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The Triangle Inequality and Its Applications in the Relative Metric Space*

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ABSTRACT

Let *C* be a plane convex body. For arbitrary points $a,b \in E^n$, denote by $|ab|$ the Euclidean length of the line-segment ab. Let a_1b_1 be a longest chord of *C* parallel to the line-segment ab. The relative distance $d_C(a,b)$ between the points a and b is the ratio of the Euclidean distance between a and b to the half of the Euclidean distance between a_1 and b_1. In this note we prove the triangle inequality in E^2 with the relative metric $d_C(\cdot,\cdot)$, and apply this inequality to show that $6 \le l(P) \le 8$, where $l(P)$ is the perimeter of the convex polygon P measured in the metric $d_P(\cdot,\cdot)$. In addition, we prove that every convex hexagon has two pairs of consecutive vertices with relative distances at least 1.

Keywords: Relative Distance; Triangle Inequality; Hexagon

We use some definitions from [1]. For arbitrary points $a,b \in E^n$, denote by *ab* the line-segment connecting the points *a* and *b*, by $|ab|$ the Euclidean length of the line-segment *ab*, and by $\overline{ab}$ the straight line passing through the points *a* and b. Let a_1b_1 be a longest chord of *C* parallel to *ab*. The *C-distance* $d_C(a,b)$ between the points a,b is defined by the ratio of $|ab|$ to $\frac{1}{2}|a_1b_1|$. If there is no confusion about C, we may use the terms *relative distance* between a and b. Observe that for arbitrary points $a,b \in E^n$ the *C*-distance between a and b is equal to their $\left[\frac{1}{2}\left(C+(-C)\right)\right]$-distance. Thus $d_C(\cdot,\cdot)$ is the metric of E^n whose unit ball is $\frac{1}{2}\left(C+(-C)\right)$. We denote by λ_n the relative distance between two consecutive vertices of the regular n-gon. It is clear that $\lambda_3 = \lambda_4 = 2, \lambda_5 = \sqrt{5}-1$, and $\lambda_6 = 1$. Doliwka and Lassak [1] proved that every convex pentagon has a pair of consecutive vertices with relative distance at least λ_5.

In this paper we first prove the triangle inequality with respect to the relative metric of a plane convex body. Then we apply this inequality to show that $6 \le l(P) \le 8$, where $l(P)$ is the perimeter of the convex polygon P measured in the metric $d_P(\cdot,\cdot)$. In the last, we prove that every convex hexagon has two pairs of consecutive vertices with relative distances at least 1.

For simplicity, if two lines $\overline{pq}$ and $\overline{rs}$ are parallel, we write $\overline{pq} \| \overline{rs}$. Denote by $x_1x_2\cdots x_n$ the polygon formed by the points $x_1,x_2,\cdots,x_n$, and by $A(P)$ the area of the polygon P. A chord pq of C is called an *affine diameter* if there is no longer chord parallel to pq in C.

Lemma 1 *Let C be a plane convex body, and* x,y,z *be arbitrary three points in* E^2. *Then the triangle inequality* $d_C(y,z) \le d_C(x,z) + d_C(x,y)$ *holds.*

Proof. By the properties of affine transformation, we may assume that the triangle xyz formed by the points x,y,z is a regular triangle. Let x_1y_1, x_2z_2, and z_1y_2 be the affine diameters of C parallel to *xy*, *xz*, *yz* re-

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spectively, and let $|x_1y_1| = \mu_1, |x_2z_2| = \mu_2, |z_1y_2| = \mu_3$. Since xyz is a regular triangle, by the definition of relative distance, we need to prove the following inequality.

$$\frac{1}{\mu_3} \le \frac{1}{\mu_1} + \frac{1}{\mu_2} \tag{1}$$

Take the lines $\overline{x_1u}$ and $\overline{x_2v}$ through the points x_1 and x_2, respectively, such that they are parallel to $\overline{z_1y_2}$, where u (resp. v) is the intersection point of the lines x_1u (resp. x_2v) and y_1z_2. Denote by μ the relative distance between the points x_1 and u. (See **Figure 1**) Since z_1y_2 is an affine diameter of C, we obtain $\mu \le \mu_3$ and

$$\frac{1}{2}\mu_2\mu_3 \sin\frac{\pi}{3} \ge \frac{1}{2}\mu_2\mu \sin\frac{\pi}{3} = A(x_1z_2ux_2) \tag{2}$$

The following equality is obvious.

$$A(x_1x_2y_1z_2) = \frac{1}{2}\mu_1\mu_2 \sin\frac{\pi}{3} \tag{3}$$

By symmetry, we may assume without loss of generality that $|x_1u| \ge |x_2v|$. Then

$$A(x_2y_1u) \le A(x_1y_1u) = \frac{1}{2}\mu_1\mu \sin\frac{\pi}{3} \le \frac{1}{2}\mu_1\mu_3 \sin\frac{\pi}{3} \tag{4}$$

By (2), (3), and (4),

$$\begin{aligned} &\frac{1}{2}\mu_2\mu_3 \sin\frac{\pi}{3} + \frac{1}{2}\mu_1\mu_3 \sin\frac{\pi}{3} \\ &\ge A(x_1z_2ux_2) + A(uy_1x_2) \\ &= A(x_1x_2y_1z_2) = \frac{1}{2}\mu_1\mu_2 \sin\frac{\pi}{3} \end{aligned}$$

from which (1) holds and the proof is complete.

Let P be a convex polygon. We denote by $bd(P)$ the boundary of P, and by $l(P)$ the perimeter of P measured in the metric $d_P(\cdot,\cdot)$.

Proposition 2 *For arbitrary convex polygon* P, *we have* $6 \le l(P) \le 8$.

From Theorem 2 in [2] we know that for every convex polygon P the perimeters of P and $\frac{1}{2}(P + (-P))$ are equal in every distance space. Thus we may assume without loss of generality that P is a centrally symmetric convex polygon. We take a point $p_1 \in bd(P)$, then there exists a point $p_4 \in bd(P)$ such that p_1p_4 passes through the center of P. And take the points $p_2, p_3 \in bd(P)$ such that $d_P(p_2, p_3) = \frac{1}{2} d_P(p_1, p_4)$ and $p_2p_3 \| p_1p_4$. Then $H = p_1p_2p_3p_4p_5p_6$ is an affine regular hexagon, where p_5, p_6 are the antipodal points of p_2, p_3, respectively. It is clear that $l(H) = 6$. Since the boundary of P is dissected into six parts by the vertices of H, we consider the part between p_1 and p_6 (the other five parts can be treated similarly). Let $v_1, v_2, \cdots, v_k$ be the vertices of P between p_1 and p_6. (See **Figure 2**) Draw the line-segments $p_1v_1, p_1v_2, \cdots, p_1v_k$. By Lemma 1, we get

$$\begin{aligned} &d_P(p_1, v_k) + d_P(v_k, p_6) \ge d_P(p_1, p_6), \\ &d_P(p_1, v_{k-1}) + d_P(v_{k-1}, v_k) \ge d_P(p_1, v_k), \cdots, \\ &d_P(p_1, v_2) + d_P(v_2, v_3) \ge d_P(p_1, v_3), \\ &d_P(p_1, v_1) + d_P(v_1, v_2) \ge d_P(p_1, v_2). \end{aligned}$$

Adding all these triangle inequalities, we obtain that

$$\begin{aligned} &d_P(p_1, v_1) + d_P(v_1, v_2) + d_P(v_2, v_3) \\ &+ \cdots + d_P(v_k, v_6) \ge d_P(p_1, p_6) \end{aligned}$$

So we get $6 = l(H) \le l(P)$.

It is clear that we may circumscribe a parallelogram $Q := efgh$ about P with the minimal area such that $p_1, p_2, p_3, p_4 \in bd(P)$ are the midpoints of the sides ef, fg, gh, he, respectively. By the properties of affine transformation we suppose without loss of generality that Q is a square. Let $v_1, v_2, \cdots, v_n$ be the vertices of P between p_1 and p_2. Let $v_i^x, 1 \le i \le n$, be the perpendicular projection of v_i onto the line segment gf, and let $v_i^y, 1 \le i \le n$, be the perpendicular projection of v_i onto the line segment ef. (See **Figure 3**) According to Lemma 1, we obtain that

$$\begin{aligned} &d_P(p_1, v_1^y) + d_P(f, v_1^x) \ge d_P(p_1, v_1), \\ &d_P(v_1^y, v_2^y) + d_P(v_1^x, v_2^x) \ge d_P(v_1, v_2), \cdots, \\ &d_P(v_n^y, f) + d_P(v_n^x, p_2) \ge d_P(v_n, p_2) \end{aligned}$$

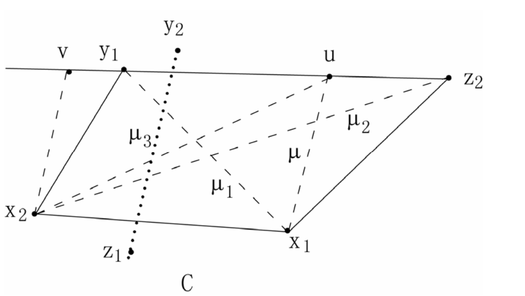

Figure 1. The figure of Lemma 1.

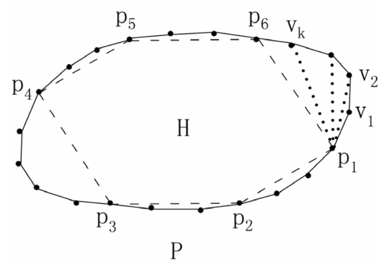

Figure 2. The figure of 6 = *l*(*H*) ≤ *l*(*P*).

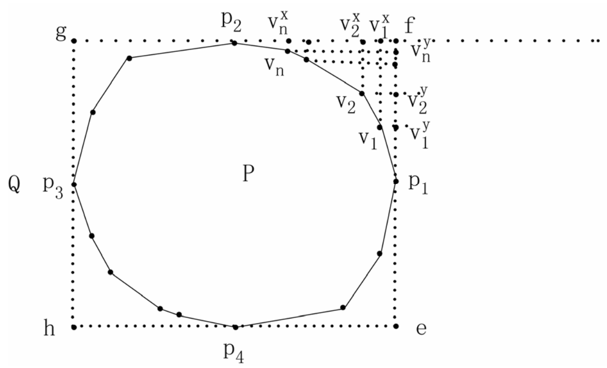

Figure 3. The figure of $l(P) \le l(Q) = 8$.

Adding all these inequalities, we have

$$d_P(p_1, f) + d_P(f, p_2)$$
$$\ge d_P(p_1, v_1) + d_P(v_1, v_2) + \cdots + d_P(v_n, p_2)$$

Similarly, we can consider the other parts of the polygon P between p_2 and p_3, p_3 and p_4, p_4 and p_1. Hence we have $l(P) \le l(Q) = 8$.

From Proposition 2 we obtain

Corollary 3 *Every convex hexagon has a pair of consecutive vertices with relative distance at least* 1 (*that is,* λ_6).

By the following Lemma [3], we give a stronger result than Corollary 3.

Lemma 4 *Let C be a plane convex body. We can circumscribe a parallelogram P about C such that the midpoints of a pair of opposite sides of P belong to C.*

Theorem 5 *Every convex hexagon has two pairs of consecutive vertices with relative distances at least* 1.

Proof. Denote by H the given convex hexagon. By Lemma 4, we can circumscribe a parallelogram P about H such that the midpoints of the opposite level sides of P belong to H. If H is a degenerate hexagon, then the result is obvious. Hence we consider the following three cases.

Case 1. The parallelogram P has two sides, each of which contains exactly two vertices of H.

This case contains two different configurations, as shown in **Figure 4**. We first consider (1) in **Figure 4**. Since the segment ac is an affine diameter of H, we get $d_H(a,c) = 2$. By Lemma 1, we obtain $d_H(a,b) + d_H(b,c) \ge d_H(a,c) = 2$. Then either $d_H(a,b) \ge 1$ or $d_H(b,c) \ge 1$. Similarly, df is an affine diameter of H, so either $d_H(d,e) \ge 1$ or $d_H(e,f) \ge 1$. Then consider (2) in **Figure 4**. Since d is the midpoint of the side yz of P, the segments dc and de are not less than half of their affine diameters, respectively. Then we obtain that $d_H(c,d) \ge 1$ and $d_H(d,e) \ge 1$.

Case 2. P has exactly one side which contains two vertices of H.

If these two vertices of H belong to xy or wz,

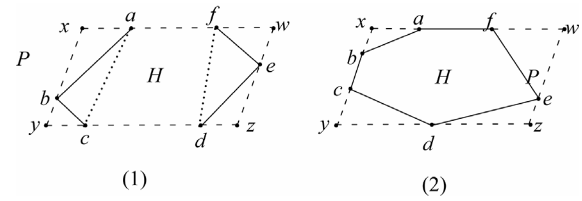

Figure 4. Case 1.

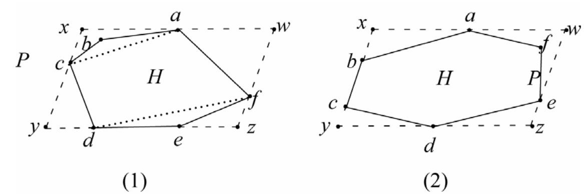

Figure 5. Case 2.

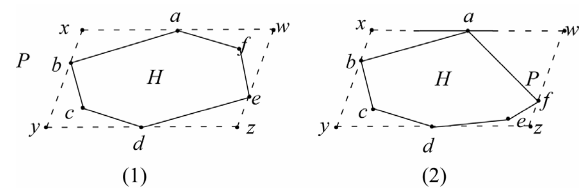

Figure 6. Case 3.

then the result is clear, see (2) in **Figure 5**. Otherwise, since P and H have five points in common, the remaining vertex of H must be located inside one of the four triangular regions bounded by P and H. See (1) in **Figure 5**. Since a is the midpoint of the side xw of P, we get $d_H(a, f) \ge 1$. Moreover, one of the segments ac and df must be an affine diameter of H, say df, then we obtain that either $d_H(e,d) \ge 1$ or $d_H(e, f) \ge 1$.

Case 3. Every side of P contains exactly one vertex of H.

There are two different configurations in this case, as shown in **Figure 6**. In (1) of **Figure 6**, since a and d are midpoints of the sides xw and yz of P, respectively, we conclude that $d_H(a,b) \ge 1$ and $d_H(d,e) \ge 1$. In (2) of **Figure 6**, since a is the midpoint of the side xw of P, we obtain that $d_H(a,b) \ge 1$ and $d_H(a, f) \ge 1$. The proof is complete.

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