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Entire Large Solutions of Quasilinear Elliptic Equations of Mixed Type*

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Abstract

In this paper, the existence and nonexistence of nonnegative entire large solutions for the quasilinear elliptic equation $div\left(|\nabla u|^{m-2}\nabla u\right) = p(x)f(u)+q(x)g(u)$ are established, where $m \ge 2$, f and g are nondecreasing and vanish at the origin. The locally Hölder continuous functions p and q are nonnegative. We extend results previously obtained for special cases of f and g.

Keywords: Entire Solutions, Large Solutions, Quasilinear Elliptic Equations

1. Introduction

In this paper, we consider the problem

$$\begin{cases} div\left(|\nabla u|^{m-2}\nabla u\right) = p(x)f(u)+q(x)g(u), \ in\ R^N (N\ge 3) \\ u(x)\to\infty, \quad as \quad |x|\to\infty \end{cases} \tag{1}$$

where $m \ge 2$, $f,g \in C([0,\infty),[0,\infty))\bigcap C^1((0,\infty),[0,\infty))$, the locally Hölder continuous functions p and q are nonnegative on $\mathbf{R}^N$. In addition, we assume that

$$\begin{gathered} f(0)=g(0)=0;\ f'(t)\ge 0, g'(t)\ge 0, \\ f(t)g(t)>0,\ for\ \forall t>0 \end{gathered} \tag{2}$$

We call nonnegative solutions of (1) entire large solutions. In fact, this problem appears in the study of non-Newtonian fluids [1,2] and non-Newtonian filtration [3,4], such problems also arise in the study of the sub-sonic motion of a gas [5], the electric potential in some bodies [6], and Riemannian geometry [7].

Large solutions of the problem

$$\begin{cases} \Delta u(x) = f(u(x)),\ \ x\in\Omega, \\ u|_{\partial\Omega}=\infty, \end{cases} \tag{3}$$

where Ω is a bounded domain in $R^N(N\ge 1)$ have been extensively studied, see [8-20]. A problem with $f(u)=e^u$ and $N=2$ was first considered by Bieberbach [13] in 1916. Bieberbach showed that if Ω is a bounded domain in R^2 such that $\partial\Omega$ is a C^2 submanifold of R^2, then there exisSts a unique $u\in C^2(\Omega)$ such that $\Delta u=e^u$ in Ω and $|u(x)-ln(d(x))^{-2}|$ is bounded on Ω. Here $d(x)$ denotes the distance from a point x to $\partial\Omega$. Rademacher [17], using the idea of Bieberbach, extended the above result to a smooth bounded domain in R^3. In this case the problem plays an important role, when $N=2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N=3$, according to [17], in the study of the electric potential in a glowing hollow metal body. Lazer and McKenna [6] extended the results for a bounded domain Ω in $R^N(N\ge 1)$ satisfying a uniform external sphere condition and the non-linearity $f=f(x,u)=p(x)e^u$, where $p(x)$ is continuous and strictly positive on $\overline{\Omega}$. Lazer and McKenna [12] obtained similar results when Δ is replaced by the Monge-Ampere operator and Ω is a smooth, strictly convex, bounded domain. Similar results were also obtained for $f=p(x)u^a$ with $a>1$. Posteraro [16], for $f(u)=-e^u$ and $N\ge 2$, proved the estimates for the solution $u(x)$ of the problem (1,2) and for the measure of Ω comparing with a problem of the same type defined in a ball. In particular, when $N=2$, Posteraro [16] obtained an explicit estimate of the minimum of $u(x)$ in terms of the measure of Ω:

$$\min_{\Omega} u(x) \ge ln(8\pi/|\Omega|).$$

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The existence, but not uniqueness, of solutions of the problem (3) with f monotone was studied by Keller [18]. For $f(u) = -u^a$ with $a > 1$, the problem (3) is of interest in the study of the sub-sonic motion of a gas when $a = 2$ (see [15]) and is related to a problem involving super-diffusion, particularly for $1 < a \le 2$ (see [21,22]). Pohozaev [15] proved the existence, but not uniqueness, for the problem (1.2) when $f(u) = -u^2$. For the case where $f(u) = -u^{(N+2)/(N-2)} (N > 2)$, Loewener and Nirenberg [20] proved that if $\partial\Omega$ consists of a disjoint union of finitely compact C^∞ manifolds, each having co-dimension less than $N/2+1$, then there exists a unique solution of the problem (3). The uniqueness was established for $f(u) = -u^a$ with $a > 3$, when $\partial\Omega$ is a C^2-submanifold and Δ is replaced by a more general second-order elliptic operator, by Kondrat'ev and Nikishkin [19]. Marcus and Veron [14] proved the uniqueness for $f(u) = -u^a$ with $a > 1$, when $\partial\Omega$ is compact and is locally the graph of a continuous function defined on an $(N-1)$-dimensional space.

In [23], the authors considered the problem of existence and nonexistence of positive entire large solutions of the semilinear elliptic equation

$$\Delta u = p(x)u^\alpha + q(x)u^\beta, \quad 0 < \alpha \le \beta.$$

Recently [24], which is to extend some of these results to a more general the problem

$$\begin{cases} \Delta u = p(x)f(u) + q(x)g(u) \quad in\ R^N,\ N \ge 3, \\ u(x) \to \infty \quad as\ |x| \to \infty. \end{cases}$$

Quasilinear elliptic problems with boundary blowup

$$\begin{cases} div\left(|\nabla u|^{m-2}\nabla u\right) = f(u(x)), \ x \in \Omega, \\ u|_{\partial\Omega} = \infty, \end{cases} \tag{4}$$

have been studied, see [9,25,26] and the references therein. Diaz and Letelier [10] proved the existence and uniqueness of large solutions to the problem (4) both for $f(u) = u^\gamma, \gamma > m-1$ (super-linear case) and $\partial\Omega$ being of the class C^2. Lu, Yang and E.H.Twizell [25] proved the existence of Large solutions to the problem (4) both for $f(u) = u^\gamma, \gamma > m-1, \Omega = R^N$ or Ω being a bounded domain (super-linear case) and $\gamma \le m-1, \Omega = R^N$ (sub-linear case) respectively.

Recently [27], which is to extend some results of [28] to the following quasilinear elliptic problem

$$\begin{cases} div\left(|\nabla u|^{m-2}\nabla u\right) = p(x)f(u), \quad in\ \ \Omega \\ u(x) \to \infty, \quad on\ \partial\Omega \end{cases} \tag{5}$$

where $\Omega \subseteq \mathbf{R}^N$, the non-negative function $p(x) \in C(\overline{\Omega})$, and the continuous function f satisfies (2) and the Keller-Osserman condition

$$\int_1^\infty [F(s)]^{-1/m} ds = \infty, \ F(s) = \int_0^s f(t)dt \tag{6}$$

then the author also consider the nonexistence for the non-negative non-trivial entire bounded radial solution on $\mathbf{R}^N$ of (5) when p satisfies

$$\int_0^\infty (tp_*(t))^{1/(m-1)} dt = \infty, \ p_*(t) = \min_{|x|=t} p(x). \tag{7}$$

On the other hand, if f does not satisfy (6), that is $\int_1^\infty [F(s)]^{-1/m} ds < \infty$, we can obtain from Lemma 2.4 in [29] that

$$\int_1^\infty \frac{1}{f^{1/(m-1)}(s)} ds < \infty \tag{8}$$

which is also shown in [30]. In this paper, we will require the above integral to be infinite, that is

$$\int_1^\infty \frac{1}{f^{1/(m-1)}(s)} ds = \infty \tag{9}$$

which is a very important condition in our main results. Furthermore, motivated by the results of [24], we also admit the following condition which is opposite to (7), that is

$$\int_0^\infty (tp^*(t))^{1/(m-1)} dt < \infty, \ p^*(t) = \max_{|x|=t} p(x). \tag{10}$$

As far as the authors know, however, there are no results which contain the existence criteria of positive solutions to the problem (1). In this paper, we prove the existence of the positive large solutions for the problem (1). When $p = 2$, the related results have been obtained by A.Lair and A.Mohammed [24]. The main results of the present paper contain extension of the results in [24] and complement of the results in [10,25,26].

The plan of the paper is as follows. In Section 2, for the convenience of the reader we give some basic lemmas that will be used in proving our results. In Section 3 we state and prove the main results. Section 4 contains some consequences of the main theorems, and other results. In Section 5 we present an Appendix where we state and prove three lemmas needed for proofs in previous sections.

2. Preliminary

In this section, we give some results that we shall use in the rest of the paper.

Lemma 2.1.(Weak comparison principle)(see [25]) Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \ge 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0,\infty) \to (0,\infty)$ is continuous and non-decreasing. Let $u_1, u_2 \in W^{1,m}(\Omega)$ satisfy

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{m-2} \nabla u_1 \nabla\psi dx + \int_\Omega \theta(u_1)\psi dx \\ &\le \int_\Omega |\nabla u_2|^{m-2} \nabla u_2 \nabla\psi dx + \int_\Omega \theta(u_2)\psi dx \end{aligned}$$

for all non-negative $\psi \in W_0^{1,m}(\Omega)$. Then the inequality

$$u_1 \le u_2 \quad on \quad \partial\Omega$$

implies that

$$u_1 \le u_2 \quad in \quad \Omega.$$

Lemma 2.2. Let f verify (9), and $\rho:[0,\infty)\to[0,\infty)$ be continuous. Then

$$\begin{cases}(r^{N-1}\Phi_m(v'(r)))' = r^{N-1}\rho(r)f(v), \quad r>0\\ v(0)=\alpha, \quad v'(0)=0\end{cases} \tag{11}$$

admits a non-negative solution v defined on $[0,\infty)$, where $\Phi_m(s)=|s|^{m-2}s$. If in addition f is nondecreasing and ρ satisfies (7), then $v(r)\to\infty$ as $r\to\infty$.

Proof. First we note that (11) has a solution $v\in C^1(0,R)$ for a maximal $0<R\le\infty$. As a consequence of (7) we claim that $R=\infty$. By way of contradiction, let us suppose that $0<R<\infty$ instead. Then we must have $v(r)\to\infty$ as $r\to R^-$. Let

$$\phi(t):=\sup\{\rho(s):0<s\le t\}, \quad t>0$$

Then ϕ is nondecreasing, and clearly $\rho(t)\le\phi(t)$ for $t>0$. Integrating Equation (11) from 0 to r yields

$$\Phi_m(v'(r)) = r^{1-N}\int_0^r s^{N-1}\rho(s)f(v(s))ds \tag{12}$$

From (12) we see that $v'(r)\ge 0$, therefore, v is a non-decreasing function and we can obtain from (12) that $\Phi_m(v'(r))\le \frac{r}{N}\varphi(r)f(v(r))$. Then we can obtain

$$\int_0^r \frac{v'(t)}{f^{1/(m-1)}(v(t))}dt \le \int_0^r \left(\frac{t}{N}\right)^{1/(m-1)}\varphi^{1/(m-1)}(t)dt$$

That is

$$\int_\alpha^{v(r)} \frac{1}{f^{1/(m-1)}(s)}ds \le \int_0^r \left(\frac{t}{N}\right)^{1/(m-1)}\varphi^{1/(m-1)}(s)ds$$

Letting $r\to R$, and recalling that $v(r)\to\infty$, we conclude that

$$\int_\alpha^\infty \frac{1}{f^{1/(m-1)}(s)}ds \le \left(\frac{R}{N}\right)^{1/(m-1)}\int_0^R \varphi^{1/(m-1)}(s)ds$$

which is an obvious contradiction. Thus, indeed v is defined on $(0,\infty)$.

We now show that $v(r)\to\infty$ as $r\to\infty$. For this we will use (7) on ρ. Integrating the equation in (11) we find

$$v(r) = \alpha + \int_0^r\left(t^{1-N}\int_0^t s^{N-1}\rho(s)f(v(s))ds\right)^{1/(m-1)}dt$$

$$\ge \left(\frac{f(\alpha)}{N}\right)^{1/(m-1)}\int_0^r (t\rho_*(t))^{1/(m-1)}dt$$

That is

$$v(r)\ge C(m,\alpha,N)\int_0^r (t\rho_*(t))^{1/(m-1)}dt, \quad r>0$$

and as a consequence of (7) we conclude that $v(r)\to\infty$ as $r\to\infty$.

3. Main Theorems

In this section, we will state the first of our main results.

Theorem 3.1. Under the following hypotheses

$$(H1)\psi(t)=\int_1^t \frac{1}{f(s)}ds, \quad t>0;$$

$$(H2)\int_0^\infty\left(\left(p^*\right)^{1/(m-1)}(t)-\left(p_*\right)^{1/(m-1)}(t)\right)$$
$$\left(tf\left(\phi\left(\overline{p}(t)\right)\right)\right)^{1/(m-1)}dt<\infty, \ \overline{p}(t)=\int_0^t sp_*(s)ds,$$

where ϕ is the inverse of ψ;

$$(H3)\int_0^\infty\left(tq^*(t)g\left(\phi\left(\overline{p}(t)\right)\right)\right)^{1/(m-1)}dt<\infty$$

Let f and g satisfy (2). Furthermore, assume that (9) and (10) hold. If p satisfies (7), then (1) admits a solution.

Proof. Let v be an entire radial large solution of $div(|\nabla v|^{m-2}\nabla v)=p_*(|x|)f(v)$ such that $v(0)=\alpha$ for some $0<\alpha<1$. This is possible by Lemma 2.2, since f satisfies (9) and p_* satisfies (7). Thus v is a super-solution of (1). We proceed to construct a sub-solution u of (1) such that $u\le v$ on $\mathbf{R}^N$. Then by the standard regularity argument for elliptic problems, it is a straight forward argument to prove that (1) would have a solution w such that $u\le w\le v$ on $\mathbf{R}^N$. For each positive integer n, let u_n be a solution of

$$\begin{cases}div(|\nabla u|^{m-2}\nabla u)\\ = p^*(|x|)f(u)+q^*(|x|)g(u), 0.4cm \ in \ B_n,\\ u(x)=v, 0.4cmon \quad \partial B_n,\end{cases} \tag{13}$$

where $B_n=B(0,n)$ is the ball of radius n centered at the origin. That such a solution exists is shown in Lemma 5.2 of Appendix. Then we note that each u_n is a radial solution and that

$$0<u_{n+1}\le u_n\le v, \quad on \quad B_n.$$

Let

$$u(x):=\lim_{n\to\infty}u_n(x), x\in R^N.$$

Since each u_n is radial, it follows that u is radial as well. By a standard argument we can show that u is a solution of the differential equation in (1). Clearly $u\le v$ on $\mathbf{R}^N$. So We only prove that u is nontrivial and that $u(x)\to\infty$ as $|x|\to\infty$.

Recalling that u_n and v are radial and that $u_n(n)=v(n)$ we see that

$$u_n(0)+\int_0^n (t^{1-N}\int_0^t s^{N-1}(p^*(s)f(u_n)+q^*(s)g(u_n))ds)^{1/(m-1)}dt$$
$$=v(0)+\int_0^n\left(t^{1-N}\int_0^t s^{N-1}p_*(s)f(v)ds\right)^{1/(m-1)}dt\ge v(0)+\int_0^n\left(t^{1-N}\int_0^t s^{N-1}p_*(s)f(u_n)ds\right)^{1/(m-1)}dt$$

for $\forall x\ge 0, m\ge 2$, we can use the inequality $(1+x)^{1/(m-1)}\le 1+x^{1/(m-1)}$, then we obtain

$$u_n(0)+\int_0^n\left(t^{1-N}\int_0^t s^{N-1}p^*(s)f(u_n)ds\right)^{1/(m-1)}dt+\int_0^n\left(t^{1-N}\int_0^t s^{N-1}q^*(s)g(u_n)ds\right)^{1/(m-1)}dt$$
$$-\int_0^n\left(t^{1-N}\int_0^t s^{N-1}p_*(s)f(u_n)ds\right)^{1/(m-1)}dt\ge v(0)=\alpha$$

that is

$$u_n(0)+\int_0^n (t^{(1-N)/(m-1)}\left((\int_0^t s^{N-1}p^*(s)f(u_n)ds)^{1/(m-1)}-(\int_0^t s^{N-1}p_*(s)f(u_n)ds)^{1/(m-1)}\right)dt$$
$$+\int_0^n\left(t^{1-N}\int_0^t s^{N-1}q^*(s)g(u_n)ds\right)^{1/(m-1)}dt\ge v(0)=\alpha$$

Since $p^*(s)$ is increasing and $p_*(s)$ is decreasing, so

$$\int_0^n t^{(1-N)/(m-1)}\left(\left(\int_0^t s^{N-1}p^*(s)f(u_n)ds\right)^{1/(m-1)}-\left(\int_0^t s^{N-1}p_*(s)f(u_n)ds\right)^{1/(m-1)}\right)dt$$
$$\le\int_0^n t^{(1-N)/(m-1)}((p^*)^{1/(m-1)}(t)\left(\int_0^t s^{N-1}f(u_n(s))ds\right)^{1/(m-1)}-(p_*)^{1/(m-1)}(t)\left(\int_0^t s^{N-1}f(u_n(s))ds)^{1/(m-1)}\right)dt$$
$$\le\int_0^n\left((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t)\right)t^{(1-N)/(m-1)}\left(\int_0^t s^{N-1}f(u_n(s)ds\right)^{1/(m-1)}dt$$
$$\le\int_0^n\left((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t)\right)\left(\int_0^t f(u_n(s))ds\right)^{1/(m-1)}dt$$
$$\le C_1(m)\int_0^n\left((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t)\right)(tf(u_n(t)))^{1/(m-1)}dt$$

and

$$\int_0^n\left(t^{1-N}\int_0^t s^{N-1}q^*(s)g(u_n)ds\right)^{1/(m-1)}dt\le C_2(m)\int_0^n\left(tq^*(t)g(u_n(t))\right)^{1/(m-1)}dt$$

Therefore we get

$$u_n(0)+C_1(m,N)\int_0^n\left((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t)\right)(tf(u_n))^{1/(m-1)}dt+C_2(m,N)\int_0^n\left(tq^*(t)g(u_n)\right)^{1/(m-1)}dt\ge\alpha \tag{14}$$

Now, let $\phi(t)$ be the inverse of the increasing function defined in (9). We note that $\phi(t)\ge 1$ for all $t\ge 0$. Furthermore, we have

$$\phi'(t)=f(\phi(t)),\phi''(t)=f'(\phi(t))f(\phi(t)),t>0.$$

Let w be an entire large solution of $\Delta w=p_*(|x|)$ such that $w(0)=0$. Set $a(x):=\phi(w(x))$. Then $\Delta a\ge p_*(|x|)f(a)$. Since $a(0)=\phi(w(0))\ge 1>\alpha=v(0)>0$, we invoke Lemma 2.1 in [24] to conclude that $v(x)\le a(x)$ for all $x\in\mathbf{R}^N$.

Moreover, $w(r)\le\overline{p}(r):=\int_0^r tp_*(t)dt,$ we have $v(x)\le\phi(\overline{p}(|x|))$.

Now, recalling that $u_n\le v$ for all $n\in\mathbf{N}$ we see that

$$tq^*(t)g(u_n(t))\le tq^*(t)g(v(t))\le tq^*(t)g(\phi(\overline{p}(t))),$$
$$((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t))(tf(u_n(t)))^{1/(m-1)}$$
$$\le((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t))(tf(v(t)))^{1/(m-1)}$$
$$\le((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t))(tf(\phi(\overline{p}(t))))^{1/(m-1)}$$

Take note of (9) and (10), we invoke the Lebesgue dominated convergence theorem to infer from (14) that

$$u(0)+C_1(m,N)\int_0^n\left((p^*)^{1/(m-1)}(t)-(p_*)^{1/(m-1)}(t)\right)(tf(u))^{1/(m-1)}dt$$
$$+C_2(m,N)\int_0^n\left(tq^*(t)g(u)\right)^{1/(m-1)}dt\ge\alpha>0.$$

This shows that u is nontrivial. Now we note that

$$u_n(r)=u_n(0)+\int_0^r\left(t^{1-N}\int_0^t s^{N-1}\left(p^*(s)f(u_n)+q^*(s)g(u_n)\right)ds\right)^{1/(m-1)}dt$$
$$\ge\int_0^r\left(t^{1-N}\int_0^t s^{N-1}p^*(s)f(u_n)ds\right)^{1/(m-1)}dt.$$

Recalling that $u_n\le v$ for all n, we invoke the Lebesgue dominated convergence theorem again, on letting $n\to\infty$

$$u(r)\ge\int_0^r\left(t^{1-N}\int_0^t s^{N-1}p^*(s)f(u)ds\right)^{1/(m-1)}dt.$$

Since u is nontrivial we see that $u(\frac{r_0}{2})>0$ for some $r_0>0$. Thus for $r>r_0$, we have

$$u(r)\ge f^{1/(m-1)}\left(u(\frac{r_0}{2})\right)\int_{r_0}^{r}\left(t^{1-N}\int_0^t s^{N-1}p^*(s)ds\right)^{1/(m-1)}dt$$

then

$$u(r)\ge f^{1/(m-1)}\left(u(\frac{r_0}{2})\right)\left(\frac{1}{N}\right)^{1/(m-1)}\int_{r_0}^{r}(tp_*(t))^{1/(m-1)}dt$$

Therefore, as a consequence of (7) we see that $u(x)\to\infty$ as $|x|\to\infty$.

To show our next main result, now we set p is c-positive on Ω (*i.e.*, for any $x_0\in\Omega$ satisfying $p(x_0)=0$, there exists a domain Ω_0 such that $x_0\in\Omega_0,\overline{\Omega}_0\subset\Omega$, and $p(x)>0$ for all $x\in\partial\Omega_0$.) we know that p is c-positive on $\mathbf{R}^N$ if and only if there is a sequence Ω_n of smooth bounded domains with $\Omega_n\subseteq\Omega_{n+1}$ for each n such that $\bigcup_{n=1}^{\infty}\Omega_n=\mathbf{R}^N$ and p is c-positive on each Ω. It is easy to see that if γ is a non-negative and locally Hölder continuous function in $\mathbf{R}^N$ that satisfies (10), then the following problem admits a positive solution.

$$\begin{cases}-div(|\nabla w|^{m-2}\nabla w)=\gamma(x), & x\in\mathbf{R}^N\\ w(x)\to 0, & |x|\to\infty\end{cases}\tag{15}$$

In fact

$$v(x)=\int_{|x|}^{\infty}\left(t^{1-N}\int_0^t s^{N-1}\gamma^*(s)ds\right)^{1/(m-1)}dt.$$

is a super-solution of (15) such that $v(x)\to 0$ as $|x|\to\infty$. On the other hand, 0 is a sub-solution of (15), (See [31], Lemma 3) the assertion follows.

Theorem 3.2. Suppose f and g satisfy (2). If (1) has a solution, f satisfies (8) and p is c-positive in $\mathbf{R}^N$, (3) admits a solution. Conversely, if $f+g$ satisfies (8), (15) admits a non-negative solution with $\gamma(x)=p(x)+q(x)$ and $\xi(x):=\min\{p(x),q(x)\}$ is c-positive, then (1) has a solution.

Proof. Let $\{\Omega_n\}$ be a sequence of bounded smooth domains in $\mathbf{R}^N$ as provided in the definition of the c-positivity of p.

Suppose (1) has a solution, say v is a solution. For each n, the problem

$$\begin{cases}div(|\nabla u|^{m-2}\nabla u)=p(x)f(u), & x\in\Omega_n\\ u(x)=\infty, & x\in\partial\Omega_n\end{cases}\tag{16}$$

has a solution(see [29]). For each positive integer n, let u_n be a solution of (16). Then by Lemma 2.1 it follows that

$$v(x)\le u_{n+1}(x)\le u_n(x),\quad x\in\Omega_n.$$

A standard procedure (for example, see [30]) can be used to show that

$$u(x):=\lim_{n\to\infty}u_n(x),\quad x\in\mathbf{R}^N,$$

is the desired solution of (3). For the converse, we let u_n be a solution of the problem

$$\begin{cases}div(|\nabla u|^{m-2}\nabla u)=p(x)f(u)+q(x)g(u), & x\in\Omega_n\\ u(x)=\infty, & x\in\partial\Omega_n\end{cases}\tag{17}$$

The existence of such a solution is demonstrated in Lemma 5.3 of Appendix. It easily follows that the sequence $\{u_n\}$ is a non-increasing sequence. Let

$$u(x)=\lim_{n\to\infty}u_n(x),\quad x\in\mathbf{R}^N.$$

A standard argument shows that u is a solution of the quasilinear equation in (17). Thus we need only show that u is nontrivial and that $u(x)\to\infty$ as $|x|\to\infty$. For this we consider the following function

$$\psi(t)=\int_t^{\infty}\frac{1}{h^{1/(m-1)}(s)}ds,\quad t>0,\tag{18}$$

where $h(t):=f(t)+g(t)$. Obviously, (18) is finite for all $t>0$. We also notice that

$$\psi'(t)=-\frac{1}{h^{1/(m-1)}(t)}<0,\ \ \psi''(t)=\frac{h'(t)}{(m-1)(h^{1/(m-1)}(t))^m}>0$$

Now fix $\varepsilon>0$, and let $v_n(x)=\psi(u_n(x)+\varepsilon),\quad x\in\Omega_n$. Note the sequence v_n is nondecreasing. Moreover, a simple computation shows that

$$\begin{aligned}&-div(|\nabla v_n|^{m-2}\nabla v_n)=|\psi'(u_n+\varepsilon)|^{m-1}div(|\nabla u_n|^{m-2}\nabla u_n)\\&-(m-1)|\psi'(u_n+\varepsilon)|^{m-2}\psi''(u_n+\varepsilon)|\nabla u_n|^m\\&\le|\psi'(u_n+\varepsilon)|^{m-1}\left(p(x)f(u_n)+q(x)g(u_n)\right)\\&=\frac{p(x)f(u_n)+q(x)g(u_n)}{h(u_n+\varepsilon)}\le p(x)+q(x)\end{aligned}$$

We can also note that $v_n=0$ on $\partial\Omega_n$. Let w be a solution of (15). Thus by Lemma 2.1 we see that $v_n\le w$ on Ω_n for all n, letting $n\to\infty$, and then $\varepsilon\to 0$ we see that $\psi(u)\le w$ on $\mathbf{R}^N$. Thus $\psi(u(x))\to 0$ as $|x|\to\infty$, that is $u(x)\to\infty$ as $|x|\to\infty$.

4. Consequences and Related Results

We can obtain some consequences of the main theorems, and other results that are of independent interest.

Theorem 4.1. Let f and g be continuous, nondecreasing functions such that $f+g$ satisfies (9), and Suppose $p+q$ is nontrivial. If there is a solution to

$$\begin{cases}div(|\nabla u|^{m-2}\nabla u)\le p(x)f(u)+q(x)g(u), & x\in R^N, N\ge 3\\ u(x)\to\infty, & as\ |x|\to\infty\end{cases}\tag{19}$$

then $p+q$ satisfies (7).

Proof. Let u be a solution of (19). Let v be a solution of the initial value (11) with $\rho=(p+q)^*$, f replaced by $f+g$ and $\alpha=\alpha_0$ where α_0 is chosen such that $\alpha_0>u(0)$. Since $f+g$ satisfies (9), we recall from Lemma 2.2 that v is defined on $[0,\infty)$. Then $w(x)=v(|x|)$ is a solution of

$$div(|\nabla w|^{m-2}\nabla w)=(p+q)^*(|x|)(f(w(x))+g(w(x))),$$

and hence $div(|\nabla w|^{m-2}\nabla w)\ge pf(w)+qg(w)$ on $\mathbf{R}^N$. Since $v'(r)>0$ we see that $v(r)\to A$ as $r\to\infty$, for some $0<A\le\infty$. Assume that $A<\infty$ so that $w(x)\le A$ for all $x\in\mathbf{R}^N$. Since $u(x)\to\infty$ as $|x|\to\infty$, we see that for some R, we have $w(x)\le A\le u(x), |x|\ge R$. Thus $w(x)\le u(x)$ on $|x|=R$ and therefore by Lemma 2.1 we find that $w(x)\le u(x)$ on $B(0,R)$. But this contradicts the choice that $w(0)>u(0)$. So we have $A=\infty$, then $w(x)\to\infty, as\ |x|\to\infty$. From Equation (11) we find

$$\begin{aligned} v'(r)&=\left(r^{1-N}\int_0^r t^{N-1}(p+q)^*(t)(f+g)(v(t))dt\right)^{1/(m-1)}\\ &\le\left(r^{1-N}(f+g)(v(r))\int_0^r t^{N-1}(p+q)^*(t)dt\right)^{1/(m-1)} \end{aligned}\tag{20}$$

Dividing (20) through by $(f+g)^{1/(m-1)}(v(r))$ and integrating the resulting inequality on $(0,r)$ we have

$$\begin{aligned}&\int_0^r\frac{v'(t)}{(f+g)^{1/(m-1)}(v(t))}dt\\ &\le\int_0^r\left(t^{1-N}\int_0^t s^{N-1}(p+q)^*(s)ds\right)^{1/(m-1)}dt\end{aligned}$$

That is

$$\begin{aligned}&\int_{\alpha_0}^{v(r)}\frac{1}{(f+g)^{1/(m-1)}(t)}dt\\ &\le\left(\frac{1}{N}\right)^{1/(m-1)}\int_0^r(t(p+q)^*(t))^{1/(m-1)}dt\end{aligned}$$

Letting $r\to\infty$ and recalling that $f+g$ satisfies (9), the claim is proved.

As a consequence of Theorem 3.1 and Theorem 4.1 we also obtain the following corollaries.

Corollary 1. Suppose (2) and (9) hold for f. Further, let p satisfy (10). (3) admits a solution if and only if p satisfies (7).

Proof. If p satisfies (7) then Theorem 3.1, with $q(x)=0$ shows that (3) has a solution. The converse follows from Theorem 4.1 on taking $q(x)\equiv 0$ again.

The next corollary provides sufficient conditions for the existence and nonexistence of solutions to (1) when both p and q satisfy (7).

Corollary 2. Suppose f and g satisfy (2) and p and q satisfy (7). If $f+g$ satisfies (9), then (1) has no solution. On the other hand, (1) admits a solution if $f+g$ satisfies (8) and the function $\xi(x)\equiv\min\{p(x),q(x)\}$ is c-positive on $\mathbf{R}^N$.

Proof. By way of contradiction,we can obtain the first statement from Theorem 4.1. Since $p+q$ satisfies (10) and the remark noted just before Theorem 3.2 shows that (15) admits a solution with $b=p+q$. Thus the second part of the corollary is an immediate consequence of Theorem 3.2.

5. Appendix

In this appendix we state and prove results that have been used in the proofs of the main results of the paper.

We start by proving the existence of a solution to the following Dirichlet problem on a bounded smooth domain Ω in $\mathbf{R}^N$.

$$\begin{cases} div(|\nabla u|^{m-2}\nabla u)=p(x)f(u)+q(x)g(u), & in\ \Omega,\\ u(x)=\varphi(x), & on\ \ \partial\Omega.\end{cases}\tag{21}$$

Lemma 5.1. Let $\Omega\subseteq\mathbf{R}^N$ be a smooth bounded domain and let f and g satisfy (2). Let $\phi\in C^2(\partial\Omega)$ be positive. If v is a positive super-solution of (21), then the problem (21) has a solution u such that $0<u\le v$ on Ω.

Proof. Let $\beta:=\min_{x\in\partial\Omega}\phi(x)$. Obviously, $\beta>0$. Now we set

$$\varphi(t)=\int_0^t(h(s))^{1/(m-1)}ds,$$

where $h(s)=f(s)+g(s)$ for all $s\ge0$. An application of L'H $\hat{o}$ pital's Rule shows that $\varphi(t)\le t$ for all $0<t<\varepsilon$ and some $\varepsilon>0$. Without of generality we can suppose that $0<\varphi(\varepsilon)\le\beta$. Finally, let z be a solution of the Dirichlet problem

$$\begin{cases} div(|\nabla z|^{m-2}\nabla z)=p(x)+q(x), & in\ \ \Omega,\\ z(x)=\varepsilon, & on\ \ \partial\Omega.\end{cases}$$

Then the maximum principle shows that $0<z(x)\le\varepsilon$ on Ω, we define $w(x):=\varphi(z(x))$ for all $x\in\overline{\Omega}$, we note that $w(x)\le z(x)$ for all $x\in\overline{\Omega}$. A simple computation shows that

$$\varphi'(t)=h^{1/(m-1)}(t)>0,\quad \varphi''(t)=\frac{h'(t)}{(m-1)(h^{1/(m-1)}(t))^{m-2}}>0$$

and

$$\begin{aligned}&div(|\nabla w|^{m-2}\nabla w)\\ &=|\varphi'|^{m-1}div(|\nabla z|^{m-2}\nabla z)+(m-1)|\varphi'|^{m-2}\varphi''|\nabla z|^m\\ &\ge|\varphi'|^{m-1}(p(x)+q(x))=(f(z)+g(z))(p(x)+q(x))\\ &\ge(f(w)+g(w))(p(x)+q(x))\ge p(x)f(w)+q(x)g(w)\end{aligned}$$

and $w(x)\le\varphi(\varepsilon)\le\beta\le\phi(x)$ for $x\in\partial\Omega$. Thus w is a sub-solution of (21) and v is a super-solution of (21) such that $w\le\varphi\le v$ on $\partial\Omega$. By the maximum principle we note that $w\le v$ on Ω. Thus by lemma 1 in [31] we conclude that (21) has a solution u such that $w \le u \le v$ which is what we want to show.

The following lemma was used in the proof of Theorem 3.1.

Lemma 5.2. Let $a, b \in C([0,\infty),[0,\infty))$ and B be a ball in $\mathbf{R}^N$ centered at the origin. If f and g are nondecreasing on $[0,\infty)$, then given a positive constant δ, there exists a radial solution to the problem

$$\begin{cases} div(|\nabla u|^{m-2}\nabla u) = a(|x|)f(u) + b(|x|)g(u), & in\ B, \\ u(x) = \delta, & on\ \ \partial B, \end{cases} \tag{22}$$

Proof. Let $\{a_k\}$ and $\{b_k\}$ be decreasing sequences of Hölder continuous functions which converge uniformly on B to a and b respectively(See [32]). Then by Lemma 5.1, for each k there exists a nonnegative solution u_k of

$$\begin{cases} div(|\nabla u_k|^{m-2}\nabla u_k) \\ = a_k(|x|)f(u_k) + b_k(|x|)g(u_k), \quad in\ \ B, \\ u_k(x) = \delta, \quad on\ \ \partial B. \end{cases}$$

Since the sequence $\{a_k\}$ and $\{b_k\}$ are decreasing,it is easy to show that $\{u_k\}$ is increasing. Of course,it is also bounded above by δ. Thus it converges, and assume $u_k \to u$. Since u_k satisfies the integral equation

$$u_k(r) = u_k(0) + \int_0^r \left(t^{1-N} \int_0^t s^{N-1} \left(a_k(s) f(u_k(s)) + b_k(s) g(u_k(s)) \right) ds \right)^{1/(m-1)} dt$$

the function u will satisfy the integral equation

$$u(r) = u(0) + \int_0^r \left(t^{1-N} \int_0^t s^{N-1} \left(a(s) f(u(s)) + b(s) g(u(s)) \right) ds \right)^{1/(m-1)} dt.$$

Since $u_k(R) = \delta$ for each k, where R is the radius of the ball B, it is clear that $u(R) = \delta$. Thus u is a nonnegative solution of problem (22) on B as claimed.

Finally we state and prove a lemma that was used in the proof of Theorem 3.2.

Lemma 5.3. Let $\Omega \subseteq \mathbf{R}^N$ be smooth. Suppose f and g satisfy (2). If f satisfies (6) and p is c-positive on Ω, then the problem

$$\begin{cases} div(|\nabla u|^{m-2}\nabla u) = p(x)f(u) + q(x)g(u), & x \in \Omega, \\ u(x) = \infty, & x \in \partial\Omega, \end{cases} \tag{23}$$

has a solution. Similarly, if instead of requiring f to satisfy (6), we require only $f+g$ to satisfy (6), and require $\xi := \min\{p, q\}$ to to be c-positive on Ω, then (23) has a solution.

Proof. Since p is c-positive and f satisfies (6), let v be a large solution of $div(|\nabla v|^{m-2}\nabla v) = p(x)f(v)$ on Ω (see [29]). Now for each positive integer k, let w_k be a solution (See Lemma 5.1) of

$$\begin{cases} div(|\nabla w|^{m-2}\nabla w) \\ = (p(x) + q(x))(f(w) + g(w)), \quad x \in \Omega, \\ w(x) = k, \quad x \in \partial\Omega, \end{cases}$$

By Lemma 2.1 we see that

$$w_k(x) \le w_{k+1}(x) \le v(x), \quad x \in \Omega$$

If $w(x) = \lim_{k\to\infty} w_k(x)$, then by a standard procedure we conclude that w is a solution of $div(|\nabla w|^{m-2}\nabla w) = (p(x)+q(x))(f(w)+g(w))$ on Ω such that $w \le v$. Since w is a sub-solution, and v is a super-solution of the differential equation in (23), we conclude that (23) has a solution u with $w \le u \le v$ (See [31]).

Similarly.We can obtain the second part by defining v in this case as a large solution of $div(|\nabla v|^{m-2}\nabla v) = \xi(x)(f(v)+g(v))$ on Ω and the argument is as the same as the previous process.

6. References

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