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Integral Mean Estimates for Polynomials Whose Zeros are within a Circle

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Abstract

Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \le K \le 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, Aziz and Ahemad (1996) proved that

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+Ke^{i\theta}\right|^{qr} d\theta\right\}^{\frac{1}{qr}} \left\{\int_0^{2\pi}\left|P'\left(e^{i\theta}\right)\right|^{pr} d\theta\right\}^{\frac{1}{pr}}.$$

In this paper, we extend the above inequality to the class of polynomials $P(z) := a_n z^n + \sum_{j=m}^{n} a_{n-j} z^{n-j}$, $1 \le m \le n$, having all its zeros in $|z| \le K \le 1$, and obtain a generalization as well as refinement of the above result.

Keywords: Derivative of a Polynomial, Integral Mean Estimates, Complex Domain Inequalities

1. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. If $P(z)$ has all its zeros in $|z| \le 1$, then it was shown by Turan [1] that

$$Max_{|z|=1}\left|P'(z)\right| \ge \frac{n}{2} Max_{|z|=1}\left|P(z)\right|. \tag{1}$$

Inequality (1) is best possible with equality for $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. As an extension of (1) Malik [2] proved that if $P(z)$ has all its zeros in $|z| \le K$, where $K \le 1$, then

$$Max_{|z|=1}\left|P'(z)\right| \ge \frac{n}{1+K} Max_{|z|=1}\left|P(z)\right|. \tag{2}$$

Malik [3] obtained a generalization of (1) in the sense that the right-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact he proved the following theorem.

Theorem A. *If* $P(z)$ *has all its zeros in* $|z| \le 1$, *then for each* $r > 0$

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} Max_{|z|=1}\left|P'(z)\right|. \tag{3}$$

The result is sharp and equality in (3) holds for $P(z) = (z+1)^n$.

If we let $r \to \infty$ in (3), we get (1).

As a generalization of Theorem A, Aziz and Shah [4] proved the following:

Theorem B. *If* $P(z) := a_n z^n + \sum_{j=m}^{n} a_{n-j} z^{n-j}$, $1 \le m \le n$ *is a polynomial of degree n having all its zeros in the disk* $|z| \le K$, $K \le 1$, *then for each* $r > 0$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+K^m e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} Max_{|z|=1}\left|P'(z)\right|. \tag{4}$$

Aziz and Ahemad [5] generalized (3) in the sense that $Max_{|z|=1}\left|P'(z)\right|$ on $|z| = 1$ on the right-hand side of (3) is replaced by a factor involving the integral mean of $\left|P'(z)\right|$ on $|z| = 1$ and proved the following:

Theorem C. *If* $P(z)$ *is a polynomial of degree n having all its zeros in* $|z| \le K \le 1$, *then for* $r > 0$, $p > 1$, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+Ke^{i\theta}\right|^{qr} d\theta\right\}^{\frac{1}{qr}}\left\{\int_0^{2\pi}\left|P'\left(e^{i\theta}\right)\right|^{pr} d\theta\right\}^{\frac{1}{pr}}. \quad (5)$$

If we let $r\to\infty$ and $p\to\infty$ (so that $q\to 1$) in (5), we get (2).

In this paper, we extend Theorem B to the class of polynomials $P(z):=a_nz^n+\sum_{j=m}^{n}a_{n-j}z^{n-j}$, $1\le m\le n$, having all the zeros in $|z|\le K\le 1$, and thereby obtain a more general result by proving the following.

Theorem 1. *If* $P(z):=a_nz^n+\sum_{j=m}^{n}a_{n-j}z^{n-j}$, $1\le m\le n$ *is a polynomial of degree n having all its zeros in the disk* $|z|\le K$, $K\le 1$, *then for each* $r>0$, $s>1$, $t>1$ *with* $\frac{1}{s}+\frac{1}{t}=1$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+\left[\frac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}\right]e^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{sr}}\left\{\int_0^{2\pi}\left|P'\left(e^{i\theta}\right)\right|^{tr} d\theta\right\}^{\frac{1}{tr}}, \quad (6)$$

If we take m = 1 in Theorem 1, we get the following:

Corollary 1. *If* $P(z):=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree n having all its zeros in the disk* $|z|\le K$, $K\le 1$, *then for each* $r>0$, $s>1$, $t>1$ *with* $\frac{1}{s}+\frac{1}{t}=1$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+\left[\frac{n|a_n|K^{2}+|a_{n-1}|}{n|a_n|+|a_{n-1}|}\right]e^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{sr}}\left\{\int_0^{2\pi}\left|P'\left(e^{i\theta}\right)\right|^{tr} d\theta\right\}^{\frac{1}{tr}}. \quad (7)$$

The next result immediately follows from Theorem 1, if we let $t\to\infty$ so that $s\to 1$

Corollary 2. *If* $P(z):=a_nz^n+\sum_{j=m}^{n}a_{n-j}z^{n-j}$, $1\le m\le n$ *is a polynomial of degree n having all its zeros in the disk* $|z|\le K$, $K\le 1$, *then for each* $r>0$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1+\left[\frac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}\right]e^{i\theta}\right|^{r} d\theta\right\}^{\frac{1}{r}} Max_{|z|=1}\left|P'(z)\right|. \quad (8)$$

Also if we let $r\to\infty$ in the Theorem 1 and note that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} = Max_{|z|=1}\left|P(z)\right|.$$

We get the following:

Corollary 3. *If* $P(z):=a_nz^n+\sum_{j=m}^{n}a_{n-j}z^{n-j}$, $1\le m\le n$ *is a polynomial of degree n having all its zeros in the disk* $|z|\le K$, $K\le 1$, *then*

$$Max_{|z|=1}\left|P'(z)\right| \ge \frac{n|a_n|K^{m-1}+m|a_{n-m}|}{n|a_n|\left(K^{2m}+K^{m-1}\right)+m|a_{n-m}|\left(1+K^{m-1}\right)} Max_{|z|=1}\left|P(z)\right|. \quad (9)$$

For K = 1, Corollary 3 reduces to Inequality (1) (the result of Turan[1]).

2. Lemmas

For the proof of this theorem, we need the following lemmas.

The first lemma is due to Qazi [6].

Lemma 1. *If* $P(z):=a_0+\sum_{j=m}^{n}a_jz^j$ *is a polynomial of degree n having no zeros in the disk* $|z|<K$, $K\ge 1$. *Then*

$$\left[\frac{n|a_0|K^{m+1}+m|a_m|K^{2m}}{n|a_0|+m|a_m|K^{m+1}}\right]\left|P'(z)\right|\le\left|Q'(z)\right|$$

for $|z|=1$, $1\le m\le n$,

where

$$Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)} \text{ and } \frac{m}{n}\left|\frac{a_m}{a_0}\right|K^m\le 1.$$

Lemma 2. *If* $P(z):=a_nz^n+\sum_{j=m}^{n}a_{n-j}z^{n-j}$ *is a polynomial of degree n having all its zeros in the disk* $|z|\le K\le 1$ s *then*

$$\left|Q'(z)\right|\le\left[\frac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}\right]\left|P'(z)\right|$$

for $|z|=1$, $1\le m\le n$.

Proof of Lemma 2

Since all the zeros of $P(z)$ lie in $|z| \le K \le 1$, therefore all the zeros of $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$ lie in $|z| \ge \frac{1}{K} \ge 1$.

Hence applying lemma 1 to the polynomial $Q(z) := \overline{a}_n + \sum_{j=m}^{n} \overline{a}_{n-j} z^j$, we get

$$\left[\frac{n|a_n|\frac{1}{K^{m+1}} + m|a_{n-m}|\frac{1}{K^{2m}}}{n|a_n| + m|a_{n-m}|\frac{1}{K^{m+1}}}\right]|Q'(z)| \le |P'(z)|.$$

Or, equivalently

$$|Q'(z)| \le \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]|P'(z)|.$$

This proves lemma 2.

Remark 1: Lemma 3 of Govil and Mc Tume [7] is a special case of this lemma when m = 1.

Proof of Theorem 1

Since $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$, therefore, we have $P(z) = z^n \overline{Q\left(\frac{1}{\overline{z}}\right)}$. This gives

$$P'(z) = nz^{n-1}\overline{Q\left(\frac{1}{\overline{z}}\right)} - z^{n-2}\overline{Q'\left(\frac{1}{\overline{z}}\right)}. \tag{10}$$

Equivalently

$$zP'(z) = nz^{n}\overline{Q\left(\frac{1}{\overline{z}}\right)} - z^{n-1}\overline{Q'\left(\frac{1}{\overline{z}}\right)} \tag{11}$$

this implies

$$|P'(z)| = |nQ(z) - zQ'(z)| \quad for\ |z| = 1. \tag{12}$$

Now by hypothesis, $P(z)$ has all its zeros in $|z| \le K \le 1$, therefore, by Lemma 2, we have for $|z| = 1$

$$|Q'(z)| \le \frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}|P'(z)|,\ 1 \le m \le n. \tag{13}$$

Using (12) in (13), we get

$$|Q'(z)| \le \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]|nQ(z) - zQ'(z)|$$
$$for\ |z| = 1,\ 1 \le m \le n. \tag{14}$$

Since $P(z)$ has all its zeros in $|z| \le K \le 1$, by Gauss-Lucas theorem all the zeros of $P'(z)$ also lie in $|z| \le 1$, therefore, it follows that the polynomial

$$z^{n-1}\overline{P'\left(\frac{1}{\overline{z}}\right)} = nQ(z) - zQ'(z) \tag{15}$$

has all its zeros in $|z| \ge \frac{1}{k} \ge 1$ and hence, we conclude that the function

$$W(z) = \left[\frac{n|a_n|K^{m-1} + m|a_{n-m}|}{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}\right] \cdot \frac{zQ'(z)}{(nQ(z) - zQ'(z))} \tag{16}$$

is analytic for $|z| < 1$, W(0) = 0 and by (14) $|W(z)| \le 1$ for $|z| = 1$. Thus the function

$$1 + \left[\frac{n|a_n|K^{m-1} + m|a_{n-m}|}{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}\right] \cdot W(z)$$

is subordinate to the function

$$1 + \left[\frac{n|a_n|K^{m-1} + m|a_{n-m}|}{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}\right] z$$

for $|z| \le 1$. Hence by a well known property of subordination [8], we have for each $r > 0$ and $0 \le \theta < 2\pi$,

$$\int_0^{2\pi}\left|1 + \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]W(e^{i\theta})\right|^r d\theta$$
$$\le \int_0^{2\pi}\left|1 + \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]e^{i\theta}\right|^r d\theta. \tag{17}$$

Also from (16), we have

$$1 + \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]W(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)}.$$

Therefore,

$$n|Q(z)| = \left|1 + \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]W(z)\right||nQ(z) - zQ'(z)|. \tag{18}$$

Using (12) and the fact that $|Q(z)| = |P(z)|$ for $|z| = 1$, we get from (18)

$$n|P(z)| = \left|1 + \left[\frac{n|a_n|K^{2m} + m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1} + m|a_{n-m}|}\right]W(z)\right||P'(z)| \quad \text{for } |z| = 1. \tag{19}$$

Combining (17) and (19), we get

$$n^r \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^r d\theta \le \int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}\right]e^{i\theta}\right|^r \left|P'\left(e^{i\theta}\right)\right|^r d\theta \text{ for } r>0. \quad (20)$$

Now applying Hölder's inequality for $s>1$, $t>1$, with $\frac{1}{s}+\frac{1}{t}=1$ to (20), we get

$$n^r \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^r d\theta \le \left\{\int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}\right]e^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{s}} \left\{\int_0^{2\pi} \left|P'\left(e^{i\theta}\right)\right|^{tr} d\theta\right\}^{\frac{1}{t}} \text{ for } r>0. \quad (21)$$

This is equivalent to

$$n\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} \left|1+\left[\frac{n|a_n|K^{2m}+m|a_{n-m}|K^{m-1}}{n|a_n|K^{m-1}+m|a_{n-m}|}\right]e^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{sr}} \left\{\int_0^{2\pi} \left|P'\left(e^{i\theta}\right)\right|^{tr} d\theta\right\}^{\frac{1}{tr}} \text{ for } r>0. \quad (22)$$

which proves the desired result.

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4. References

[1] P. Turan, "Über die Ableitung von Polynomen," *Compositio Mathematica*, Vol. 7, 1939, pp. 89-95.

[2] M. A. Malik, "On the Derivative of a Polynomial," *Journal of the London Mathematical Society*, Vol. 2, No. 1, 1969, pp. 57-60. doi:10.1112/jlms/s2-1.1.57

[3] M. A. Malik, "An Integral Mean Estimate for Polynomials," *Proceedings of the American Mathematical Society*, Vol. 91, No. 2, 1984, pp. 281-284.

[4] A. Aziz and W. M. Shah, "An Integral Mean Estimate for Polynomials," *Indian Journal of Pure and Applied Mathematics*, Vol. 28, No. 10, 1997, pp. 1413-1419.

[5] A. Aziz and N. Ahemad, "Integral Mean Estimates for Polynomials Whose Zeros Are within a Circle," *Glasnik Matematicki Series, III*, Vol. 31, No. 2, 1996, pp. 229-237.

[6] M. A. Qazi, "On the Maximum Modulus of Polynomials," *Proceedings of the American Mathematical Society*, Vol. 115, 1990, pp. 337-343. doi:10.1090/S0002-9939-1992-1113648-1

[7] N. K. Govil and G. N. McTume, "Some Generalizations Involving the Polar Derivative for an Inequality of Paul Turan," *Acta Mathematica Thunder*, Vol. 104, No. 1-2, 2004, pp. 115-126. doi:10.1023/B:AMHU.0000034366.66170.01

[8] E. Hille, "Analytic Function Theory, Vol. II," *Introduction to Higher Mathematics*, Ginn and Company, New York, 1962.