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General Boundary Value Problems for Nonlinear Uniformly Elliptic Equations in Multiply Connected Infinite Domains

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ABSTRACT

This article discusses the general boundary value problem for the nonlinear uniformly elliptic equation of second order $u_{z\bar z}=F(z,u,u_z,u_{zz})+G(z,u,u_z)$ in D, (0.1) and the boundary condition $\frac{\partial u}{\partial\nu}+2c_1(z)u=2c_2(z)$ on Γ, (0.2) in a multiply connected infinite domain D with the boundary Γ. The above boundary value problem is called Problem G. Problem G extends the work [8] in which the equation (0.1) includes a nonlinear lower term and the boundary condition (0.2) is more general. If the complex equation (0.1) and the boundary condition (0.2) meet certain assumptions, some solvability results for Problem G can be obtained. By using reduction to absurdity, we first discuss a priori estimates of solutions and solvability for a modified problem. Then we present results on solvability of Problem G.

Keywords: General Boundary Value Problems; Nonlinear Elliptic Equations; Multiply Connected Infinite Domains

1. Formulation of Elliptic Equations and Boundary Value Problems

Let D be an $(N+1)$-connected domain which includes the infinite point and has the boundary

$\Gamma=\bigcup_{j=0}^{N}\Gamma_j$ in $\mathbb{C}$, where $\Gamma\in C_\mu^2(0<\mu<1)$.

Without loss of generality, we assume that D is a circular domain in $|z|>1$, where the boundary consists of $N+1$ circles $\Gamma_0=\Gamma_{N+1}=\{|z|=1\}$, $\Gamma_j=\{|z-z_j|=r_j\}, j=1,\cdots,N$ and $z=\infty\in D$. Note that this article uses the same notations as in references [1-8]. We consider the nonlinear uniformly elliptic equation of second order

$$\begin{cases} u_{z\bar z}=F(z,u,u_z,u_{zz})+G(z,u,u_z),\\ F=\operatorname{Re}[Qu_{zz}+A_1u_z]+A_2u+A_3,\\ G=G(z,u,u_z), Q=Q(z,u,u_z,u_{zz}),\\ A_j=A_j(z,u,u_z), j=1,2,3. \end{cases}\tag{1.1}$$

This is the complex form of the nonlinear real equation

$$\Phi(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=0\tag{1.2}$$

with certain conditions (see [3]). We suppose that the Equation (1.1) satisfies Condition C, as described below.

Condition C 1) $Q(z,u,w,U), A_j(z,u,w)(j=1,2,3)$ are measurable in $z\in D$ for all continuous functions $u(z),w(z)$ in $\bar D$ and all measurable functions $U(z)\in L_{p_0,2}(\bar D)$, and satisfy

$$L_{p,2}[A_1(z,u,w),\bar D]\le k_0, L_{p,2}[A_2(z,u,w),\bar D]\le\varepsilon k_0,$$
$$L_{p,2}[A_3(z,u,w),\bar D]\le k_1, A_2(z,u,w)\ge 0 \text{ in } D,\tag{1.3}$$

in which $p_0,p(2<p_0\le p),k_0,k_1,\varepsilon(\le 1)$ are non-negative constants.

2) The above functions are continuous in $u\in\mathbb{R}, w\in\mathbb{C}$ for almost every $z\in D, U\in\mathbb{C}$, and $Q=0, A_j=0(j=1,2,3)$ for $z\notin D$.

3) The Equation (1.1) satisfies the uniform ellipticity condition

$$|F(z,u,w,U_1)-F(z,u,w,U_2)|\le q_0|U_1-U_2|,\tag{1.4}$$

for almost every point $z\in D$, any functions $u(z),w(z)\in C(\bar D)$ and $U_1,U_2\in\mathbb{C}$, where $q_0(<1)$

is a non-negative constant.

4) The function $G(z,u,w)$ possesses the form

$$G(z,u,w)=B_1|w|^{\sigma}+B_2|u|^{\tau} \text{ in } D, \quad (1.5)$$

where $u(z), w(z)$ are continuous functions in $\bar{D}$, $0<\sigma,\tau<\infty, L_{p,2}\left[B_j,\bar{D}\right]\le k_0\left(j=1,2,2<p_0\le p\right)$ for a positive constant k_0.

According to [7], we introduce the general boundary value problem for the Equation (1.1) in $\bar{D}$ as follows.

Problem G Find a continuously differentiable solution $u(z)$ of the second order Equation (1.1) in $\bar{D}$ satisfying the boundary conditions

$$\begin{aligned}&\frac{\partial u}{\partial \nu}+2c_1(z)u=2c_2(z),\\ &\textit{i.e.}\ \mathrm{Re}\left[\overline{\lambda(z)}u_z\right]+c_1(z)u=c_2(z), z\in\Gamma.\end{aligned} \quad (1.6)$$

Here ν is a given unit vector at the point $z\in\Gamma$, and $\lambda(z)=\cos(\nu,x)-i\cos(\nu,y)$, $\sigma(z)$ and $\tau(z)$ are real functions. We assume λ, c_1 and c_2 satisfy the conditions

$$C_\alpha[\lambda,\Gamma]\le k_0, C_\alpha[c_1,\Gamma]\le\varepsilon k_0, C_\alpha[c_2,\Gamma]\le k_2, \quad (1.7)$$

and

$$c_1(z)\cdot\cos(\nu,n)\ge 0,\ z\in\Gamma,$$

in which $\alpha(1/2<\alpha<1), \varepsilon, k_0, k_2$ are non-negative constant, and n is the unit outer normal at $z\in\Gamma$. If $\cos(\nu,n)=0, c_1(z)=0$ on $\Gamma_j, 1\le j\le N,$ then we assume that

$$\int_{\Gamma_j}c_2(z)\mathrm{d}z=0, u\left(1/a_j^*\right)=b_j^*, \left|b_j^*\right|\le k_2, 1\le j\le N, \quad (1.8)$$

in which a_j^* is a point on Γ_j and $b_j^*(j=1,\cdots,N)$ are real constants. There is no harm in assuming that $\cos(\nu,n)=0, c_1(z)=0$ on $\Gamma^*=\Gamma_1\cup\cdots\cup\Gamma_{N_0}(N_0\le N)$, $\cos(\nu,n)$ and $c_1(z)$ do not both vanish identically on $\Gamma^{**}=\Gamma_{N_0+1}\cup\cdots\cup\Gamma_N$.

We can see that the above boundary conditions include some irregular oblique derivative boundary conditions. If $\cos(\nu,n)>0$ on Γ, then Problem G is the regular oblique derivative problem (Problem III). If $\cos(\nu,n)=0$ and $c_1=0$ on Γ, then Problem G is the first boundary value problem, *i.e.*, the Dirichlet boundary value problem (Problem D), in which the boundary condition is

$$\begin{aligned}&u(z)=r(z)\\ &=\int_{1/a_j^*}^{z}c_2(z)\mathrm{d}s+b_j^*, r\left(1/a_j^*\right)=b_j^*, j=1,\cdots,N+1.\end{aligned} \quad (1.9)$$

One problem regarding the well posed-ness of Problem G for (1.1) can be formulated as follows:

Problem H Find a system of continuous functions $\left[u(z),w(z)\right]$ of the equation

$$\begin{cases}w_{\bar{z}}=F(z,u,w,w_z)+G(z,u,w),\\ F=\mathrm{Re}\left[Qw_z+A_1w\right]+A_2u+A_3,\\ G=G(z,u,w), Q=Q(z,u,w,w_z),\\ A_j=A_j(z,u,w), j=1,2,3, w=u_z,\end{cases} \quad (1.10)$$

satisfying the modified boundary conditions

$$\begin{aligned}&\frac{\partial u}{\partial \nu}+2c_1(z)u=2\left[c_2(z)+h(z)\right],\\ &\textit{i.e.}\ \mathrm{Re}\left[\overline{\lambda(z)}u_z\right]+c_1(z)u=c_2(z)+h(z), z\in\Gamma,\end{aligned} \quad (1.11)$$

and the point conditions:

$$u\left(1/a_j\right)=b_j, j=0,1,\cdots,m, a_0\in\Gamma_0, a_0\ne a_j\ (j=1,\cdots,m). \quad (1.12)$$

An explanation of the above conditions is given as follows. The boundary Γ can be divided into two parts: $\Gamma^+\subset\left\{\cos(\nu,n)\ge 0, c_1(z)\ge 0\right\}$ and $\Gamma^-\subset\left\{\cos(\nu,n)\le 0, c_1(z)\le 0\right\}$, such that $\Gamma^+\cup\Gamma^-=\Gamma, \Gamma^+\cap\Gamma^-=\varnothing, \overline{\Gamma^+}\cap\overline{\Gamma^-}=E=\left\{a_1,\cdots,a_m,a_1',\cdots,a_l'\right\}$, every component of Γ^+ and Γ^- includes its initial point, but does not include the terminal point, and there is at least one point on each component of Γ^+,Γ^- so that $\cos(\nu,n)\ne 0$. The points $a_j(j=1,\cdots,m)$ and $a_j'(j=1,\cdots,l)$ possess the following property. $a_j\in\Gamma^+$ and $a_j'\in\Gamma^-$, when the direction of ν at a_j, a_j' is the same as the direction of Γ. $a_j\in\Gamma^-$ and $a_j'\in\Gamma^+$, when the direction of ν at a_j, a_j' is opposite to the direction of Γ. And $\cos(\nu,n)$ changes the sign once on the two components of Γ^+,Γ^- with the end point a_j or a_j'. And $b_j(j=0,1,\cdots,m)$ in (1.12) are real constants satisfying the condition: $\left|b_j\right|\le k_3$, herein k_3 is a non-negative constant. Moreover, the undetermined function $h(z)$ in (1.11) can be written as

$$h(z)=h_j\eta_j(z),\ z\in\Gamma_j',\ j=0,1,\cdots,l. \quad (1.13)$$

In (1.13) $\Gamma_j'\subset\Gamma_j\setminus\Gamma^*(j=0,1,\cdots,l)$ are non-degenerate, multiply disjointed arcs, each of which consists of inner points of $\Gamma_j'(j=0,1,\cdots,l)$, such that $\cos(\nu,n)=0, \sigma(z)=0$ on $\Gamma_j'(j=1,\cdots,l), a_0\in\Gamma_0', \Gamma_0'\cap E=\varnothing$. In addition, $h_j(j=0,1,\cdots,l)$ are unknown real constants to be determined appropriately, and $\eta_j(z)$ is a positive function on Γ_j' and $\eta_j(z)=0$ on $\Gamma\setminus\Gamma_j'$ and $C_\alpha\left[\eta_j(z),\Gamma\right]\le k_0, j=0,1,\cdots,l,$ in which $\alpha(1/2<\alpha<1)$ and k_0 are non-negative constants. It is not difficult to see that the index of Problem H is given by

$$K=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda(z)=N-1+\frac{m-l}{2}. \quad (1.14)$$

If $\cos(\nu,n)\ge 0, c_1(z)\ge 0$ on Γ, then $\Gamma^+=\Gamma, \Gamma^-=\varnothing, E=\varnothing$. In this case, Problem H for (1.1) is called Problem O or Problem IV, which includes the Dirichlet problem, the Neumann problem and the regular oblique derivative problem as its special cases. We note that except the case where $\cos(\nu,n)=0$ and $c_1(z)=0$ on Γ, the conditions (1.12) and (1.13) can be replaced by

$$\begin{aligned}&u(1/a_j)=b_j,\ j=0,1,\cdots,m,\\&h(z)=h_j\eta_j(z), z\in\Gamma_0',\ j=1,\cdots,l.\end{aligned}\tag{1.15}$$

with

$$|b_j|\le k_3,\ j=0,1,\cdots,m,\tag{1.16}$$

in which k_3 is a non-negative constant. Also note that [4,7] discuss the corresponding problem for the equation (1.1) with $G(z,u,u_z)=0$ in the bounded domains.

2. A Priori Estimates of Solutions of Boundary Value Problems

We first give a priori estimates of solutions of Problem H.

Theorem 2.1 Suppose the second order nonlinear Equation (1.10) satisfies Condition C, and ε in (1.3), (1.7) is small enough. Then any solution $[u(z),w(z)]=[u(z),u_z]$ of Problem H for (1.10) with $G(z,u,w)=0$ satisfies the estimates

$$S(u)=C^1_\beta\left[u(z),\bar D\right]+L_{p_0,2}\left[|u_{zz}|+|u_{z\bar z}|,\bar D\right]\le M_1,\tag{2.1}$$

$$S(u)\le M_2k_*=M_2(k_1+k_2+k_3),$$

in which $\beta=\min(\alpha,1-2/p_0)$, $2<p_0\le p$, $M_1=M_1(q_0,p_0,k,\alpha,K,D)$, $k=(k_1,k_2,k_3)$, $M_2=M_2(q_0,p_0,k_0,\alpha,K,D)$.

Proof First of all, we prove that the solution $u(z)$ of Problem H satisfies the estimate

$$S_1=C^1\left[u(z),\bar D\right]\le M_3=M_3(q_0,p_0,k,\alpha,K,D).$$

Suppose that the estimate (2.3) is not true. Then there exist sequences of coefficients $\{Q^n\},\{A_1^n\},\{A_2^n\},\{A_3^n\},\{\lambda_n\},\{c_{1n}\},\{c_{2n}\},\{b_{jn}^*\},\{b_{jn}\}$ of (1.10), (1.11), (1.12) and (1.15) satisfying the same conditions of $Q,A_1,A_2,A_3,\lambda,c_1,c_2,b_j^*,b_j$, such that $\{Q^n\}$, $\{A_1^n\}$, $\{A_2^n\}$, $\{A_3^n\}$ in D weakly converge to Q^0,A_1^0,A_2^0,A_3^0 respectively, and $\{\lambda_n\},\{c_{1n}\},\{c_{2n}\},\{b_{jn}^*\}$, $\{b_{jn}\}$ on Γ uniformly converge to $\lambda_0,c_{10},c_{20},b_{j0}^*,b_{j0}$ respectively, and the corresponding boundary value problems

$$u_{z\bar z}-\mathrm{Re}\left[Q^nu_{zz}+A_1^nu_z\right]-A_2^nu=A_3^n,\ A_2^n\ge 0\text{ in }D,\tag{2.4}$$

$$\begin{aligned}&\frac{\partial u}{\partial\nu_n}+2c_{1n}u=2c_{2n}+2h_n,\\&c_{1n}(z)\cdot\cos(\nu_n,n)\ge 0\text{ on }\Gamma,\int_{\Gamma_j}c_{2n}\mathrm{d}s=0,\end{aligned}\tag{2.5}$$

$$\begin{aligned}&u(1/a_j^*)=b_{jn}^*,\ j=1,\cdots,N_0,\\&u(1/a_j)=b_{jn},\ j=0,1,\cdots,m,n=1,2,\end{aligned}\tag{2.6}$$

have the continuously differentiable solutions $u_n(z)(n=1,2,\cdots)$ with the property that $\tilde H_n=C^1\left[u_n,\bar D\right]\to\infty$ as $n\to\infty$. There is no harm in assuming that $\tilde H_n\ge 1, n=1,2,\cdots$ Denote $U_n=u_n/\tilde H_n, n=1,2,\cdots$ It is clear that the function $w_n(z)=U_{nz}$ is a solution of the following Riemann-Hilbert boundary value problem

$$w_{n\bar z}-\mathrm{Re}\left[Q^nw_{nz}+A_1^nw_n\right]=A^n, A^n=A_1^nu_n+A_3^n\text{ in }D,\tag{2.7}$$

$$\begin{aligned}&u_n(1/a_j^*)=b_{jn}^*,\ j=1,\cdots,N_0,\\&u_n(1/a_j)=b_{jn},\ j=0,1,\cdots,m,n=1,2,\end{aligned}\tag{2.8}$$

where the index of $\lambda_n(z)$ is $K=N-1+(m-l)/2$, and $C\left[w_n(z),\bar D\right]\le 1$ showing that $w_n(z)$ on $\bar D$ is bounded. According to the method in the proof of Theorem 4.7, Chapter I [4], we can obtain that $w_n(z)$ satisfies the estimate

$$L(w_n)=C_\beta\left[w_n,\bar D\right]+L_{p_0,2}\left[|w_{nz}|+|w_{n\bar z}|,\bar D\right]\le M_4,\tag{2.9}$$

in which $M_4=M_4(q_0,p_0,k,\alpha,K,D)$, and then

$$U_n(z)=-2\,\mathrm{Re}\int_{1/a_j^*}^z\frac{w_n(z)}{z^2}\mathrm{d}z+u_0(z)/\tilde H_n$$

satisfies

$$S(U_n)=C^1_\beta\left[U_n,\bar D\right]+L_{p_0,2}\left[|U_{nzz}|+|U_{nz\bar z}|,\bar D\right]\le M_5,\tag{2.10}$$

where $M_5=M_5(q_0,p_0,k,\alpha,K,D)$. Hence from $\{U_n(z)\}$ and $\{U_{nz}\}$, we can choose the subsequences $\{U_{n_k}(z)\}$ and $\{U_{n_kz}\}$, which uniformly converge to $U_0(z)$ and U_{0z} in $\bar D$ respectively, such that $U_0(z)$ is a solution of the following boundary value problem

$$U_{z\bar z}-\mathrm{Re}\left[Q^0U_{zz}+A_1^0U_z\right]-A_2^0U=0,\ A_2^0\ge 0\text{ in }D,\tag{2.11}$$

$$\frac{\partial U}{\partial\nu_0}+2c_{10}u=2h_0,\ c_{10}(z)\cdot\cos(\nu_0,n)\ge 0\text{ on }\Gamma,\tag{2.12}$$

$$U(1/a_j^*)=0,\ j=1,\cdots,N_0,U(1/a_j)=0,\ j=0,1,\cdots,m.\tag{2.13}$$

By the uniqueness of solutions of Problem H (see Theorem 2.3 below), we see that $U(z)=0$ on $\bar D$. However from $C^1\left[U_n(z),\bar D\right]=1$, it can be derived that $C^1\left[U_0(z),\bar D\right]=1$. This contradiction proves that (2.3) is true. Afterwards, using the method of deriving (2.9) from $C^1\left[U_n,\bar D\right]=1$, we can obtain the estimate (2.1). The estimate (2.2) can be concluded from (2.1).

Theorem 2.2 Let the Equation (1.1) satisfy Condition

C and ε in (1.3), (1.7) be a sufficiently small positive constant. Then any solution $\left[w(z),u(z)\right]$ of Problem H for (1.10) satisfies the estimates

$$C_\beta\left[w(z),\bar{D}\right]+C_\beta\left[u(z),\bar{D}\right]\le M_6k_*,\quad(2.14)$$

$$L_{p_0,2}\left[|w_{\bar{z}}|+|w_z|,\bar{D}\right]+L_{p_0,2}\left[u_z,\bar{D}\right]\le M_7k_*,\quad(2.15)$$

where β, p_0 are as stated in Theorem 2.1, $M_j=M_j\left(q_0,p_0,k_0,\alpha,K,D\right), j=6,7,$

$$k_*=k_1+k_2+k_3+k_0\left\{\left[C\left(w,\bar{D}\right)\right]^\sigma+\left[C\left(u,\bar{D}\right)\right]^\tau\right\}.$$

Proof It is easy to see that $\left[w(z),u(z)\right]$ of Problem H for (1.10) satisfies the following equation and boundary conditions:

$$w_{\bar{z}}-\operatorname{Re}\left[Qw_z\right]+A_1w=A_2u+A_3+G,\ z\in D,\quad(2.16)$$

$$\operatorname{Re}\left[\overline{\lambda(z)}w(z)\right]=-c_1u+c_2(z)+h(z),\ z\in\Gamma,\quad(2.17)$$

$$\begin{aligned}&u\left(1/a_j^*\right)=b_{jn}^*,\ j=1,\cdots,N_0,\\&u\left(1/a_j\right)=b_{jn},\ j=0,1,\cdots,m,n=1,2,\end{aligned}\quad(2.18)$$

By using the same method as in the proof of Theorem 2.1, we can obtain the estimates (2.14) and (2.15).

Now we discuss the uniqueness of solutions of Problem H for the nonlinear elliptic Equation (1.1) with $G(z,u,w)=0$. For this, we need to consider the following condition

$$\begin{cases}F\left(z,u_1,u_{1z},U\right)-F\left(z,u_2,u_{2z},U\right)=\operatorname{Re}\left[\tilde{A}_1\left(u_1-u_2\right)_z\right]+\tilde{A}_2\left(u_1-u_2\right),\\ \tilde{A}_j=\tilde{A}_j\left(z,u_1,u_2,U\right), j=1,2,\ L_{p_0,2}\left[\tilde{A}_j,\bar{D}\right]\le k_0,\ 2<p_0\le p,\end{cases}\quad(2.19)$$

for any continuously differentiable functions $u_j(z)\in C_\beta^1\left(\bar{D}\right), j=1,2$ and any measurable function $U(z)\in L_{p_0,2}\left(\bar{D}\right)$, where $\beta=\left[\min\left(\alpha,1-2/p_0\right)\right]$, $p_0\left(2<p_0\le p\right)$, k_0 are constants as stated in Section 1. We can prove the uniqueness of solutions of Problem H for (1.1).

Theorem 2.3 Let the second order nonlinear Equation (1.1) satisfy Condition C and (2.19) with $\tilde{A}_2\ge0$ in D. Then the solution of Problem H for (1.10) with $G(z,u,u_z)=0$ is unique.

Proof Let $u_1(z),u_2(z)$ be two solutions of Problem H for (1.10). By the above conditions, we see that $u(z)=u_1(z)-u_2(z)$ is a solution of the following boundary value problem Problem

$$u_{z\bar{z}}-\operatorname{Re}\left[\tilde{Q}u_{zz}+\tilde{A}_1u_z\right]-\tilde{A}_2u=0,\ z\in D,\quad(2.20)$$

$$\frac{\partial u}{\partial\nu}+2c_1(z)u(z)=2H(z),\ z\in\Gamma,\quad(2.21)$$

$$u\left(1/a_j^*\right)=0,\ j=1,\cdots,N_0,u\left(1/a_j\right)=0,h=0,1,\cdots,m,\quad(2.22)$$

with

$$\begin{cases}\operatorname{Re}\left[\tilde{Q}\left(u_1-u_2\right)_{zz}\right]=F\left(z,u_1,u_{1z},u_{1zz}\right)-F\left(z,u_1,u_{1z},u_{2zz}\right),\\ \operatorname{Re}\left[\tilde{A}_1\left(u_1-u_2\right)_z\right]=F\left(z,u_1,u_{1z},u_{2zz}\right)-F\left(z,u_1,u_{2z},u_{2zz}\right),\\ \tilde{A}_2=\begin{cases}\dfrac{F\left(z,u_1,u_{2z},u_{2zz}\right)-F\left(z,u_2,u_{2z},u_{2zz}\right)}{u_1-u_2} & \text{for } u_1(z)\ne u_2(z),\\ 0 & \text{for } u_1(z)=u_2(z),\ z\in D,\end{cases}\\ \left|\tilde{Q}\right|\le q_0<1, L_{p_0,2}\left[\tilde{A}_j,\bar{D}\right]<\infty,\ j=1,2,\ \tilde{A}_2\ge0 \text{ in } D,\end{cases}$$

where q_0,p_0,k_1 are non-negative constants. According to the proof of Theorem 2.6, Chapter I, [4], and using the extremum principle of solutions for (2.20) (see Chapter 3, [3]), we can prove that $u(z)=0$ in D, and then $u_1(z)=u_2(z)$ in D.

3. Solvability of Boundary Value Problems

We first prove a lemma.

Lemma 3.1. If $G(z,u,w)$ satisfies the condition stated in Condition C, then the nonlinear mapping T: $C\left(\bar{D}\right)\times C\left(\bar{D}\right)\to L_{p,2}\left(\bar{D}\right)$ defined by $G=G\left[z,u(z),w(z)\right]$ is coninuous and bounded with

$$\begin{aligned}L_{p,2}\left[G\left(z,u(z),w(z)\right),\bar{D}\right]&\le L_{p,2}\left[B_1,\bar{D}\right]\\ &\left[C\left(w,\bar{D}\right)\right]^\sigma+L_{p,2}\left[B_2,\bar{D}\right]\left[C\left(u,\bar{D}\right)\right]^\tau,\end{aligned}\quad(3.1)$$

where $p=p_0>2$.

Proof In order to prove that the mapping T: $C\left(\bar{D}\right)\times C\left(\bar{D}\right)\to L_{p,2}\left(\bar{D}\right)$ defined by $G=G\left[z,u(z),w(z)\right]$ is continuous, we choose any sequence of functions $\left[w_n(z),u_n(z)\right]\left(w_n(z),u_n(z)\in C\left(\bar{D}\right),n=0,1,2,\cdots\right)$

such that $C\left[w_n-w_0,\bar{D}\right]+C\left[u_n-u_0,\bar{D}\right]\to 0$ as $n\to\infty$. Similarly to Lemma 2.2.1 [5], we can prove that $C_n=G\left(z,u_n,w_n\right)-G\left(z,u_0,w_0\right)$ possesses the property that

$$L_{p,2}\left[C_n,\bar{D}\right]\to 0 \text{ as } n\to\infty. \quad (3.2)$$

And the inequality (3.1) is obviously true.

Theorem 3.2. Let the complex Equation (1.1) satisfy Condition C, and the positive constant ε in (1.3) and (1.7) be small enough.

1) When $0<\sigma,\tau<1$, Problem H for the Equation (1.10) has a solution $\left[w(z),u(z)\right]$, where $w(z),u(z)\in W^1_{p_0,2}(D)$ with the constant $p_0\left(2<p_0\le p\right)$ as stated before.

2) When $\min(\sigma,\tau)>1$, Problem H for (1.10) has a solution $\left[w(z),u(z)\right]$, where $w(z)\in W^1_{p_0,2}(D)$, provided that

$$M_8=L_{p_0,2}\left[A_3,\bar{D}\right]+C_\alpha\left[c_2,\Gamma\right]+\sum_{j=0}^{m}\left|b_j\right| \quad (3.3)$$

is sufficiently small.

Proof 1) In this case, the algebraic equation for t becomes

$$M_9\left\{L_{p,2}\left[A_3,\bar{D}\right]+L_{p,2}\left[B_1,\bar{D}\right]t^\sigma+L_{p,2}\left[B_2,\bar{D}\right]t^\tau+L_\alpha\left[c_2,\Gamma\right]+\sum_{j=0}^{m}\left|b_j\right|\right\}=t, \quad (3.4)$$

with $M_9=M_6+M_7$, where M_6,M_7 are constants as stated in (2.14) and (2.15). Because $0<\sigma,\tau<1$, Equation (3.4) has a unique solution $t=M_{10}>0$. Now we introduce a bounded, closed and convex subset B^* of the Banach space $C\left(\bar{D}\right)\times C\left(\bar{D}\right)$, whose elements are of the form $\left[w(z),u(z)\right]$ satisfying the condition

$$w(z),u(z)\in C\left(\bar{D}\right), C\left[w(z),\bar{D}\right]+C\left[u(z),\bar{D}\right]\le M_{10}. \quad (3.5)$$

We choose a pair of functions $\left[\tilde{w}(z),\tilde{u}(z)\right]\in B^*$ and substitute it into the appropriate positions of $F\left(z,u,w,w_z\right),G\left(z,u,w\right)$ in (1.10) and the boundary condition (1.11) to obtain

$$w_{\bar{z}}=\tilde{F}\left(z,u,w,\tilde{u},\tilde{w},w_z\right)+G\left(z,\tilde{u},\tilde{w}\right), \quad (3.6)$$

$$\operatorname{Re}\left[\overline{\lambda(z)}w(z)\right]=-c_1(z)\tilde{u}+c_2(z),\ z\in\Gamma, \quad (3.7)$$

where

$$\begin{aligned}&\tilde{F}\left(z,u,w,\tilde{u},\tilde{w},w_z\right)\\ &=\operatorname{Re}\left[Q\left(z,\tilde{u},\tilde{w},w_z\right)w_z+A_1\left(z,\tilde{u},\tilde{w}\right)w\right]\\ &\quad+A_2\left(z,\tilde{u},\tilde{w}\right)u+A_3\left(z,\tilde{u},\tilde{w}\right).\end{aligned}$$

In accordance with the method in the proof of Theorem 1.2.5 [5], we can prove that the boundary value problem (3.6), (3.7) and (1.15) has a unique solution $\left[w(z),u(z)\right]$. Denote by $[w,u]=T\left[\tilde{w}(z),\tilde{u}(z)\right]$ the mapping from $\left[\tilde{w}(z),\tilde{u}(z)\right]$ to $\left[w(z),u(z)\right]$. Noting that

$$L_{p,2}\left[A_2u,\bar{D}\right]\le\varepsilon M_{10}k_0,\ C_\alpha\left[-c_1u,\Gamma\right]\le\varepsilon M_{10}k_0,$$

provided that the positive number ε is sufficiently small, and noting that the coefficients of complex Equation (3.6) satisfy the same conditions as in Condition C, from Theorem 2.2, we can obtain

$$\begin{aligned}&C\left[w,\bar{D}\right]+L_{p_0,2}\left[\left|w_{\bar{z}}\right|+\left|w_z\right|,\bar{D}\right]+C\left[u,\bar{D}\right]+L_{p_0,2}\left[u_z,\bar{D}\right]\le M_9\left\{L_{p,2}\left[A_3,\bar{D}\right]+C_\alpha\left[c_2,\Gamma\right]+\sum_{j=0}^{m}\left|b_j\right|+L_{p,2}\left[G,\bar{D}\right]\right\}\\ &\le M_9\left\{M_8+L_{p,2}\left[B_1,\bar{D}\right]C\left[\tilde{w},\bar{D}\right]^\sigma+L_{p,2}\left[B_2,\bar{D}\right]C\left[\tilde{u},\bar{D}\right]^\tau\right\}\le M_9\left\{M_8+L_{p,2}\left[B_1,\bar{D}\right]M_{10}^\sigma+L_{p,2}\left[B_2,\bar{D}\right]M_{10}^\tau\right\}=M_{10}.\end{aligned} \quad (3.8)$$

This shows that T maps B^* onto a compact subset in B^*. Next, we verify that T in B^* is a continuous operator. In fact, we arbitrarily select a sequence $\left\{\tilde{w}_n(z),\tilde{u}_n(z)\right\}$ in B^*, such that

$$C\left(\tilde{w}_n-\tilde{w}_0,\bar{D}\right)+C\left(\tilde{u}_n-\tilde{u}_0,\bar{D}\right)\to 0 \text{ as } n\to\infty. \quad (3.9)$$

By Lemma 3.1, we can see that

$$\begin{aligned}&L_{p,2}\left[A_j\left(z,\tilde{u}_n,\tilde{w}_n\right)-A_j\left(z,\tilde{u}_0,\tilde{w}_0\right),\bar{D}\right]\\ &\quad\to 0\left(j=1,2,3\right) \text{ as } n\to\infty.\end{aligned} \quad (3.10)$$

Moreover, from $\left[w_n,u_n\right]=T\left[\tilde{w}_n,\tilde{u}_n\right],\left[w_0,u_0\right]=T\left[\tilde{w}_0,\tilde{u}_0\right]$, it is clear that $\left[w_n-w_0,u_n-u_0\right]$ is a solution of Problem H for the following equation:

$$\begin{aligned}\left(w_n-w_0\right)_{\bar{z}}&=\tilde{F}\left(z,u_n,w_n,\tilde{u}_n,\tilde{w}_n,w_{nz}\right)\\ &\quad-\tilde{F}\left(z,u_0,w_0,\tilde{u}_0,\tilde{w}_0,w_{0z}\right)\\ &\quad+G\left(z,\tilde{u}_n,\tilde{w}_n\right)-G\left(z,\tilde{u}_0,\tilde{w}_0\right) \text{ in } D,\end{aligned} \quad (3.11)$$

$$\begin{aligned}&\operatorname{Re}\left[\overline{\lambda(z)}\left(w_n-w_0\right)\right]\\ &=-c_1(z)\left(\tilde{u}_n-\tilde{u}_0\right)+h(z) \text{ on } \Gamma,\end{aligned} \quad (3.12)$$

$$u_n\left(1/a_j^*\right)-u_0\left(1/a_j^*\right)=0,\ j=1,\cdots,N_0,$$
$$u_n\left(1/a_j\right)-u_0\left(1/a_j\right)=0,\ j=0,1,\cdots,m. \quad (3.13)$$

In accordance with the method in proof of Theorem 2.2, we can obtain the estimate

$$\begin{aligned}&C\left[w_n-w_0,\bar{D}\right]+L_{p_0,2}\left[\left|\left(w_n-w_0\right)_{\bar{z}}\right|+\left|\left(w_n-w_0\right)_z\right|,\bar{D}\right]\\&+C\left[u_n-u_0,\bar{D}\right]+L_{p_0,2}\left[\left(u_n-u_0\right)_z,\bar{D}\right]\\&\le M_{11}\left\{\varepsilon L_{p,2}\left[A_2\left(z,\tilde{u}_n,\tilde{w}_n\right)\tilde{u}_n-A_2\left(z,\tilde{u}_0,\tilde{w}_0\right)\tilde{u}_0,\bar{D}\right]\right.\\&\quad+L_{p,2}\left[A_3\left(z,\tilde{u}_n,\tilde{w}_n\right)-A_3\left(z,\tilde{u}_0,\tilde{w}_0\right),\bar{D}\right]\\&\quad\left.+L_{p,2}\left[G\left(z,\tilde{u}_n,\tilde{w}_n\right)-G\left(z,\tilde{u}_0,\tilde{w}_0\right),\bar{D}\right]+\varepsilon C_\alpha\left[c_1(z)\left(\tilde{u}_n-\tilde{u}_0\right),\Gamma\right]\right\},\end{aligned} \quad (3.14)$$

in which $M_{11}=M_{11}\left(q_0,p_0,k_0,\alpha,K,D\right)$. From (3.9), (3.10) and the above estimate, we obtain $C\left[w_n-w_0,\bar{D}\right]+C\left[u_n-u_0,\bar{D}\right]\to 0$ as $n\to\infty$. On the basis of the Schauder fixed-point theorem, there exists a function $\left[w(z),u(z)\right]\left(w(z),u(z)\in C(\bar{D})\right)$ such that $\left[w(z),u(z)\right]=T\left[w(z),u(z)\right]$. And from Theorem 2.2, it is easy to see that $w(z),u(z)\in W^1_{p_0,2}(D)$, and $\left[w(z),u(z)\right]$ is a solution of Problem H for the Equation (1.10) with the condition $0<\sigma,\tau<1$.

In addition, using a method similar to the above, we see that if $G(z,u,w)=\operatorname{Re}B_1w+B_2|u|^\tau$ in D, where $0<\tau<1, L_{p,2}\left[B_j,\bar{D}\right]\le k_0<\infty, j=1,2,$ then the above solvability result still holds.

2) Secondly, we discuss the case, where $\min(\sigma,\tau)>1$. In this case, (3.4) has the solution $t=M_{10}$ provided that M_8 in (3.3) is small enough. We consider a closed and convex subset B_* in the Banach space $C(\bar{D})\times C(\bar{D})$, *i.e.*,

$$B_*=\left\{w(z),u(z)\in C(\bar{D}),C\left[w,\bar{D}\right]+C\left[u,\bar{D}\right]\le M_{10}\right\}.$$

Applying a similar method as before, we can verify that there exists a solution $\left[w(z),u(z)\right]\in W^1_{p_0,2}(D)\times W^1_{p_0,2}(D)$ of Problem H for (1.10) with the condition $\min(\sigma,\tau)>1$.

Moreover, if $G(z,u,w)=\operatorname{Re}B_1w+B_2|u|^\tau$ in D, where $1<\tau<\infty$, $L_{p,2}\left[B_j,\bar{D}\right]\le k_0<\infty, j=1,2,$ then under the same condition, we can derive the above solvability result by a similar method.

From the above theorem, the next result can be derived.

Theorem 3.3 Under the same conditions as in Theorem 3.2, Problem G has $l+1$ solvability conditions, and the general solution $u(z)$ includes $m+1$ arbitrary real constants.

Proof Let the solution $\left[w(z),u(z)\right]$ of Problem H for (1.10) be substituted into the boundary condition (1.11). If the function $h(z)=0,\ z\in\Gamma$, *i.e.* $h_j=0, z\in\Gamma', j=0,1,\cdots,l,$ then we have $w(z)=u_z$ in D and the function $u(z)$ is just a solution of Problem G for (1.1). Hence the total number $l+1$ of above equalities is just the number of solvability conditions of Problem G.

Also note that the real constants $b_j\left(j=0,1,\cdots,m\right)$ in (1.12) and (1.15) are arbitrarily chosen. This shows that the general solution of Problem G for (1.1) includes the $m+1$ arbitrary real constants as stated in the theorem.

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