarrow \infty$, or $\|u\|_{\partial} \leq M_2$, where M_2 is a positive constant.

Case 1 As $\|u\|_{\partial} \leq M_2$, by Equation (3.5), we obtain

$$\begin{aligned} \Phi(u) &\geq \frac{1}{2} \|u\|_c^2 - \frac{1}{2} \left(\frac{\lambda}{\lambda_1} - \frac{\varepsilon}{\lambda_1} \right) \|u\|_c^2 - \frac{1}{2}(\mu - \varepsilon) \int_{\partial\Omega} u^2 d\sigma - \int_{\Omega} f_0 dx - \int_{\partial\Omega} g_0 d\sigma \\ &\geq \left(1 - \frac{\lambda}{\lambda_1} + \frac{\varepsilon}{\lambda_1} \right) \|u\|_c^2 - \frac{1}{2}(\mu - \varepsilon) \int_{\partial\Omega} u^2 d\sigma - \int_{\Omega} f_0 dx - \int_{\partial\Omega} g_0 d\sigma. \end{aligned}$$

Hence, we obtain that $\Phi(u)$ is coercive on $H^1(\Omega)$ since $\lambda < \lambda_1$.

Case2 As $\|u\|_{\partial} \rightarrow \infty$, we have

$$\begin{aligned} \Phi(u) &\geq \left(1 - \frac{\lambda}{\lambda_1} + \frac{\varepsilon}{\lambda_1} \right) \|u\|_c^2 - \frac{1}{2}(\mu - \varepsilon) \int_{\partial\Omega} u^2 d\sigma - \int_{\Omega} f_0 dx - \int_{\partial\Omega} g_0 d\sigma \\ &\geq \left(1 - \frac{\lambda}{\lambda_1} - \frac{\mu}{\mu_1} + \frac{\varepsilon}{\lambda_1} + \frac{\varepsilon}{\mu_1} \right) \|u\|_c^2 - \int_{\Omega} f_0 dx - \int_{\partial\Omega} g_0 d\sigma \end{aligned}$$

By H3), we obtain $\mu_1 \lambda_1 - \mu_1 \lambda - \mu \lambda_1 > 0$, then $\Phi(u)$ is coercive on $H^1(\Omega)$.

Hence we obtain that the functional $\Phi(u)$ is bounded from below, and every (PS) sequence $\{u_n\}$ is bounded in $H^1(\Omega)$. From Lemma 3.1, we obtain that $\Phi(u)$ satisfies (PS) condition and is bounded from below.

The Proof of Theorem 2.2 We write $L^2(\Omega) = V \oplus V^\perp$, where

$$V = E(\lambda_1) \oplus E(\lambda_2) \oplus \dots \oplus E(\lambda_k) \quad \text{and} \quad V^\perp = \overline{E(\lambda_{k+1}) \oplus E(\lambda_{k+2}) \oplus \dots}$$

Hence, we have

$$H^1(\Omega) = (V \cap H^1(\Omega)) \oplus (V^\perp \cap H^1(\Omega)).$$

Since $(V \cap H^1(\Omega))$ is a finite dimensional space, by [2], we obtain that for given $\delta > 0$, there is a $\rho_1 > 0$ such that

$$u \in (V \cap H^1(\Omega)), \|u\|_c \leq \rho_1 \Rightarrow |u(x)| \leq \delta, \text{ a.e. } x \in \Omega.$$

Hence, for each $u \in (V \cap H^1(\Omega)), \|u\|_c \leq \rho_1$, by H4), we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|u\|_c^2 - \int_{\Omega} F(x, u) dx - \oint_{\partial\Omega} G(x, u) d\sigma \leq \frac{1}{2} \|u\|_c^2 - \frac{1}{2} (\alpha - \varepsilon) \int_{\Omega} u^2 dx - \frac{1}{2} (a - \varepsilon) \oint_{\partial\Omega} u^2 d\sigma \\ &\leq \frac{1}{2} \left(1 - \frac{a}{\mu_1} + \frac{\varepsilon}{\mu_1} \right) \|u\|_c^2 - \frac{1}{2} (\alpha - \varepsilon) \int_{\Omega} u^2 dx \leq \frac{1}{2} \left(1 - \frac{a}{\mu_1} - \frac{\alpha}{\lambda_k} + \frac{\varepsilon}{\mu_1} + \frac{\varepsilon}{\lambda_k} \right) \|u\|_c^2 \end{aligned}$$

We have for $\varepsilon > 0$ sufficiently small,

$$\frac{1}{2} \left(1 - \frac{a}{\mu_1} - \frac{\alpha}{\lambda_k} + \frac{\varepsilon}{\mu_1} + \frac{\varepsilon}{\lambda_k} \right) < 0.$$

Therefore, we obtain that $\Phi(u) \leq 0$.

On the other hand, let $\Omega_1 = \{x \in \bar{\Omega} : |u(x)| \leq \delta\}$ and $\Omega_2 = \{x \in \bar{\Omega} : |u(x)| > \delta\}$, then for every $u \in (V^\perp \cap H^1(\Omega))$ with $|u(x)| \leq \delta$, by H4), we obtain that

$$\int_{\Omega} \left(\frac{1}{2} (\beta + \varepsilon) u^2 - F(x, u) \right) dx \geq 0, \text{ and } \oint_{(\partial\Omega \cap \Omega_1)} \left(\frac{1}{2} (b + \varepsilon) u^2 - G(x, u) \right) dx \geq 0. \tag{3.6}$$

Combining Equation (3.4), Equation (3.6), H1) and H2), we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|u\|_c^2 - \int_{\Omega} F(x, u) dx - \oint_{\partial\Omega} G(x, u) d\sigma \\ &= \frac{1}{2} \|u\|_c^2 - \int_{\Omega_1} F(x, u) dx - \oint_{(\partial\Omega \cap \Omega_1)} G(x, u) d\sigma - \int_{\Omega_2} F(x, u) dx - \oint_{(\partial\Omega \cap \Omega_2)} G(x, u) d\sigma \\ &\geq \frac{1}{2} \|u\|_c^2 - \frac{(\beta + \varepsilon)}{2} \int_{\Omega} u^2 dx - \frac{(b + \varepsilon)}{2} \oint_{\partial\Omega} u^2 d\sigma - c_1 \|u\|_c^{q_1} - c_2 \|u\|_c^{q_2} - \int_{\Omega} f_0 dx - \oint_{\partial\Omega} g_0 d\sigma \\ &\geq \frac{1}{2} \left(1 - \frac{b}{\mu_1} - \frac{\varepsilon}{\mu_1} \right) \|u\|_c^2 - \frac{1}{2} (\beta + \varepsilon) \int_{\Omega} u^2 dx - c_1 \|u\|_c^{q_1} - c_2 \|u\|_c^{q_2} - \int_{\Omega} f_0 dx - \oint_{\partial\Omega} g_0 d\sigma \\ &\geq \frac{1}{2} \left(1 - \frac{b}{\mu_1} - \frac{\beta}{\lambda_{k+1}} - \frac{\varepsilon}{\mu_1} - \frac{\varepsilon}{\lambda_{k+1}} \right) \|u\|_c^2 - c_1 \|u\|_c^{q_1} - c_2 \|u\|_c^{q_2} - \int_{\Omega} f_0 dx - \oint_{\partial\Omega} g_0 d\sigma \end{aligned}$$

From Equation (2.6), we have, for $\varepsilon > 0$ sufficiently small

$$\frac{1}{2} \left(1 - \frac{b}{\mu_1} - \frac{\beta}{\lambda_{k+1}} - \frac{\varepsilon}{\mu_1} - \frac{\varepsilon}{\lambda_{k+1}} \right) > 0.$$

Since $q_1, q_2 > 2$, we can choose $\rho_2 > 0$ sufficiently small and $u \in (V^\perp \cap H^1(\Omega)), \|u\|_c \leq \rho_2$, such that the functional $\Phi(u) \geq 0$.

By Lemma 3.2, we obtain that $\Phi(u)$ satisfies (PS) condition and is bounded from below. If $\inf_{u \in H^1(\Omega)} \Phi(u) < 0$, then by Lemma 2.1, Φ possesses two nonzero critical point. From (3.3), we obtain that there exist two nontrivial weak solutions for the problem (1.1).

4. Conclusion

Using Local Linking Theorem, we obtain the existence of two nontrivial weak solutions for the problem (1.1)

which the nonlinearities $f(x,u)$ and $g(x,u)$ are compared with higher Neumann eigenvalue and the first Steklov eigenvalue.

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