

Existence of Positive Solutions for Boundary Value Problem of Nonlinear Fractional q -Difference Equation*

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ABSTRACT

In this paper, we investigate the existence of positive solutions for a class of nonlinear q -fractional boundary value problem. By using some fixed point theorems on cone, some existence results of positive solutions are obtained.

Keywords: Fractional q -Difference Equation; Positive Solution; Fixed Point Theorems on Cone

1. Introduction

Considering the following boundary value problem of nonlinear fractional q -difference equation:

$$\begin{cases} (D_q^\alpha y)(x) = -f(x, y(x)), & 0 < x < 1, 3 < \alpha \leq 4, \\ y(0) = (D_q y)(0) = (D_q^2 y)(0) = 0, (D_q^3 y)(1) = 0, \end{cases} \quad (\text{P})$$

where f is a nonnegative continuous function and D_q^α is the fractional q -derivative of the Riemann-Liouville type.

Fractional differential calculus is a discipline to which many researchers are dedicating their time, perhaps because of its demonstrated applications in various fields of science and engineering [1]. Recently, there are many papers dealing with the boundary value problem of fractional differential equations, see [2-5] and references therein.

The q -difference calculus or quantum calculus is an old subject that was initially developed by Jackson [6,7], and basic definitions and properties of q -difference calculus can be found in [8]. The fractional q -difference calculus had its origin in the works by Al-Salam [9] and Agarwal [10]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q -difference calculus were made, see [11,12].

The question of the existence of positive solutions for

fractional q -difference boundary value problems is in its infancy, see [13-16]. No contributions exist, as far as we know, concerning the existence of positive solutions for problem (P).

This paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we discuss the existence of positive solutions for problem (P).

2. Preliminaries

Let $q \in (0, 1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \quad (2.1)$$

The q -analogue of the power function $(a - b)^n$ with $n \in \mathbb{N}_0$ is

$$(a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, a, b \in \mathbb{R}. \quad (2.2)$$

More generally, if $a \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}. \quad (2.3)$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q -gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\} \quad (2.4)$$

and satisfies $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$. The q -derivative of a function f is here defined by

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$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x \rightarrow 0} (D_q f)(x) \tag{2.5}$$

and q -derivative of higher order by

$$(D_q^0 f)(x) = f(x), \quad (D_q^n f)(x) = D_q (D_q^{n-1} f)(x), n \in \mathbb{N}. \tag{2.6}$$

The q -integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, x \in [0, b]. \tag{2.7}$$

If $a \in [0, b]$ and f defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \tag{2.8}$$

Remark 2.1. (see [17]) If $a = bq^n, n \in \mathbb{N}$ and $f(t) \geq g(t)$ on $[a, b]$, then

$$\int_a^b f(t) d_q t \geq \int_a^b g(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x), \quad (I_q^n f)(x) = I_q (I_q^{n-1} f)(x), n \in \mathbb{N}. \tag{2.9}$$

The fundamental theorem of calculus applies to these operators I_q and D_q , i.e.,

$$(D_q I_q f)(x) = f(x) \tag{2.10}$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0). \tag{2.11}$$

Basic properties of the two operators can be found in [14]. We now point out three formulas that will be used later (${}_i D_q$ denotes the derivative with respect to variable i)

$$\begin{aligned} [a(t-s)]^\alpha &= a^\alpha (t-s)^{(\alpha)}, \\ {}_i D_q (t-s)^{(\alpha)} &= [\alpha]_q (t-s)^{(\alpha-1)}, \end{aligned}$$

$$\left(-x D_q \int_0^x f(x,t) d_q t \right)(x) = \int_0^x -x D_q f(x,t) d_q t + f(qx, x).$$

Remark 2.2. (see [14]) We note that if $\alpha > 0$ and $a \leq b \leq t$, then

$$(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}. \tag{2.12}$$

Definition 2.3. (see [10]) Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q -integral of the Riemann-Liouville type is $(I_q^\alpha f)(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \tag{2.13}$$

$\alpha > 0, x \in [0, 1].$

Definition 2.4. (see [14-16]) The fractional q -derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0, \tag{2.14}$$

where m is the smallest integer greater than or equal to α .

Next, we list some properties that are already known in the literature.

Lemma 2.5. (see [14-16]) Let $\alpha, \beta \geq 0$ and f be a function defined on $[0, 1]$, Then, the next formulas hold:

- 1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x),$
- 2) $(D_q^\alpha I_q^\alpha f)(x) = f(x).$

Lemma 2.6. (see [14-16]) Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0). \tag{2.15}$$

Let $p = 4$, in view of Lemma 2.5 and Lemma 2.6, we see that

$$\begin{aligned} (D_q^\alpha y)(x) &= -f(x, y(x)) \\ \Leftrightarrow (I_q^\alpha D_q^4 I_q^{4-\alpha} y)(x) &= -I_q^\alpha f(x, y(x)) \\ \Leftrightarrow y(x) &= c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + c_3 x^{\alpha-3} \\ &\quad + c_4 x^{\alpha-4} - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t, y(t)) d_q t \end{aligned}$$

for some constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Using the boundary condition $y(0) = 0$ we have $c_4 = 0$. Differentiating both side of the above equality, one gets

$$\begin{aligned} D_q y &= [\alpha-1]_q c_1 x^{\alpha-2} + [\alpha-2]_q c_2 x^{\alpha-3} + [\alpha-3]_q c_3 x^{\alpha-4} \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^x [\alpha-1]_q (x-qt)^{(\alpha-2)} f(t, y(t)) d_q t. \end{aligned} \tag{2.16}$$

Using the boundary condition $(D_q y)(0) = 0$, we have $c_3 = 0$. similarly, we have $c_2 = 0$. From

$$\begin{aligned} (D_q^3 y) &= [\alpha-1]_q [\alpha-2]_q [\alpha-3]_q c_1 x^{\alpha-4} \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^x [\alpha-1]_q [\alpha-2]_q [\alpha-3]_q \\ &\quad \cdot (x-qt)^{(\alpha-4)} f(t, y(t)) d_q t \end{aligned} \tag{2.17}$$

and boundary value problem $(D_q^3)y(1) = 0$, one can obtain

$$c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qt)^{(\alpha-4)} f(t, y(t)) d_q t. \quad (2.18)$$

Putting all things together we finally have

$$\begin{aligned} yx &= \left(\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qt)^{(\alpha-4)} f(t, y(t)) d_q t \right) x^{\alpha-1} \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t, y(t)) d_q t \\ &= \frac{1}{\Gamma_q(\alpha)} \left[\int_0^x \left((1-qt)^{(\alpha-4)} x^{\alpha-1} - (x-qt)^{(\alpha-1)} \right) f(t, y(t)) d_q t \right] \\ &\quad + \int_x^1 (1-qt)^{(\alpha-4)} x^{\alpha-1} f(t, y(t)) d_q t. \end{aligned} \quad (2.19)$$

If we define a function G by

$$G(x, t) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-t)^{(\alpha-4)} x^{\alpha-1} - (x-t)^{(\alpha-1)}, & 0 \leq t \leq x \leq 1, \\ (1-t)^{(\alpha-4)} x^{\alpha-1}, & 0 \leq x \leq t \leq 1. \end{cases} \quad (2.20)$$

Hence, in order to solve the problem (P), it is sufficient to find positive solutions of the following integral equation

$$y(t) = \int_0^1 G(x, qt) f(t, y(t)) d_q t. \quad (2.21)$$

Some properties of the function G needed in the sequel are now stated and proved.

Lemma 2.7. Function G defined above satisfies the following conditions:

$$G(x, qt) \geq 0, \quad G(x, qt) \leq G(1, qt), \quad (2.22)$$

for all $0 \leq x, t \leq 1$,

$$G(x, qt) \geq x^{\alpha-1} G(1, qt), \text{ for all } 0 \leq x, t \leq 1. \quad (2.23)$$

Proof. Let

$$g_1(x, t) = (1-t)^{(\alpha-4)} x^{\alpha-1} - (x-t)^{(\alpha-1)}, \quad 0 \leq t \leq x \leq 1 \quad (2.24)$$

and

$$g_2(x, t) = (1-t)^{(\alpha-4)} x^{\alpha-1}, \quad 0 \leq x \leq t \leq 1. \quad (2.25)$$

It is clear that $g_2(x, qt) \geq 0$. Now, $g_2(0, qt) = 0$. For $x \neq 0$, in view of (2.3) and Remark 2.2, we have

$$\begin{aligned} g_1(x, qt) &= (1-qt)^{(\alpha-4)} x^{\alpha-1} - x^{\alpha-1} \left(x - \frac{qt}{x} \right)^{(\alpha-1)} \\ &\geq x^{\alpha-1} \left[(1-qt)^{(\alpha-4)} - (1-qt)^{(\alpha-1)} \right] \geq 0. \end{aligned} \quad (2.26)$$

Therefore, $G(x, t) \geq 0$. Moreover,

$$\begin{aligned} &{}_x D_q g_1(x, qt) \\ &= (1-qt)^{(\alpha-4)} [\alpha-1]_q x^{\alpha-2} - [\alpha-1]_q x^{\alpha-2} \left(x - \frac{qt}{x} \right)^{(\alpha-2)} \\ &\geq [\alpha-1]_q x^{\alpha-2} \left[(1-qt)^{(\alpha-4)} - (1-qt)^{(\alpha-2)} \right] \geq 0. \end{aligned} \quad (2.27)$$

i.e., $g_1(x, qt)$ is an increasing function of x . Obviously, $g_2(x, qt)$ is increasing in x , therefore, $G(x, qt)$ is an increasing function of x for fixed $t \in [0, 1]$. This concludes the proof of (2.22).

Suppose now that $x \geq qt$, then

$$\begin{aligned} \frac{G(x, qt)}{G(1, qt)} &= \frac{(1-qt)^{(\alpha-4)} x^{\alpha-1} - (x-qt)^{(\alpha-1)}}{(1-qt)^{(\alpha-4)} - (1-qt)^{(\alpha-1)}} \\ &= \frac{x^{\alpha-1} \left[(1-qt)^{(\alpha-4)} - \left(1 - \frac{qt}{x} \right)^{(\alpha-1)} \right]}{(1-qt)^{(\alpha-4)} - (1-qt)^{(\alpha-1)}} \\ &\geq \frac{x^{\alpha-1} \left[(1-qt)^{(\alpha-4)} - (1-qt)^{(\alpha-1)} \right]}{(1-qt)^{(\alpha-4)} - (1-qt)^{(\alpha-1)}} = x^{\alpha-1}. \end{aligned} \quad (2.28)$$

If $x \leq qt$, then

$$\frac{G(x, qt)}{G(1, qt)} = x^{\alpha-1} \quad (2.29)$$

and this finishes the proof of (2.23).

Let $\mathfrak{S} = C[0, 1]$ be the Banach space endowed with norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$. Define the cone C by

$$C = \{u \in \mathfrak{S} : u(x) \geq x^{\alpha-1} \|u\|\}.$$

It follows from the non-negativeness and continuity of G and f that the operator $T : C \rightarrow \mathfrak{S}$ defined by

$$(Tu)(x) = \int_0^1 G(x, qt) f(t, u(t)) d_q t \quad (2.30)$$

is completely continuous [18]. Moreover, for $u \in C$, in view of (2.22) and (2.23), we have $(Tu)(x) \geq 0$ on $[0, 1]$ and

$$\begin{aligned} (Tu)(x) &= \int_0^1 G(x, qt) f(t, u(t)) d_q t \\ &\geq x^{\alpha-1} \int_0^1 G(1, qt) f(t, u(t)) d_q t \\ &\geq x^{\alpha-1} \|Tu\|, \end{aligned} \quad (2.31)$$

that is $T(C) \subset C$.

Lemma 2.8. (see [19]) Let E be a Banach space, $P \subset E$ a cone, and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ is a completely continuous op-

erator such that either

$$1) \|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1 \text{ and } \|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2,$$

or

$$2) \|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1 \text{ and } \|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2 \text{ holds. Then } T \text{ has a fixed point in } P \cap (\Omega_2 \setminus \Omega_1).$$

3. Main Results

Let $\Omega_i = \{y \in C[0, 1] : \|y\| \leq r_i\}$, $i = 1, 2, 3, 4$, where $r_i > 0$ will be defined later.

Theorem 3.1. Suppose that $f(t, u)$ is a nonnegative continuous function on $[0, 1] \times [0, \infty)$. In addition, suppose that one of the following two conditions holds:

$$(H1) \quad \lim_{y \rightarrow 0} \min_{x \in [0, 1]} \frac{f(x, y)}{y} = \infty, \\ \lim_{y \rightarrow \infty} \min_{x \in [0, 1]} \frac{f(x, y)}{y} = 0;$$

$$(H2) \quad \lim_{y \rightarrow 0} \min_{x \in [0, 1]} \frac{f(x, y)}{y} = 0, \\ \lim_{y \rightarrow \infty} \min_{x \in [0, 1]} \frac{f(x, y)}{y} = \infty.$$

Then problem (P) has at least one positive solution.

Proof. Note that the operator $T(C) \subset C$ is completely continuous. Now, assume that condition (H1) holds. Since

$\lim_{y \rightarrow 0} \min_{x \in [0, 1]} \frac{f(x, y)}{y} = \infty$, there exists an $r_1 > 0$ such that

$$f(x, y) \geq \varepsilon_1 y, \text{ for } x \in [0, 1], 0 \leq y \leq r_1, \quad (3.1)$$

where the constant $\varepsilon_1 > 0$ such that

$$\varepsilon_1 \int_0^1 G(1, qt) t^{\alpha-1} d_q t > 1. \quad (3.2)$$

Thus,

$$f(x, y(x)) \geq \varepsilon_1 y, \text{ for } y \in C \cap \partial\Omega_1, x \in [0, 1].$$

This, together with the definitions of C and Lemma 2.7, implies that for any $y \in C \cap \partial\Omega_1$,

$$\begin{aligned} \|Ty\| &= \max_{x \in [0, 1]} \int_0^1 G(x, qt) f(t, y(t)) d_q t \\ &= \int_0^1 G(1, qt) f(t, y(t)) d_q t \\ &\geq \varepsilon_1 \int_0^1 G(1, qt) y(t) d_q t \\ &\geq \varepsilon_1 \int_0^1 G(1, qt) t^{\alpha-1} d_q t \|y\| \\ &\geq \|y\|. \end{aligned} \quad (3.3)$$

That is, for any $y \in C \cap \partial\Omega_1$, $\|Ty\| \geq \|y\|$.

On the other hand, from $\lim_{y \rightarrow \infty} \max_{x \in [0, 1]} \frac{f(x, y)}{y} = 0$,

it follows that there exists a $L_1 > 0$ such that

$$f(x, y) \leq \varepsilon_2 y \text{ for } x \in [0, 1], y \geq L_1, \quad (3.4)$$

where the constant $\varepsilon_2 > 0$ satisfies

$$\varepsilon_2 \int_0^1 G(1, qt) d_q t < \frac{1}{2}. \quad (3.5)$$

Let $L_2 = \max_{x \in [0, 1], y \in [0, L_1]} \{f(x, y)\}$. Then we have

$$f(x, y) \leq \varepsilon_2 y + L_2, \text{ for } x \in [0, 1], y \geq 0. \quad (3.6)$$

Let $r_2 = \max\{2r_1, 2L_2 \int_0^1 G(1, qt) d_q t\}$ and $y \in C \cap \partial\Omega_2$, then

$$\begin{aligned} \|Ty\| &= \max_{x \in [0, 1]} \int_0^1 G(x, qt) f(t, y(t)) d_q t \\ &= \int_0^1 G(1, qt) f(t, y(t)) d_q t \leq \int_0^1 G(1, qt) (\varepsilon_2 y + L_2) d_q t \\ &\leq \varepsilon_2 \int_0^1 G(1, qt) d_q t \|y\| + L_2 \int_0^1 G(1, qt) d_q t \leq \|y\|. \end{aligned} \quad (3.7)$$

Thus, the operator T satisfies condition of Lemma 2.8. Consequently, the operator T has at least one fixed point $y \in C \cap ((\bar{\Omega})_2 \setminus \Omega_1)$, which is one positive solution of the problem (P).

Next, we suppose that condition (H2) holds. The proof is similar to that of the case in which (H1) holds and will only be sketched here. Let $\tau = q^n$ with $n \in \mathbb{N}$. Select two positive constants $\varepsilon_3, \varepsilon_4$ with $\varepsilon_3 \int_0^1 G(1, qt) d_q t < 1$ and $\varepsilon_4 \tau^{\alpha-1} \int_\tau^1 G(1, qt) d_q t > 1$, respectively. Then, there exist two positive constants r_3 and L_3 such that

$$f(x, y) \leq \varepsilon_3 y, \text{ for } x \in [0, 1], 0 \leq y \leq r_3, \quad (3.8)$$

$$f(x, y) \geq \varepsilon_4 y, \text{ for } x \in [0, 1], y \geq L_3. \quad (3.9)$$

It follows from that for $y \in C \cap \partial\Omega_3$,

$$\begin{aligned} \|Ty\| &= \int_0^1 G(1, qt) f(t, y(t)) d_q t \leq \varepsilon_3 \int_0^1 G(1, qt) y(t) d_q t \leq \|y\|. \end{aligned} \quad (3.10)$$

In addition, let $r_4 = \max\{2r_3, L_3 \tau^{1-\alpha}\}$. If $y \in C \cap \partial\Omega_4$, then

$$y(x) \geq \tau^{\alpha-1} r_4 \geq L_3, \text{ for } x \in [\tau, 1] \quad (3.11)$$

and

$$f(x, y(x)) \geq \varepsilon_4 y, \text{ for } x \in [\tau, 1], y \in C \cap \partial\Omega_4. \quad (3.12)$$

In view of Remark 2.1, for $y \in C \cap \partial\Omega_4$, we have

$$\begin{aligned} \|Ty\| &= \max_{x \in [0, 1]} \int_0^1 G(x, qt) f(t, y(t)) d_q t \\ &= \int_0^1 G(1, qt) f(t, y(t)) d_q t \geq \varepsilon_4 \int_\tau^1 G(1, qt) y(t) d_q t \\ &\geq \varepsilon_4 \int_\tau^1 G(1, qt) t^{\alpha-1} d_q t \|y\| \geq \varepsilon_4 \tau^{\alpha-1} \int_\tau^1 G(1, qt) d_q t \|y\| \\ &\geq \|y\|. \end{aligned} \quad (3.13)$$

Thus, $\|Ty\| \geq \|y\|$ for $y \in C \cap \partial\Omega_4$. Consequently, the operator T has at least one fixed point $y \in C \cap (\bar{\Omega}_4 \setminus \Omega_3)$, which is one positive solution of the problem (P).

Example 3.2

$$\begin{cases} (D_{0.5}^{3.5}y)(x) = -f(x, y(x)), & 0 < x < 1, \\ y(0) = (D_{0.5}y)(0) = (D_{0.5}^2y)(0) = 0, (D_{0.5}^3y)(1) = 0, \end{cases}$$

where $f(x, y(x)) = y^{\frac{1}{2}}(x) + \log(1 + y(x))$. Obviously,

$$\lim_{y \rightarrow 0} \max_{x \in [0,1]} \frac{f(x, y)}{y} = \infty,$$

$$\lim_{y \rightarrow \infty} \max_{x \in [0,1]} \frac{f(x, y)}{y} = 0.$$

Thus, by the first part of Theorem 3.1, we can get that the problem above has at least one positive solution.

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