

# Observer for Linear Distributed-Parameter Systems with Application to Isothermal Plug-Flow Reactor

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## ABSTRACT

This paper presents a conception of an exponential observer for a class of linear distributed-parameter systems (DPSs), in which the dynamics are partially unknown. The given distributed-parameter observer ensures asymptotic state estimator with exponentially decay error, based on the theory of  $C_0$ -semigroups in a Hilbert space. The theoretical observer developed is applied to a chemical tubular reactor, namely the isothermal Plug-Flow reactor basic dynamical model for which measurements are available at the reactor output only. The process is described by Partial differential equations with unknown initial states. For this application, performance issues are illustrated in a simulation study.

**Keywords:** Distributed-Parameter Systems;  $C_0$ -semigroup; Exponential Observers; Perturbed Systems; Tubular Reactor

## 1. Introduction

In many physical systems (e.g., bio-reactor, vibrations problems in mechanics, diffusion problems), the states of the mathematical model depend on spatial variable, which is a position in a one-dimensional or multi-dimensional space. This kind of system is called distributed-parameter system(s) (DPSs) (see [1]). A powerful tool in the analysis of DPSs is the theory of  $C_0$ -semigroups (see [2,3]).

For the state feedback control, the exact and full knowledge of states of the system is important. However, the presence of spatial variables makes the state not available for direct measurements and that imposes limitations to the design. In such case, the states can be estimated using state estimators (observers). For this purpose, the classical theory of the Luenberger observer [4] has been extended from linear finite-dimensional systems to a large class of DPSs by many authors, (we mention as examples [5-10] and the references within). However, research on efficient and practicable observer design for DPSs has not been so extensive as in the case of finite-dimensional systems, and papers on distributed-parameter observers are scattered in the literature.

In This paper, we focus on estimating the states of linear DPSs with partially unknown dynamics. The paper

presents an exponential distributed-parameter observer design which is an asymptotic state estimator with exponentially decay error. The proposed conception is efficient and suitable in practice applications; for instance, it is appropriate to be applied to estimate the states of a chemical tubular reactor, namely the isothermal Plug-Flow Reactor basic dynamical model, with unknown initial state, in the case where measurements may occur at the reactor output only.

## 2. Infinite Dimensional Observer Design

Let consider the linear Infinite-Dimensional system given by

$$\begin{cases} \dot{x} = Ax(t) + Bu(t) \\ y(t) = Cx(t), x(0) = x_0 \end{cases} \quad (1)$$

Here,  $A$  is the infinitesimal generator of a  $C_0$ -semigroup on a real Hilbert space  $H$  with inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\| \cdot \|$ ,  $y(t) \in Y$  is the known output function associated to the unknown initial condition  $x(0)$ ,  $Y$  is another real Hilbert space and  $C$  is a bounded linear operator from  $H$  into  $Y$ . The dot over a variable means the time derivative of the variable.

The purpose is to design a dynamic system (observer or state estimator) for the system (1) by using  $y$  as the input such that the output of this designed dynamic system is used as an estimate of the current state  $x(t)$  of (1).

The initial state  $x(0)$  of (1) is unknown while the initial state  $\hat{x}(0)$  of the observer can be assigned arbitrarily. Thus, the error between  $x(0)$  and  $\hat{x}(0)$  is still an unknown quantity even if we know  $\hat{x}(0)$ . As a basic requirement in observer design, we require that if  $\hat{x}(0) = x(0)$  then  $\hat{x}(t) = x(t)$  for all  $t \geq 0$ . So, the observer for (1) can be expressed in the following form

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \\ \hat{x}(0) = \hat{x}_0. \end{cases} \tag{2}$$

where  $L$  is the observer gain operator.

**Proposition 2.1:** Given the linear infinite-dimensional system (1). Suppose that there exists a bounded linear operator  $L$  from  $H$  into  $H$  such that the linear operator  $A - LC$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $H$ , such that there exist constants  $M > 0$  and  $\rho > 0$  and a time  $t_\rho$  satisfying:

$$\|T(t)\| \leq M \exp(-\rho t), \forall t \geq t_\rho$$

then the dynamic system (2) with  $\hat{x}(0)$  arbitrary chosen is an exponential observer for system (1) and

$$\|\hat{x}(t) - x(t)\| \leq k \exp(-\rho t)$$

for all  $t \geq t_\rho$ , where  $k$  is a positive number depending on  $\hat{x}(0) - x(0)$ .

**Proof 1** Let consider the estimation error  $e(t) = \hat{x}(t) - x(t)$ , for all  $e(0) \in H$ , the evolution system,

$$\begin{cases} \dot{e}(t) = (A - LC)e(t) \\ e(0) = \hat{x}_0 - x_0 \end{cases} \tag{3}$$

has a unique mild solution on the interval  $[0, +\infty[$ , given by:  $e(t) = T(t)e(0)$ , for all  $t \geq 0$  (see [2,3]).

Hence,

$$\|e(t)\| \leq \|T(t)\| \|e(0)\|, \forall t \geq 0$$

Thus, the error satisfies:

$$\|e(t)\| \leq \|T(t)\| \exp(-\rho t), \forall t \geq t_\rho$$

and that implies that the norm of the difference  $\hat{x}(t) - x(t)$  will decrease exponentially to zero. This competes the proof.

Let now suppose that linear operator  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T_A(t))_{t \geq 0}$  and  $D$  is linear bounded operator on  $H$ . The following result will be needed in the sequel.

**Theorem 2.2:** [3] The operator  $A + D$  is the infinitesimal

generator of a  $C_0$ -semigroup  $(T_{A+D}(t))_{t \geq 0}$  which is the unique solution of the equation

$$T_{A+D}(t)x_0 = T_A(t)x_0 + \int_0^t T_A(t-s)DT_{A+D}(s)x_0 ds$$

for all  $x_0 \in H$ . If in addition,  $\|T_A(t)\| \leq M \exp(\omega t)$ , then,

$$\|T_{A+D}(t)\| \leq M \exp((\omega + M\|D\|)t)$$

### 3. Application to the Plug-Flow Reactor Basic Model

The theory presented in the linear infinite-dimensional sitting in Section 2, is applied to reconstruct the states of a chemical tubular reactor with the following chemical reaction:



where  $C_1$  is the reactant,  $C_2$  the product, and  $b > 0$  is the stoichiometric coefficient of the reaction. The dynamics of the process in a tubular reactor without axial dispersion are given, for all time  $t \geq 0$  and for all  $z \in [0, L]$  where  $L$  is the reactor length, by mass balance equations (see [11]):

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= -v \frac{\partial x_1}{\partial z} - k_0 x_1 + v \Delta_w x_{in}(t) \\ \frac{\partial x_2}{\partial t} &= -v \frac{\partial x_2}{\partial z} + b k_0 x_1 \end{aligned} \tag{5}$$

with the boundary conditions:

$$x_1(z=0, t) = 0, \quad x_2(z=0, t) = 0 \tag{6}$$

and the initial conditions:

$$x_1(z, t=0) = x_1^0, \quad x_2(z, t=0) = x_2^0 \tag{7}$$

where  $x_1(z, t), x_2(z, t), x_{in}(z, t)$  and  $v$  are the concentrations of  $C_1$  and  $C_2$  (mol/l), the influent reactant concentration (mol/l) and the fluid superficial velocity (m/s). We assume that the kinetics depend only on the reactant concentration  $x_1$  and we consider a reaction rate model of the form  $k_0 x_1$ , where  $k_0$  is the kinetic constant ( $s^{-1}$ ).  $\Delta_w(\cdot)$  denotes a finite unit impulse of window width  $w$ , i.e. an approximate Dirac delta distribution at  $z = 0$ , given by

$$\begin{cases} \Delta_w(z) = 1, & \text{if } z \in [0, w) \\ \Delta_w(z) = 0, & \text{elsewhere.} \end{cases}$$

The initial states  $x_1^0, x_2^0$  are supposed to be unknown hereafter.

Throughout the sequel, we assume  $H = L^2[0, L] \times L^2[0, L]$ , the Hilbert space with the usual inner product

$$\langle\langle (x_1, x_2), (y_1, y_2) \rangle\rangle = \langle x_1, y_1 \rangle_{L^2} + \langle x_2, y_2 \rangle_{L^2}$$

and the induced norm

$$\|(x_1, x_2)\| = \sqrt{\|x_1\|_{L^2}^2 + \|x_2\|_{L^2}^2}$$

for all  $(x_1, x_2)^T$  and  $(y_1, y_2)^T$  in  $H$ .

A partial-differential equations like (5)-(7), when expanded with an output equation, can also be expressed as an abstract state space equation on the Hilbert space  $H$  :

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), x(0) = x_0 \end{cases} \quad (8)$$

associated to the unknown initial condition

$x_0 := (x_1^0, x_2^0)^T$ . Where,  $\dot{x}$  stands for the time derivative of the state  $x(t) = (x_1(.,t), x_2(.,t))^T$ , and the control  $u(t) = x_{in}(t)$ . The control operator  $B$  is a bounded linear operator from  $IR^2$  to  $H$ , which is defined by

$$B = \begin{pmatrix} \nu \\ 0 \end{pmatrix} \Delta_w(\cdot)$$

The output trajectory  $C$  is a bounded linear operator, and the linear operator  $A$  is defined on its domain,

$$D(A) = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H : x \text{ is absolutely continuous, } \frac{dx}{dz} \in H \text{ and } x_{i=1,2}(0) = 0 \right\}$$

by,

$$Ax = \begin{pmatrix} -\nu \frac{d}{dz} - k_0 & 0 \\ bk_0 & -\nu \frac{d}{dz} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The operator  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T_A(t))_{t \geq 0}$  on  $H$ , exponentially stable, i.e., there exist constants  $M, k_1 \in IR^{*+}$  such that,

$$\|T_A(t)\| \leq M \exp(-k_1 t), \quad \forall t \geq 0$$

Satisfying,

$$T_A(t) = \begin{pmatrix} T_{11}(t) & 0 \\ T_{21}(t) & T_{22}(t) \end{pmatrix}$$

such that for all  $(x_1, x_2) \in H$ , (see [11] and the references within for more detail),

$$(T_{11}(t)x_1)(z) = \begin{cases} \exp(-k_0 t)x_1(z - \nu t) & \text{if } z \geq \nu t \\ 0 & \text{if } z < \nu t, \end{cases}$$

$$(T_{22}(t)x_2)(z) = \begin{cases} x_2(z - \nu t) & \text{if } z \geq \nu t, \\ 0 & \text{if } z < \nu t, \end{cases}$$

and

$$T_{21}(t)x_1 = \int_0^t T_{22}(t-s)bk_0T_{11}(s)ds.$$

**Remark 3.1** [11] It can be observed that the  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  is such that  $T(t)x=0$  for all  $x \in H$ , for all  $t > L/\nu$ , whence the stability bound is rather conservative and is only of interest for  $t$  small.

More specifically, Remark 3.1 could lead us to deduce intuitively that the growth bound  $\omega_0$  of the  $C_0$ -semigroup  $(T(t))_{t \geq 0}$ , defined as follows:

$$\omega_0 = \inf_{t \geq 0} \frac{\log \|T(t)\|}{t}$$

is equal to  $-\infty$ .

### 3.1. Observer Design

Hereafter we consider measurements of the state vector  $x(t)$  are available at the reactor output only. In this case, the output function  $y(\cdot)$  is defined as follows: we consider a (very small) finite interval at the reactor output  $[1-\omega, 1]$  such that:

$$\begin{aligned} y(t) &= (Cx)(t) \\ &:= \int_0^1 \chi_{[1-\omega, 1]}(a)x(a, t) da, \quad \forall t \in IR^+ \end{aligned} \quad (9)$$

where,  $\chi_{[1-\omega, 1]}(a) = 1$  if  $a \in [1-\omega, 1]$  and  $\chi_{[1-\omega, 1]}(a) = 0$  elsewhere, with  $0 < \omega < 1$  is a small number.

The observer operator  $C : H \rightarrow IR^2$  is linear bounded. For all  $x, y \in H \times IR^2$ ,

$$\begin{aligned} \langle Cx, y \rangle_{IR^2} &= \left\langle \int_0^1 \chi_{[1-\omega, 1]}(a)x(a, \cdot) da, y \right\rangle_{IR^2} \\ &= \int_0^1 \langle x(a, \cdot), \chi_{[1-\omega, 1]}(a)y \rangle_{IR^2} da \end{aligned}$$

The adjoint operator  $C^*$  of  $C$  is defined for all  $(z, t) \in [0, 1] \times IR^+$  by:

$$(C^*y)(z) = \chi_{[1-\omega, 1]}(z)y$$

For all  $x \in H$ ,

$$\begin{aligned} \|C^*Cx\|^2 &= \int_0^1 \left( \chi_{[1-\omega]}(z) \left( \int_0^1 \chi_{[1-\omega, 1]}(a)x(a, \cdot) da \right) \right)^2 dz \\ &= \omega \left( \int_0^1 \chi_{[1-\omega, 1]}(a)x(a, \cdot) da \right)^2 \\ &\leq \omega \|\chi_{[1-\omega, 1]}\|^2 \|x\|^2 = \omega^2 \|x\|^2 \end{aligned}$$

Then,  $\|C^*C\| \leq \omega$

A candidate observer for the system (5)-(7), is obtained as the output of the following dynamic system

$$\begin{aligned} \frac{\partial \hat{x}_1}{\partial t} &= -v \frac{\partial \hat{x}_1}{\partial z} - k_0 \hat{x}_1 + v \Delta_\omega x_{in}(t) + g C_1^* (C_1 x_1 - C_1 \hat{x}_1) \\ \frac{\partial \hat{x}_2}{\partial t} &= -v \frac{\partial \hat{x}_2}{\partial z} + b k_0 \hat{x}_1 + g C_2^* (C_2 x - C_2 \hat{x}_2) \end{aligned} \tag{10}$$

with the boundary conditions:

$$\begin{aligned} \hat{x}_1(z=0, t) &= 0, \\ \hat{x}_2(z=0, t) &= 0 \end{aligned} \tag{11}$$

and the initial conditions:

$$\begin{aligned} \hat{x}_1(z, t=0) &= \hat{x}_1^0, \\ \hat{x}_2(z, t=0) &= \hat{x}_2^0 \end{aligned} \tag{12}$$

With  $C = (C_1 \ C_2)^T$  defined by (9) and  $g$  the is a positive number.

The system (10)-(12) can be written on its compact form

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + GC^*C(x(t) - \hat{x}(t)) \\ \hat{y}(t) = C\hat{x}(t), \hat{x}(0) = \hat{x}_0 \end{cases} \tag{13}$$

where, the linear operator  $G$  is the observer gain, satisfy

$$G := \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} = gI \tag{14}$$

with  $I$  is the identity operator of the Hilbert  $H$ .

**Corollary 3.1:** Given the isothermal Plug-Flow reactor basic dynamical model (5)-(7). Suppose that there exists a bounded linear operator  $G = gI$ , where  $g$  is a positive number, such that  $g < k_0\omega$ , the dynamic system (10)-(12) is an exponential observer for the system (5)-(7).

**Proof 2** Let consider the linear operator  $G = gI$ , where  $g$  is a positive number. The operator  $GC^*C$  is a bounded linear operator on  $H$ , such that

$$\|GC^*C\| \leq g\omega.$$

On the other hand, from Remark 3.1 and the definition of the growth bound, there exists a time  $t_{k_0}$  such that

$$\frac{\log \|T(t)\|}{t} < -k_0, \quad \forall t \geq t_{k_0}$$

Hence,

$$\|T(t)\| \leq \exp(-k_0 t), \quad \forall t \geq t_{k_0}$$

Now, by the Theorem 2.2, the linear operator  $A - GC^*C$  is the infinitesimal generator of a  $C_0$ -semi-group  $(T_{A-GC^*C}(t))_{t \geq 0}$  satisfying, for all  $t \geq t_{k_0}$ :

$$\begin{aligned} \|T_{A-GC^*C}(t)\| &\leq \exp\left(\left(-k_0 + \|GC^*C\|\right)t\right) \\ &\leq \exp\left(\left(-k_0 + g\omega\right)t\right) \end{aligned}$$

If  $g \leq \frac{k_0}{\omega}$ , it follows by application of the Theorem 2.1,

that the dynamic system (10)-(12) is an exponential observer for the system (5)-(7). More precisely the reconstruction error  $\hat{x}(t) - x(t)$  satisfies, for all  $t \geq t_{k_0}$ ,

$$\|\hat{x}(t) - x(t)\| \leq \|\hat{x}(0) - x(0)\| \exp\left(\left(-k_0 + g\omega\right)t\right)$$

### 3.2. Simulation Result

In order to test the performance of the proposed observer, numerical simulations will be given. The equations have been integrated by using a backward finite difference approximation for the first-order space derivative  $\partial/\partial z$ . The adopted numerical values for the process parameters are taken from the **Table 1** (see [12]).

**Figures 1 and 2** show respectively the evolution in time and space of the error on the reactant concentration  $e_1 = \hat{x}_1 - x_1$  and the evolution in time and space of the error on the product concentration  $e_2 = \hat{x}_2 - x_2$ , related to the observer (10)-(12).

The measurements are taken on the length interval  $[3 \times L/4, L]$  i.e.,  $\omega = 3 \times L/4$ , and the process model has been arbitrary initialized with the constant profiles  $x_1(0, z) = 1, x_2(0, z) = 0, \hat{x}_1(0, z) = 0$ , and  $\hat{x}_2(0, z) = 1$ . In order to response to the assumption of the Corollary 3.1, we set  $g = \frac{1}{2} \frac{k_0}{\omega}$  for the observer design parameter.

### 4. Conclusions and Prospects

In this paper we present a conception of a state estimator for linear distributed-parameter systems, which ensures that the estimation error converges exponentially to zero. The theory developed is applied to reconstruct the state of the isothermal Plug-Flow reactor basic dynamical model, and performed by a simulation study in which the parameters can be tuned by the user to satisfy specific needs in terms of convergence rate.

One of the purposes in designing an observer is to obtain an efficient and practicable feedback control that stabilizes the original system around a desired profile. So the investigation of the stability of the overall closed-loop system (which is composed of original system, ob-

**Table 1. Process parameters for numerical simulations.**

Process parameters	Numerical values
$v$	0.025 m/s
$L$	1 m
$k_0$	$10^6 \text{ s}^{-1}$
$b$	2
$x_{in}$	0.02 mol/L

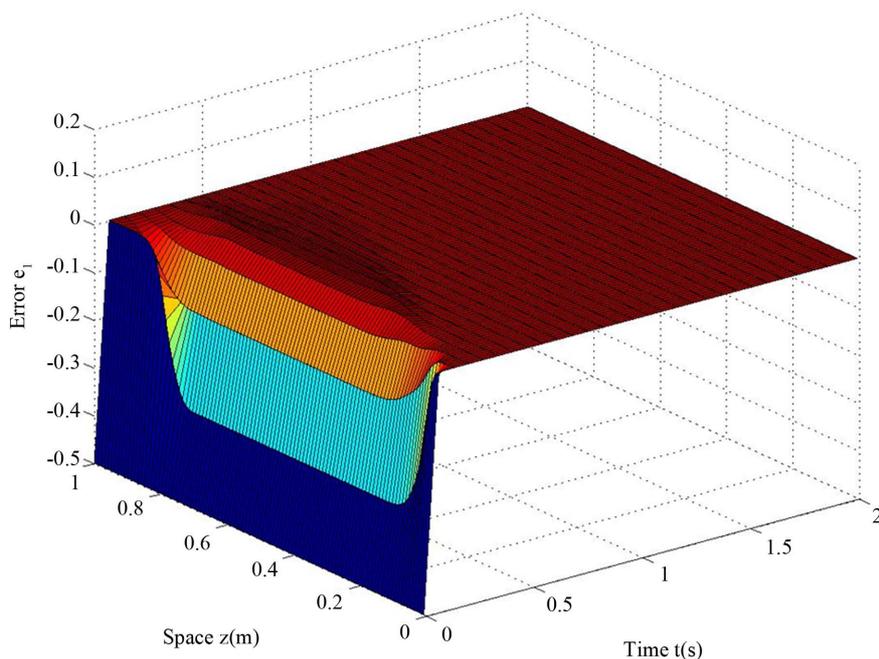


Figure 1. Error evolution of the reactant concentration.

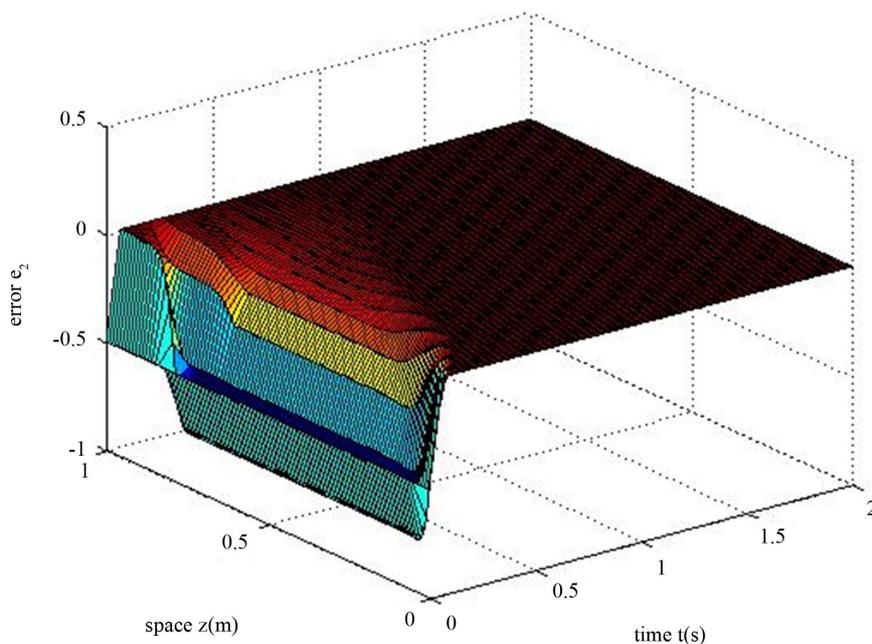


Figure 2. Error evolution of the product concentration.

server, and feedback controller), is an interesting topic for future research.

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