Open Journal of Applied Sciences
Supplement： 2012 world Congress on Engineering and Technology

Description of the derived categories of tubular algebras in terms of dimension vectors

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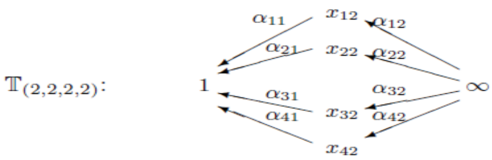
Abstract In this paper, we give a description of the derived category of a tubular algebra by calculating the dimension vectors of the objects in it.

Keywords: derived categories; tubular algebra; dimension vectors

1. Introduction

Let Λ be a basic connected algebra over an algebraically closed field k. We denote by mod Λ the category of all finitely generated right Λ-modules and by ind Λ a full subcategory of mod Λ containing exactly one representative of each isomorphism class of indecomposable Λ-modules. For a Λ-module M, we denote the dimension vector by dim M. The bounded derived category of mod Λ is denoted by $D^b(\Lambda)$. We denote the Grothendieck group of Λ by $K_0(\Lambda)$, Auslander-Reiten translation by τ, the Cartan matrix by C_Λ. Let $\hat{\Lambda}$ be the repetitive algebra of Λ, mod $\hat{\Lambda}$ the stable module category. When the global dimension of Λ is finite, C_Λ is invertible by [1], and $D^b(\Lambda)$ is equivalent to mod $\hat{\Lambda}$ as triangulated categories by [2].

By [1], a tubular extension A of a tame-concealed algebra of extension type T = (2, 2, 2, 2), (3, 3, 3), (4, 4, 2) or (6, 3, 2) is called a tubular algebra. For example, the canonical tubular algebras of T (2, 2, 2, 2) is determined by the following quiver with relations.



$$\alpha_{11}\alpha_{12}+\alpha_{21}\alpha_{22}+\alpha_{31}\alpha_{32}=0,$$
$$\alpha_{11}\alpha_{12}+\lambda\alpha_{21}\alpha_{22}+\alpha_{41}\alpha_{42}=0,$$
$$\lambda\notin\{0,1\}$$

By [1], global dimension of a tubular algebra A is 2, then $D^b(A)$ is equivalent to mod $\hat{A}$. And tubular algebras of the same extension type are tilt-cotilt equivalent, see [3]. Then we only consider the derived categories of canonical tubular algebras, whose structures are given in [4].

$$D^b(A)=\bigvee_{r\in Q} T_r$$

where (1) for any $r\in Q$, T_r is the standard stable $P_1(k)$-tubular family of type T;

(2) for any $r\in Q$, T_r is separating $\bigvee_{s<r} T_s$ from $\bigvee_{r<u} T_u$.

Based on the results above, we give a description of the derived category of a canonical tubular algebra by calculating the dimension vectors of the objects in it.

2. Description of The Derived Categories of Tubular Algebras In Terms Of Dimension Vectors

In this section, let A be a canonical tubular algebra of type T.

Definition 1.1. ([1]) Let n be the rank of Grothendieck group $K_0(A)$, C_A the Cartan matrix of A. Then

(1) The Coxeter matrix Φ_A is defined by $-C_A^{-T}C_A$;

(2) The quadratic form χ_A in $\mathbb{Z}_n$ is defined by

$$\chi_A(\alpha)=\frac{1}{2}\alpha(-C_A^{-T}+C_A^{-1})\alpha^T$$

for any α in $\mathbb{Z}_n$.

(3) Let h_0, h_∞ be the positive generators of rad χ_A. For an A-module M, define

$$index(M)=-\frac{l_0(\underline{\dim M})}{l_\infty(\underline{\dim M})}$$

where

$$l_0(\underline{\dim M})=h_0C_A^{-T}(\underline{\dim M})^T,$$
$$l_\infty(\underline{\dim M})=h_\infty C_A^{-T}(\underline{\dim M})^T.$$

In particular, for any $\alpha\in$ rad χ_A, $\alpha=r_0h_0+r_\infty h_\infty$, where $r_0,r_\infty\in\mathbb{Z}$. Then, $\text{index}(\alpha)=\frac{r_\infty}{r_0}$.

(4) $\alpha \in \mathrm{rad}\,\chi_A$ is called a real (respectively, imaginary) root,
if $\chi_A(\alpha) = \frac{1}{2}\alpha(-C_A^{-T} + C_A^{-1})\alpha^T = 1$ (respectively, = 0).

It is well known that there exists a "minimal " imaginary root δ such that rad $\chi_A = \mathbb{Z}\delta.$

Now we recall some results in [4]. Let Λ be a finite dimensional $k-$algebra and $\widehat{\Lambda}$ the repetitive algebra. Denote
by $P(\widehat{\Lambda})$ the subgroup of $K_0(\widehat{\Lambda})$ generated by the dimension vectors of indecomposable projective $\widehat{\Lambda}-$ modules.

Lemma 1.2. $K_0(\widehat{\Lambda}) = K_0(\Lambda) + P(\widehat{\Lambda}).$

Definition 1.3. Let $\pi_\Lambda : K_0(\widehat{\Lambda}) \to K_0(\Lambda)$ be the projective morphism. Define $\underline{\dim}^\Lambda : \mathrm{mod}\,\widehat{\Lambda} \to K_0(\Lambda)$ where for any $\widehat{\Lambda}$-module *X*, $\underline{\dim}^\Lambda X = \pi_\Lambda(\underline{\dim} X).$

Lemma 1.4. Let Φ_Λ be the Coxeter matrix of Λ, $\hat{\tau}$ the Auslander-Reiten translation of $\widehat{\Lambda}$. Then

$$\underline{\dim}^\Lambda \hat{\tau} X = (\underline{\dim} X)\Phi_\Lambda.$$

Note that if Λ has finite global dimension, we have a triangulated equivalence: $\eta : D^b(\Lambda) \to \underline{\mathrm{mod}}\widehat{\Lambda}$. For an object $X^\bullet \in D^b(\Lambda)$, define $\underline{\dim} X^\bullet = \sum_i (-1)^i \underline{\dim} X^i.$

Then we have

Lemma 1.5. $\underline{\dim} X^\bullet = \underline{\dim}^\Lambda \eta(X^\bullet).$

By representation theory of Auslander-Reiten quivers in [5]
and the results above, we have a method to describing the derived category of a canonical tubular algebra in terms of dimension vectors.

Theorem 1.6. Let A be a canonical tubular algebra of type
T, the rank of $K_0(A)$ be *n*. Then

(1) Let $\hat{\delta}$ be the minimal imaginary root in $K_0(\widehat{A})$ corresponding the $\mathrm{P}_1(k)$- tubular family T_r , and let $\delta = \underline{\dim}^A(\hat{\delta})$. Then δ is determined by $\chi_A(\delta) = 0.$

(2) Let *X* be an object in the bottom of a tube of rank *r* in T_r. Then $\underline{\dim}^A X$ is determined by the following:

$$(*)\begin{cases} \chi_A(\underline{\dim}^A X) = \frac{1}{2}\underline{\dim}^A X(-C_A^{-T} + C_A^{-1})(\underline{\dim}^A X)^T = 1 \\ \underline{\dim}^A X + (\underline{\dim}^A X)\Phi_A + \cdots + (\underline{\dim}^A X)\Phi_A^{r-1} = \delta. \end{cases}$$

Proof. (1) By [4], $\delta = r_0 h_0 + r_\infty h_\infty \in \mathrm{rad}\,\chi_A$, and thus $\chi_A(\delta) = 0.$ Since $\mathrm{index}(\delta) = \frac{r_\infty}{r_0} \in \mathbb{Q},$ where $r_0, r_\infty \in \mathbb{Z},$ and $(r_0, r_\infty) = 1,$ it suffices to calculating r_0 and r_∞.

(2) Directly from Lemma 1.4 and 1.5.

Example 1.7. Now let *A* be a canonical tubular algebra of type T(2, 2, 2, 2). The Cartan matrix and Coxeter matrix are as following:

$$C_A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \Phi_A = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For each object $X \in \underline{\mathrm{mod}}\widehat{A},$ denote

$$\underline{\dim}^A X = (x_1, x_2, \cdots, x_6),$$

Then $\chi_A(\underline{\dim}^A X) = \sum_{i=2}^{5}(x_i - \frac{x_1 + x_6}{2})^2$.

(1) Description of the minimal imaginary root δ .

Case 1. If $r_0 + r_\infty \equiv 1(\mathrm{mod}\,2)$

$$\delta_1 = (2r_0, r_0 + r_\infty, r_0 + r_\infty, r_0 + r_\infty, r_0 + r_\infty, 2r_\infty).$$

Case 2. If $r_0 + r_\infty \equiv 0(\mathrm{mod}\,2)$

$$\delta_2 = (r_0, \frac{r_0 + r_\infty}{2}, \frac{r_0 + r_\infty}{2}, \frac{r_0 + r_\infty}{2}, \frac{r_0 + r_\infty}{2}, r_\infty).$$

(2) Description of $\underline{\dim}^A X$ where *X* is an object in the bottom of a tube of rank 2.

If $\delta = \delta_1$, that is $r_0 + r_\infty \equiv 1(\mathrm{mod}\,2),$ (*) in Theorem 1.6 should be as follows:

$$\begin{cases} \sum_{i=1}^{6} x_i^2 - \sum_{i=2}^{5} x_1 x_i - \sum_{i=2}^{5} x_6 x_i + 2x_1 x_6 = 1 \\ \sum_{i=2}^{5} x_i - 2x_6 = 2r_0 \\ \sum_{i=2}^{5} x_i - x_1 - x_6 = r_0 + r_\infty \\ \sum_{i=2}^{5} x_i - 2x_1 = 2r_\infty \end{cases}$$

Then, $\sum_{i=2}^{5}(x_i - \frac{r_0 + r_\infty}{2})^2 = 1.$

case 1. We have four different tubes of rank 2.

(i)

$$\underline{\dim}^A X = (r_0, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty+1}{2}, r_\infty)$$

$$\underline{\dim}^A \hat{\tau} X = (r_0, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty-1}{2}, r_\infty)$$

(ii)

$$\underline{\dim}^A X = (r_0, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty+1}{2}, r_\infty)$$

$$\underline{\dim}^A \hat{\tau} X = (r_0, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty-1}{2}, r_\infty)$$

(iii)

$$\underline{\dim}^A X = (r_0, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty-1}{2}, r_\infty)$$

$$\underline{\dim}^A \hat{\tau} X = (r_0, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty+1}{2}, r_\infty)$$

(iv)

$$\underline{\dim}^A X = (r_0+1, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty+1}{2}, \frac{r_0+r_\infty+1}{2}, r_\infty+1)$$

$$\underline{\dim}^A \hat{\tau} X = (r_0-1, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty-1}{2}, \frac{r_0+r_\infty-1}{2}, r_\infty-1)$$

If $\delta = \delta_2$, (*) in Theorem 1.6 should be as follows:

$$\begin{cases} \sum_{i=1}^{6} x_i^2 - \sum_{i=2}^{5} x_1 x_i - \sum_{i=2}^{5} x_6 x_i + 2x_1 x_6 = 1 \\ \sum_{i=2}^{5} x_i - 2x_6 = r_0 \\ \sum_{i=2}^{5} x_i - x_1 - x_6 = \frac{r_0+r_\infty}{2} \\ \sum_{i=2}^{5} x_i - 2x_1 = r_\infty \end{cases}$$

Then, $\sum_{i=2}^{5} (x_i - \frac{r_0+r_\infty}{4})^2 = 1.$

case 2. When $r_0+r_i \equiv 2 \pmod 4$, we have four different tubes of rank 2.

(i)

$$\underline{\dim}^A X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_\infty-1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_\infty+1}{2})$$

(ii)

$$\underline{\dim}^A X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_\infty-1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_\infty+1}{2})$$

(iii)

$$\underline{\dim}^A X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_\infty-1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_\infty+1}{2})$$

(iv)

$$\underline{\dim}^A X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_\infty-1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty+2}{4}, \frac{r_0+r_\infty-2}{4}, \frac{r_\infty+1}{2})$$

case 3. When $r_0+r_1 \equiv 0 \pmod 4$, we have four different tubes of rank 2.

(i)

$$\underline{\dim}^A X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty+4}{4}, \frac{r_\infty+1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty-4}{4}, \frac{r_\infty-1}{2})$$

(ii)

$$\underline{\dim}^A X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty+4}{4}, \frac{r_0+r_\infty}{4}, \frac{r_\infty+1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty-4}{4}, \frac{r_0+r_\infty}{4}, \frac{r_\infty-1}{2})$$

(iii)

$$\underline{\dim}^A X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty+4}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_\infty+1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty-4}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_\infty-1}{2})$$

(iv)

$$\underline{\dim}^A X = (\frac{r_0+1}{2}, \frac{r_0+r_\infty+4}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_\infty+1}{2})$$

$$\underline{\dim}^A \hat{\tau} X = (\frac{r_0-1}{2}, \frac{r_0+r_\infty-4}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_0+r_\infty}{4}, \frac{r_\infty-1}{2})$$

3. Acknowledgment

The authors would like to thank the referee for his or her valuable suggestions and comments. The first-named author thanks NSF of China (Grant No. 11126300) and of Shandong Province (Grant No. ZR2011AL015) for support.

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