Applied Mathematics, **2013, 4, 969-971**
http://dx.doi.org/10.4236/am.2013.47133 Published Online July 2013 (http://www.scirp.org/journal/am)

Second Descendible Self-Mapping with Closed Periodic Points Set

Gengrong Zhang[1], Zhanjiang Ji[1], Fanping Zeng[1,2]
[1]College of Mathematics and Information Science, Guangxi University, Nanning, China
[2]Department of Mathematics and Computer Science, Liuzhou Teachers College, Liuzhou, China
Email: zgrzaw@gxu.edu.cn

Received February 21, 2013; revised March 22, 2013; accepted March 31, 2013

Copyright © 2013 Gengrong Zhang *et al*. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ABSTRACT

Let $X_n=\prod_{i=1}^{n} I_i\left(I_i=I\right)$ and $f:X_n\to X_n$ be a continuous map. If *f* is a second descendible map, then $P(f)$ is closed if and only if one of the following hold: 1) $pp(f)\subset\left\{2^k:k\ge 0\right\}$; 2) For any $z\in R(f)$, there exists a $y\in w(z,f)\cap P(f)$ such that every point of the set $orb(y,f)$ is a isolated point of the set $w(z,f)$; 3) For any $z\in R(f)$, the set $w(z,f)$ is finite; 4) For any $z\in R(f)$, the set $w'(z,f)$ is finite. The consult give another condition of *f* with closed periodic set other than [1].

Keywords: Periodic Point; Recurrent Point; *w*-Limit Point; Second Descendible Map

1. Introduction

In this paper, let X_n denote $\prod_{i=1}^{i=n} I_i\left(I_i=I\right)$, X denote compact metric space, $C^0(X,X)$ denote all continuous self-maps on *X*. The concepts of periodic point, *w*-limit point of *z* and the orbit of *z* are showed by [2]. Denote by $P(f)$ the sets of periodic points of *f*, denote by $w(z,f)$ the *w*-limit points of *z*, and denote by $orb(z,f)$ the orbit of *z*. A point $x\in X$ is said to be recurrent point if for any neighborhood $V(x)$ of *x*, there exists a positive integer *m* such that $f^m(x)\in V(x)$. Let $R(f)$ denote the set of recurrent points.

In recent years, many authors studied equivalent conditions of closed periodic points set. Gengrong Zhang [3], Xiong Jincheng [4] and Wang Lidong [5] studied respectively anti-triangular map of X_2, continuous self-map of the closed interval and continuous self-map of the circle. They showed equivalent conditions of closed periodic points set (see more detail for [3-5]). Du Ruijin [1] given five equivalent conditions of closed periodic points set if *f* is a second descendible map of X_n. 1) $P(f)=R(f)$; 2) $P(f)=W(f)$; 3) $P(f)=\Omega(f)$; 4) $P(f)=CR(f)$; 5) $P(f)=AP(f)$.

In this paper, we will continue to study new equivalent conditions about that the set $P(f)$ is closed. The following theorem are given.

Main Theorem Let $f:X_n\to X_n$ be a continuous map. If *f* is a second descendible map, then the following properties are equivalent:

1) The set $P(f)$ is closed; 2) $pp(f)\subset\left\{2^k:k\ge 0\right\}$; 3) For any $z\in R(f)$, there exists a $y\in w(z,f)\cap P(f)$ such that every point of the set $orb(y,f)$ is a isolated point of the set $w(z,f)$; 4) For any $z\in R(f)$, the set $w(z,f)$ is finite; 5) For any $z\in R(f)$, the set $w'(z,f)$ is finite.

2. Definition and Lemma

Definition 1 For any $i\in\{1,2,\cdots,n\}$, let $p_i:X_n\to I$, define: $p_i(x)=x_i, x=\left(x_1,x_2,\cdots,x_n\right)\in X_n$, then p_i is said to be canonical projection.

Definition 2 Let $f\in C^0\left(X_n,X_n\right)$, the map *f* is said to be second descendible if for any $i\in\{1,2,\cdots,n\}$, there exists $F_i\in C^0(I,I)$ such that $p_i\bullet f=F_i\bullet p_i\left(i=1,2,\cdots,n\right)$. In this case F_i is a descendible group of *f*.

Lemma 1 [6] Let $f\in C^0\left(X_n,X_n\right)$. Then the following properties are equivalent:

1) F_i is a descendible group of *f*;
2) $f=F_1\times F_2\times\cdots\times F_n$.

Lemma 2 Let $f \in C^0(X_n, X_n)$. If *f* is a second descendible map and F_i is a descendible group of *f*, then any $z = (z_1, z_2, \cdots, z_n) \in X_n$, we have

$$w(z, f) \subset \prod_{i=1}^{i=n} w(z_i, F_i).$$

Proof. Suppose $y = (y_1, y_2, \cdots, y_n) \in w(z, f)$. There exists a positive integer sequence $\{m_k\}$ such that $f^{m_k}(z) \to y$. By Lemma 1, we can get $f^{m_k}(z) = (F_1^{m_k}(z_1), F_2^{m_k}(z_2), \cdots, F_n^{m_k}(z_n))$. Hence for any $i \in \{1, 2, \cdots, n\}$, we have $F_i^{m_k}(z_i) \to y_i$. Thus $y_i \in \prod_{i=1}^{i=n} w(z_i, F_i)$. This complete the proof.

Lemma 3 Let $f \in C^0(X_n, X_n)$. Then $z \in R(f)$ if and only if $z \in w(z, f)$.

Proof. Suppose $z \in R(f)$. For any positive integer *k*, there exists a positive integer sequence $\{m_k\}$ such that $f^{m_k}(z) \in V\left(z, \frac{1}{k}\right)$. Hence $z \in w(z, f)$. Assume $z \in w(z, f)$. Then there exists a positive integer sequence $\{m_k\}$ such that $f^{m_k}(z) \to z$. By definition, $z \in R(f)$. Hence we complete the proof.

Lemma 4 [5] Let $f \in C^0(X_n, X_n)$. Then

1) For any $z \in X$, the set $w(z, f)$ is periodic orbit if and only if the set $w(z, f)$ is finite.

2) Let $y \in w(z, f) \cap F(f)$. If *y* is a isolated point of the set $w(z, f)$, then we have $w(z, f) = \{y\}$.

Lemma 5 Let $f \in C^0(X_n, X_n)$ and $y \in w(z, f) \cap P(f)$. If all points of the set $orb(y, f)$ are isolated points of the set $w(z, f)$, then we have $orb(y, f) = w(z, f)$.

Proof. Suppose $y \in w(z, f) \cap P(f)$. Then there exists a positive integer *l* and a sequence $\{m_k\}$ such that $f^l(y) = y$ and $f^{m_k}(z) \to y$. Hence for any $i \in \{1, 2, \cdots, l\}$, we have $f^i(y) \in F(f^l) \cap w(f^i(z), f^l)$. By assumption, for any $i \in \{1, 2, \cdots, l\}$, the point of $\{f^i(y)\}$ is a isolated point of the set $w(z, f)$. Thus for any $i \in \{1, 2, \cdots, l\}$, there exists a neighborhood $V(f^i(y))$ of $f^i(y)$ such that $V(f^i(y)) \cap w(z, f) = \{f^i(y)\}$. Using the equation of $w(z, f) = \bigcup_{i=1}^{i=l} w(f^i(z), f^l)$, we have $V(f^i(y)) \cap w(f^i(z), f^l) = \{f^i(y)\}$.

By 2) of Lemma 4, we can get that $w(f^i(z), f^l) = \{f^i(y)\}$. Hence we have that $orb(y, f) = w(z, f)$.

Lemma 6 Let $f \in C^0(X_n, X_n)$ and the set $w(z, f)$ is infinite. Then any $k \neq m \geq 0$, we can get that $f^k(z) \neq f^m(z)$.

Proof. Assume on the contrary that there exists $k > m \geq 0$ such that $f^k(z) = f^m(z)$. Thus $f^{k-m}(f^m(z)) = f^m(z)$. Hence the point $f^m(z)$ is a periodic point. Therefore the set $orb(z, f)$ is finite, which is impossible. Thus the lemma is proved.

Lemma 7 [5] Let $f \in C^0(X_n, X_n)$ and for any $z \in R(f)$, the set $w'(z, f)$ is finite. Then we have $w'(z, f) \subset P(f)$.

Lemma 8 Let $f \in C^0(X_n, X_n)$. If *f* is a second descendible map and F_i is a descendible group of *f*, and the set $P(f)$ is closed. Then any $z = (z_1, z_2, \cdots, z_n) \in X_n$, we have the set $w(z, f)$ is periodic orbit.

Proof. According to [6], we can get that $P(f) = \prod_{i=1}^{i=n} P(F_i)$. By assumption, the set $P(f)$ is closed. Hence for any $i \in \{1, 2, \cdots, n\}$, the set $P(F_i)$ is closed. Let $g \in C^0(I, I)$. According to [4], the set $P(g)$ is closed if and only if for any $x \in I$, the set $w(x, g)$ is periodic orbit. Hence for any $x \in I$ and any $i \in \{1, 2, \cdots, n\}$, the set $w(x, F_i)$ is periodic orbit. Using 1) of Lemma 4, for any $x \in I$ and any $i \in \{1, 2, \cdots, n\}$, the set $w(x, F_i)$ is finite. The set $w(z, f)$ is finite since $w(z, f) \subset \prod_{i=1}^{i=n} w(z_i, F_i)$. Therefore we have the set $w(z, f)$ is periodic orbit.

3. The Proof of Main Theorem

Main Theorem Let $f: X_n \to X_n$ be a continuous map. If *f* is a second descendible map, then the following properties are equivalent:

1) The set $P(f)$ is closed;

2) $pp(f) \subset \{2^k : k \geq 0\}$;

3) For any $z \in R(f)$, there exists a $y \in w(z, f) \cap P(f)$ such that every point of the set $orb(y, f)$ is a isolated point of the set $w(z, f)$;

4) For any $z \in R(f)$, the set $w(z, f)$ is finite;

5) For any $z \in R(f)$, the set $w'(z, f)$ is finite.

Proof. 1) $\Rightarrow$ 2) First we will show that the set $P(f)$ is closed if and only if for any $i \in \{1, 2, \cdots, n\}$, $pp(F_i) \subset \{2^k : k \geq 0\}$ (*).

According to [6], we can get that $P(f) = \prod_{i=1}^{i=n} P(F_i)$. Hence the set $P(f)$ is closed if and only if for any $i \in \{1, 2, \cdots, n\}$, the set $P(F_i)$ is closed. Let $g \in C^0(I, I)$. It is obvious that the set $P(g)$ is closed if and only if $pp(g) \subset \{2^k : k \geq 0\}$. Thus we complete the proof of (*).

Assume $z = (z_1, z_2, \cdots, z_n) \in P(f)$. Then there exists a integer $l \geq 0$ such that $F_i^{2^l}(z_i) = z_i$ for any $i \in \{1, 2, \cdots, n\}$. Hence $f^{2^l}(z) = z$. Therefore 1) implies 2).

2) $\Rightarrow$ 1) Suppose $pp(f) \subset \{2^k : k \geq 0\}$. For any $i \in \{1, 2, \cdots, n\}$, $\forall z_i \in P(F_i)$. Let $z = (z_1, z_2, \cdots, z_n)$. According to [6], we can get that $P(f) = \prod_{i=1}^{i=n} P(F_i)$. Hence $z \in P(f)$. Then there exists a integer $l \geq 0$ such that $f^{2^l}(z) = z$. Thus for any $i \in \{1, 2, \cdots, n\}, F_i^{2^l}(z_i) = z_i$.

By (*), the set $P(f)$ is closed.

1) $\Rightarrow$ 3) By assumption and according to [1], $P(f)=R(f)$. For any $z\in R(f)$, let $y=z$. Thus $y\in w(z,f)\cap P(f)$. By assumption and Lemma 8, the set $w(z,f)$ is periodic orbit. Using 1) of Lemma 4, the set $w(z,f)$ is finite. Hence the set $w'(z,f)$ is empty. Thus 1) implies 3).

3) $\Rightarrow$ 4) By assumption, for any $z\in R(f)$, there exists a $y\in w(z,f)\cap P(f)$ such that every point of the set $orb(y,f)$ is a isolated point of the set $w(z,f)$. By Lemma 5, $orb(y,f)=w(z,f)$. Hence the set $w(z,f)$ is finite.

4) $\Rightarrow$ 5) It is obvious that 4) implies 5).

5) $\Rightarrow$ 1) For any $z\in R(f)$, we have $z\in w(z,f)$.

Case 1: Suppose that the set $w(z,f)$ is finite. Using 1) of Lemma 4, the set $w(z,f)$ is periodic orbit. So $z\in P(f)$. Thus $P(f)=R(f)$.

Case 2: Assume that the set $w(z,f)$ is infinite. Then exists a sequence $\{m_k\}$ such that the sequence $\{f^{m_k}(z)\}$ converges to z and by Lemma 6, all points of the set $orb(z,f)$ are different. Hence $z\in w'(z,f)$. By assumption that the set $w'(z,f)$ is finite and Lemma 7, we have that $z\in w'(z,f)\subset P(f)$. Thus $P(f)=R(f)$.

According to [1], the set $P(f)$ is closed. Thus we complete the proof of the theorem.

4. Acknowledgements

This work was supported by the NSF of China (No. 11161029), NSF of Guangxi (2010GXNSFA013109, 2012GXNSFDA276040, 2013GXNSFBA019020).

REFERENCES

[1] R.-J. Du, Y.-G. Jin and M.-X. Li, "On the Periodic Point Set of a n-Dimensional Self-Mapping," *Journal of Chongqing Technology Business University* (*Natural Science Edition*), Vol. 23, No. 1, 2006, pp. 12-14 (in Chinese).

[2] J.-C. Xiong, "Continous Self-Maps of the Closed Interval Whose Periodic Points Form a Closed Set," *Journal of China University of Science and Technology*, Vol. 11, No. 4, 1981, pp. 14-22.

[3] G. R. Zhang, R. H. Li and C. L. Yang, "Anti-Triangular Maps with Closed Periodic Sets," *Far East Journal of Dynamical Systems*, Vol. 19, No. 1, 2012, pp. 1-11.

[4] J. C. Xiong, "Dynamical Systems of Interval Maps: Nonwandering Set, Topological Entropy and Chaos," *Advances in Mathematics*, Vol. 17, No. 1, 1988 (in Chinese).

[5] L. D. Wang, "The Necessary and Sufficient Condition of a Closed Periodic Points Set of the Continuous Self-Maps of the Circle," *Journal of Shanxi Teacher's University* (*Natural Science Edition*), Vol. 4, No. 2, 1990 (in Chinese).

[6] B.-G. Yan, "The Chaotic Set and Asymptotic Periodic Point of Descendible Mapping," *Journal of Hebei University of Engineering* (*Natural Science Edition*), Vol. 24, No. 3, 2007, pp. 102-103 (in Chinese).

Copyright © 2013 SciRes.