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Asymptotic Value of the Probability That the First Order Statistic Is from Null Hypothesis

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ABSTRACT

When every element of a random vector $\underline{X} = (X_1, X_2, \cdots, X_n)$ assumes the cumulative distribution function F_0 and F_1 with probability p and $1-p$, respectively, we have shown that the probability $\mathcal{S}_0$ that the first order statistic of $\underline{X}$ is originally under F_0 can be expressed as $\mathcal{S}_0 = np\int_{-\infty}^{\infty}\{1-(1-p)F_1(x)-pF_0(x)\}^{n-1}\mathrm{d}F_0(x)$. We have also shown that $\lim_{n\to\infty}\mathcal{S}_0 = \frac{p}{p+(1-p)\xi}$, where $\xi = \lim_{x\to\bar{x}}\frac{F_1(x)}{F_0(x)}$ and $\bar{x} = \max(x_0, x_1)$ with $[x_i, \infty]$ the support of $F_i(x)$. Applications and implications of the results are discussed in the performance of wideband spectrum sensing schemes.

Keywords: Energy; Order Statistic; Probability; Spectrum Sensing

1. Introduction

In wideband spectrum sensing (WSS) of wireless communications with a multiple of receive antennas, the primary goal is to find vacant subbands in a wideband channel composed of a multitude of frequency bands [1-4]. It is beneficial in the WSS to have an accurate estimate of the noise variance. For example, in the detection scheme proposed in [5], estimation of the noise variance in a subband is performed based on the observations in all the subbands. The accuracy of the estimate of the noise variance can be shown to depend on the distributions of the observations under the null and alternative hypotheses. The key parameter in the estimation is the probability that the subband with the lowest energy is under the null hypothesis.

In this paper, we focus on the asymptotic value of the probability and discuss its implications in the performance of WSS schemes.

2. The Probability

Let F_i be an absolutely continuous cumulative distribution function (cdf) for $i=0,1$ and p be a number in the open interval $(0,1)$. Consider a vector $\underline{X} = (X_1, X_2, \cdots, X_n)$ of independent random variables, each of which assumes the cdf F_0 and F_1 with probability p and $1-p$, respectively: On the average, np random variables of the sample $\underline{X}$ has population cdf F_0 and the rest has F_1 as the population cdf. Denote by $\mathcal{S}_i = \Pr\{X_{(1)} \leftarrow F_i\}$ the probability that the first order statistic $X_{(1)}$ of $\underline{X}$ is one of the random variables [6,7] having the cdf F_i by for $i=0$ and 1.

Lemma 1. *The probability* $\mathcal{S}_i$ *can be expressed as*

$$\mathcal{S}_0 = np\int_{-\infty}^{\infty}\{1-pF_0(x)-(1-p)F_1(x)\}^{n-1} f_0(x)\mathrm{d}x \quad (1)$$

and

$$\mathcal{S}_1 = n(1-p)\int_{-\infty}^{\infty}\{1-pF_0(x)-(1-p)F_1(x)\}^{n-1}\cdot f_1(x)\mathrm{d}x, \quad (2)$$

where $f_i(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_i(x)$ denotes the probability density function (pdf) corresponding to the cdf F_i for $i=0,1$.

Proof 1. *By* E_k *let us denote the event that the cdf of*

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k random variables among the n random variables of $\underline{X}$ is F_0 and that of the rest $n-k$ random variables is F_1. Then, since $\underline{X}=(X_1,X_2,\cdots,X_n)$ is an independent random vector, we easily get [8]

$$\Pr\{E_k\}={}_nC_k p^k(1-p)^{n-k}. \quad (3)$$

We can assume that $X_j \sim F_0$ for $j=1,2,\cdots,k$ and $X_j \sim F_1$ for $j=k+1,k+2,\cdots,n$ under E_k without loss of generality. Then we easily get

$$\begin{aligned}\Pr\left\{X_{(1)} \leftarrow F_0 \middle| E_k\right\} &= \sum_{j=1}^{k}\Pr\left\{X_j = X_{(1)}, X_j \sim F_0 \middle| E_k\right\} \\ &= k\Pr\left\{X_1 = X_{(1)}, X_1 \sim F_0 \middle| E_k\right\}.\end{aligned} \quad (4)$$

Denoting the pdf of X_i and the joint pdf of $\underline{X}$ under E_k by $f_{X_i}(x_i|E_k)$ and $f_{\underline{X}}(\underline{x}|E_k)$, respectively, we have $f_{\underline{X}}(\underline{x}|E_k)=\prod_{i=1}^{k} f_0(x_i)\prod_{i=k+1}^{n} f_1(x_i)$, where $\underline{x}=(x_1,x_2,\cdots,x_n)$. Therefore,

$$\begin{aligned}&\Pr\left\{X_1 = X_{(1)}, X_1 \sim F_0 \middle| E_k\right\} = \Pr\left\{X_1 < X_2, X_1 < X_3, \cdots, X_1 < X_n \middle| E_k\right\} \\ &= \int_{-\infty}^{\infty}\int_{x_2=x_1}^{\infty}\int_{x_3=x_1}^{\infty}\cdots\int_{x_n=x_1}^{\infty}\left\{\prod_{i=1}^{k} f_0(x_i)\cdot\prod_{i=k+1}^{n} f_1(x_i)\right\}\mathrm{d}x_n\mathrm{d}x_{n-1}\cdots\mathrm{d}x_1 = \int_{-\infty}^{\infty}\{1-F_0(x)\}^{k-1}\{1-F_1(x)\}^{n-k}\cdot f_0(x)\mathrm{d}x.\end{aligned} \quad (5)$$

Thus, from (4) and (5), we have

$$\Pr\left\{X_{(1)} \leftarrow F_0 \middle| E_k\right\} = k\int_{-\infty}^{\infty}\{1-F_0(x)\}^{k-1}\{1-F_1(x)\}^{n-k}\cdot f_0(x)\mathrm{d}x. \quad (6)$$

Finally, recollecting that $\sum_{k=0}^{n} k\,{}_nC_k x^{k-1} = n(1+x)^{n-1}$ [7,9] and combining (3) and (6), we get (1) as

$$\begin{aligned}\mathcal{S}_0 &= \sum_{k=0}^{n}\Pr\left\{X_{(1)} \leftarrow F_0 \middle| E_k\right\}\Pr\{E_k\} = p\int_{-\infty}^{\infty}\left(\sum_{k=0}^{n} k\,{}_nC_k\left[p\{1-F_0(x)\}\right]^{k-1}\left[(1-p)\{1-F_1(x)\}\right]^{n-k}\right)f_0(x)\mathrm{d}x \\ &= np\int_{-\infty}^{\infty}\{1-pF_0(x)-(1-p)F_1(x)\}^{n-1}\cdot f_0(x)\mathrm{d}x.\end{aligned} \quad (7)$$

Following similar steps, we can show (2).

It is straightforward to see that

$$\begin{aligned}\mathcal{S}_0+\mathcal{S}_1 &= n\int_{-\infty}^{\infty}\{1-pF_0(x)-(1-p)F_1(x)\}^{n-1} \\ &\quad\cdot\{pf_0(x)+(1-p)f_1(x)\}\mathrm{d}x = 1\end{aligned} \quad (8)$$

irrespective of the values of n and p, by letting $1-pF_0(x)-(1-p)F_1(x)=t$, and therefore, $\mathrm{d}t=-\{pf_0(x)+(1-p)f_1(x)\}\mathrm{d}x$.

Next, we have

$$0 \le 1-pF_0(x)-(1-p)F_1(x) \le 1-pF_0(x) \quad (9)$$

since $1-p \ge 0$, $F_i(x) \ge 0$, and $pF_0(x)+(1-p)F_1(x) \le \max(F_0(x),F_1(x)) \le 1$. Thus, noting that

$$\begin{aligned}\overline{I}_n(a,b) &\overset{\Delta}{=} np\int_a^b\{1-pF_0(x)\}^{n-1} f_0(x)\mathrm{d}x \\ &= \{1-pF_0(a)\}^n-\{1-pF_0(b)\}^n,\end{aligned} \quad (10)$$

we get $0 \le \mathcal{S}_0 \le \overline{I}_n(-\infty,\infty)$ or

$$0 \le \mathcal{S}_0 \le 1-(1-p)^n \quad (11)$$

from (1) and (9) since $F_0(-\infty)=0$ and $F_0(\infty)=1$.

3. Asymptotic Value and Its Implications

3.1. Asymptotic Value of $\mathcal{S}_0$

Let us now obtain the value $\lim_{n\to\infty}\mathcal{S}_0$ more specifically.

Lemma 2. *Define*

$$I_n(a,b)=np\int_a^b\{1-pF_0(x)-(1-p)F_1(x)\}^{n-1}\cdot f_0(x)\mathrm{d}x, \quad (12)$$

where $a \le b$. Then, we have

$$\lim_{n\to\infty} I_n(a,b)=0 \text{ if } F_0(a)=F_0(b) \text{ or } F_0(a)>0. \quad (13)$$

Proof 2. *Recollecting that the pdf $f_0(x)$ is non-negative and* (9) *holds at any point x, it is clear that*

$$0 \le I_n(a,b) \le \overline{I}_n(a,b). \quad (14)$$

Now, since

$$\lim_{n\to\infty}\overline{I}_n(a,b)=\begin{cases}0, & \text{if } F_0(a)=F_0(b) \text{ or } F_0(a)>0, \\ 1, & \text{if } F_0(a)=0<F_0(b)\end{cases} \quad (15)$$

from (10), we immediately have (13) from (14).

Theorem 1. *For $i=0,1$, let the support of the cdf $F_i(x)$ be $[x_i,\infty]$. Then, we have*

$$\lim_{n\to\infty}\mathcal{S}_0=\frac{p}{p+(1-p)\xi}, \quad (16)$$

where

$$\xi = \lim_{x \to \bar{x}} \frac{F_1(x)}{F_0(x)} \tag{17}$$

assumes a non-negative value with $\bar{x} = \max(x_0, x_1)$.

Remark 1. *It is clear from* (16) *that* $\lim_{n\to\infty} \mathcal{S}_0 \ge p$ *if* $0 \le \xi \le 1$, $\lim_{n\to\infty} \mathcal{S}_0 = p$ *if* $\xi = 1$, *and* $\lim_{n\to\infty} \mathcal{S}_0 \le p$ *if* $\xi \ge 1$.

Remark 2. *When* $x_0 < x_1$ *and* $x_0 > x_1$, *we have* $\xi = 0$ *and* $\xi \to \infty$, *respectively. On the other hand, when* $x_0 = x_1$, *the value of the non-negative parameter* ξ *depends on* F_0 *and* F_1. *Based on this observation,* (16) *can be expressed as*

$$\lim_{n\to\infty} \mathcal{S}_0 = \begin{cases} 1, & \text{if } x_0 < x_1, \\ \dfrac{p}{p + (1-p)\xi}, & \text{if } x_0 = x_1, \\ 0, & \text{if } x_0 > x_1. \end{cases} \tag{18}$$

Proof 3. (Proof of Theorem) *Assume* $x_0 < x_1$, *in which case we have* $\xi = 0$. *Since* $F_1(x) = 0$ *for all* $x \le x_1$, *we have*

$$\begin{aligned} \mathcal{S}_0 &= np\int_{-\infty}^{x_T} \{1 - pF_0(x)\}^{n-1} f_0(x)\,\mathrm{d}x \\ &\quad + np\int_{x_T}^{\infty} \{1 - pF_0(x) - (1-p)F_1(x)\}^{n-1} f_0(x)\,\mathrm{d}x \\ &= \bar{I}_n(-\infty, x_T) + I_n(x_T, \infty), \end{aligned} \tag{19}$$

where x_T is a number in the interval $(x_0, x_1]$. Now, we have $\lim_{n\to\infty} \bar{I}_n(-\infty, x_T) = 1$ from (15) since $F_0(-\infty) = 0 < F_0(x_T)$, and we have $\lim_{n\to\infty} I_n(x_T, \infty) = 0$ from (13) since $F_0(x_T) > 0$, resulting in $\lim_{n\to\infty} \mathcal{S}_0 = 1$: This result and (8) will after some steps provide us with $\lim_{n\to\infty} \mathcal{S}_0 = 1$, and consequently, $\lim_{n\to\infty} \mathcal{S}_0 = 0$, when $x_0 > x_1$. Here, recollect that $x_0 > x_1$ implies $\xi \to \infty$.

Next, when $x_0 = x_1$ and $\xi = \lim_{x \to x_0} \frac{F_1(x)}{F_0(x)}$ are both finite, we can approximate $F_1(x)$ as $F_1(x) \approx \xi F_0(x)$ for a sufficiently small interval $[x_0, \bar{x}_0]$ of x, where $\bar{x}_0 > x_0$. Then, we can rewrite $\mathcal{S}_0$ as

$$\begin{aligned} \mathcal{S}_0 &= np\int_{x_0}^{\bar{x}_0} \{1 - pF_0(x) - (1-p)F_1(x)\}^{n-1} \\ &\quad \cdot f_0(x)\,\mathrm{d}x + I_n(\bar{x}_0, \infty) \\ &\approx np\int_{x_0}^{\bar{x}_0} \{1 - \theta F_0(x)\}^{n-1} f_0(x)\,\mathrm{d}x + I_n(\bar{x}_0, \infty) \\ &= \frac{p}{\theta}\left[1 - \{1 - \theta F_0(\bar{x}_0)\}^n\right] + I_n(\bar{x}_0, \infty) \end{aligned} \tag{20}$$

using (10) since $F_0(x_0) = 0$, where

$$\theta = p + (1-p)\xi \tag{21}$$

is a number larger than p. Now, choosing the number $\bar{x}_0$ in the open interval $\left(x_0, F_0^{-1}\left(\frac{1}{\theta}\right)\right)$, we will have $0 < 1 - \theta F_0(\bar{x}_0) < 1$. Then, we get

$$\lim_{n\to\infty} \mathcal{S}_0 = \frac{p}{p + (1-p)\xi} \tag{22}$$

from (20) by noting that $\lim_{n\to\infty} I_n(\bar{x}_0, \infty) = 0$ from (13) since $F_0(\bar{x}_0) > 0$.

When $x_0 = x_1 \to -\infty$ and ξ is finite, we can similarly show that (22) holds by employing the approximation $F_1(x) \approx \xi F_0(x)$ over an interval $(-\infty, \bar{x}_0]$, where $\bar{x}_0$ is now a sufficiently small negative, yet finite, number satisfying $-\infty < \bar{x}_0 < F_0^{-1}\left(\frac{1}{\theta}\right)$.

Finally, following steps similar to those leading to (22) obtained when $x_0 = x_1$ and ξ is finite, we can show that

$$\lim_{n\to\infty} \mathcal{S}_1 = \frac{1-p}{1 - p + p\dfrac{1}{\xi}} = 1 \tag{23}$$

quite immediately by symmetry when $x_0 = x_1$ and ξ is infinite: Combining this result with the relation $\mathcal{S}_0 + \mathcal{S}_1 = 1$ shown in (8), we have $\lim_{n\to\infty} \mathcal{S}_0 = 0$ when $x_0 = x_1$ and ξ is infinite.

Example 1. *Assume that* $F_0(x) = xu(x) - (x-1)\cdot u(x-1)$ *and* $F_1(x) = \frac{1}{2}\{xu(x) - (x-2)u(x-2)\}$, *where* $u(x) = 1$ *for* $x \ge 0$ *and* 0 *for* $x < 0$ *is the unit step function. Then*

$$\begin{aligned} \mathcal{S}_0 &= np\int_0^1 \left\{1 - px - (1-p)\frac{x}{2}\right\}^{n-1} \mathrm{d}x \\ &= \frac{2p}{1+p}\left\{1 - \left(\frac{1-p}{2}\right)^n\right\} \end{aligned} \tag{24}$$

Thus we have $\lim_{n\to\infty} \mathcal{S}_0 = \frac{2p}{1+p}$, which can also be obtained directly from (16) as $\frac{p}{p + \frac{1}{2}(1-p)} = \frac{2p}{1+p}$ using $\xi = \lim_{x\to 0} \frac{F_1(x)}{F_0(x)} = \frac{1}{2}$, is larger than or equal to p.

Example 2. *Assume that* $f_0(x) = \frac{1}{2}\mathrm{e}^{-|x|}$, $F_0(x) = \frac{1}{2}\mathrm{e}^{x}u(-x) + \left(1 - \frac{1}{2}\mathrm{e}^{-x}\right)u(x)$, *and* $F_1(x) = F_0(x - \mu)$. *When* $\mu = 0$, *we will obviously* *have* $\mathcal{S}_0 = p$. *Denoting* $\mu_0 = \min(0,\mu)$, *we next have*

$$\mathcal{S}_0 = np\int_{-\infty}^{\mu_0}\left[1-\left\{p+(1-p)\mathrm{e}^{-\mu}\right\}\frac{\mathrm{e}^x}{2}\right]^{n-1}\frac{\mathrm{e}^x}{2}\mathrm{d}x + I_n(\mu_0,\infty)$$
$$= \frac{p}{p+(1-p)\mathrm{e}^{-\mu}}\cdot\left(1-\left[1-\frac{1}{2}\left\{p+(1-p)\mathrm{e}^{-\mu}\right\}\mathrm{e}^{\mu_0}\right]^n\right)$$
$$+ I_n(\mu_0,\infty). \tag{25}$$

Since $0<\frac{1}{2}\left\{p+(1-p)\mathrm{e}^{-\mu}\right\}\mathrm{e}^{\mu_0}<1$ from $0<\mathrm{e}^{\mu_0}\le 1$ and $0<\mathrm{e}^{-\mu+\mu_0}\le 1$, we have $\lim_{n\to\infty}\mathcal{S}_0 = \frac{p}{p+(1-p)\mathrm{e}^{-\mu}}$: This value, which can also be obtained directly from (22) and (23) using

$$\lim_{x\to-\infty}\frac{F_1(x)}{F_0(x)} = \lim_{x\to-\infty}\frac{\mathrm{e}^{x-\mu}}{\mathrm{e}^x} = \mathrm{e}^{-\mu}, \tag{26}$$

is larger and smaller than p when μ is larger and smaller than zero, respectively, and converges to one and zero as $\mu\to\infty$ and $\mu\to-\infty$, respectively.

3.2. Discussion

In communication systems, we usually have $F_1(x)=F_0(x-\mu)$, where F_0 denotes the cdf of noise and $\mu>0$ can be regarded as a measure of the signal to noise ratio (SNR). Now, under the Gaussian, Cauchy, double exponential, and logistic [10] noise environments, from (16) we have $\lim_{n\to\infty}\mathcal{S}_0 = 1$, p, $\frac{p}{p+(1-p)\mathrm{e}^{-\mu}}$, and $\frac{p}{p+(1-p)\mathrm{e}^{-\mu}}$, respectively, since $\xi = \lim_{x\to-\infty}\frac{\exp(2\mu x)}{\exp(\mu^2)} = 0$, $\xi = \lim_{x\to-\infty}\frac{x^2}{(x-\mu)^2} = 1$, and $\xi = \lim_{x\to-\infty}\frac{\mathrm{e}^{x-\mu}}{\mathrm{e}^x} = \mathrm{e}^{-\mu}$. These observations imply that

1) we can estimate the noise variance correctly $\left(\lim_{n\to\infty}\mathcal{S}_0 = 1\right)$ by simply increasing the sample size at any positive SNR in the Gaussian case,

2) we can estimate the noise variance correctly by increasing both the sample size and SNR in the double-exponential and logistic cases, but

3) we cannot estimate the noise variance correctly with probability higher than p by increasing the sample size or SNR in the Cauchy case.

4. Summary

We have derived the probability that the first order statistic of a number of independent random variables is originally under the null hypothesis. We have also obtained the asymptotic value of the probability as the sample size tends to infinity, and then we discuss an application and implications of the results in the performance of wideband spectrum sensing schemes.

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