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Algebras of Hamieh and Abbas Used in the Dirac Equation*

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ABSTRACT

Hamieh and Abbas [1] propose using a 3-dimensional real algebra in a solution of the Dirac equation. We show that this algebra, denoted by $G\mathbb{C}$, belongs to a large class of quadratic Jordan algebras with subalgebras isomorphic to the complex numbers and that the spinor matrices associated with the solution of the Dirac equation generate a six-dimensional real noncommutative Jordan algebra.

Keywords: Dirac Equation; Jordan Algebra; Quadratic Algebra

Non-associative algebras have long been used in the mathematical description of physical phenomena; first appearing as the "*r* number algebra" in the seminal paper by Jordan, Wigner and von Neumann [2] of 1934. The *r* number algebra became known as a Jordan algebra from a 1946 paper by Albert [3]. The interested reader is referred to the books on non-associative algebras in physics Lõhmas, Paal and Sorgsepp [4], Okubo [5]. A concise history of non-associative algebra is to be found in Tomber [6]; the standard introduction to non-associative algebra is the book by Schafer [7].

Hamieh and Abbas [1] present a "description of an algebra which can be used in a possible extension of local complex quantum field theories". We further expand their description and show that these algebras are Type D Jordan algebras (see Jacobson [8]).

We construct a large family of quadratic Jordan algebras that contains the three-dimensional real algebra, the so called $G\mathbb{C}$ algebra, the generalized complex numbers, of Hamieh and Abbas [1], and show that the spinor matrices that arises from using the $G\mathbb{C}$ in a formulation of the Dirac equation generate a six-dimensional noncommutative quadratic Jordan algebra.

1. Introduction

Let $\mathfrak{A}$ be an algebra over a field *F* not of characteristic two. The associator is a trilinear mapping

$$(x,y,z)=(xy)z-x(yz)$$

of $\mathfrak{A}\times\mathfrak{A}\times\mathfrak{A}$ into $\mathfrak{A}$ that measures the lack of associativity in $\mathfrak{A}$.

One scheme of classifying nonassociative algebras involves placing conditions on the associator of certain sets of elements. Some of the better known algebras are:

1) Alternative algebras. In this variety of algebras, all elements *x* and *y* satisfy

$$(x,x,y)=(x,y,y)=0$$

for all elements *x* and *y*. The octonion division ring is an alternative algebra. An interesting variation is psuedo-octonion algebra (Okubo [5,9]).

2) Jordan algebras. These are commutative algebras in which all *x* and *y* satisfy

$$(x,y,x^2)=0.$$

A Type D Jordan algebra is the Jordan algebra of the symmetric bilinear form *q* on a vector space $\mathfrak{B}$. Albert [3] has shown that any algebra of Type D has a basis $\{e,b_1,b_2,\cdots,b_n\}$ with multiplication given by

$$eb_i=b_ie=b_i,\ \text{for all}\ 1\le i\le n,$$

$$b_ib_j=\delta_{ij}\alpha_ie\ \text{for}\ 1\le i,j\le n.$$

The algebra will be semisimple if $\alpha_i\ne 0$ for all $1\le i\le n$.

3) Noncommutative Jordan algebras. A generalization of the alternative and Jordan algebras that requires all *x* and *y* satisfy a generalization of the commutative law

$$(x,y,x)=0,$$

that is, the algebras are flexible, and

$$(x,y,x^2)=0.$$

*This paper is in final form and no version of it will be submitted for publication elsewhere.

The book by Zhevlakov, Slin'ko, Shestakov and Shirshov [10] provides a detailed analysis of the alternative and Jordan rings.

The above algebras are all power associative since each element a generates an associative subalgebra; equivalently, $a^m a^n = a^{n+m}$ for positive integers m, n. In any power associative algebra $\mathfrak{A}$ with unit element we can introduce the series

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

for $x \in \mathfrak{A}$ ignoring the question of convergence.

An algebra $\mathfrak{A}$ over a field F is called quadratic if, for every x in $\mathfrak{A}$

$$x^2 - 2t(x) + q(x)e = 0$$

where $t(x), q(x)$ are in F and e is the identity of $\mathfrak{A}$. The quantities $t(x)$ and $q(x)$ are called the trace and norm of the element x, respectively. The trace is a linear functional on $\mathfrak{A}$ see Schafer [7]. The norm $q(x)$ defines a symmetric bilinear form $q(x,y)$ on $\mathfrak{A}$ via

$$q(x,y) = q(x+y) - q(x) - q(y).$$

Say $q(x)$ is nondegenerate if $q(x,y)$ is. Any quadratic algebra is power associative and any flexible, quadratic algebra is a noncommutative Jordan algebra.

A quadratic algebra $\mathfrak{A}$ is flexible if and only if the trace is associative; that is, $t((xy)z) = t(x(yx))$ for all x, y, z in $\mathfrak{A}$. If $\mathfrak{A}$ is flexible then the mapping $x \to \bar{x} = 2t(x)e - x$ is an involution in $\mathfrak{A}$ (see Braun and Koecher [11], p. 216).

Lemma 1. *The Hamiltonian division ring is a quadratic algebra.*

Proof. Let $x = a + bI + cJ + dK$ be an element of the Hamitonian division ring. Direct computation shows that

$$x^2 - 2ax + a^2 + b^2 + c^2 + d^2 = 0. \qquad \square$$

Example 1. *The octonion division ring is a quadratic algebras.*

Example 2. *Domokos and Kövesi-Domokos* [12] *propose a quadratic algebra*, *the* "*algebra of color*" *as a candidate for the algebra obeyed by a quantized field describing quarks and leptons* (*see also Wene* [13,14], *and Schafer* [15]).

2. Construction of the Algebras

The elements of the algebra $G\mathbb{C}$ are the elements of the real vector space with basis $\{e, I, J\}$. The addition is the vector space addition and multiplication is defined by $IJ = JI = 0$, $I^2 = J^2 = -e$, e is the identity and the distributive laws. We note that the algebra is commutative and has divisors of zero.

An immediate generalization of this algebra has a basis $\{e, b^i, i = 1, \cdots, n\}$, $n \geq 2$ over the field $\mathbb{R}$ of real numbers and multiplication defined by $b^i b^j = -\delta_{ij}$ where e is the identity. For want of a better name called these the Abbas algebras. As noted above, these algebras are Type D Jordan algebras. Note that the $G\mathbb{C}$ algebra is the construction for $n = 2$; the results for the Abbas algebras apply to the $G\mathbb{C}$. Each Abbas algebra contains a copy of the complex numbers.

Lemma 2. *The Abbas algebras are quadratic algebras.*

Proof. Let H denote a Abbas algebra. Then if $x \in H$, $x = \alpha_0 e + \alpha_i b^i$, Einstein summation convention where $i = 1, 2, \cdots, n.$ Then

$$x^2 = \alpha_0 x + \alpha_0 \alpha_i b^i - (b^i b^i)$$

$$-2\alpha_0 x = -2\alpha_0^2 - 2\alpha_0 \alpha_i b^i$$

Adding both sides gives

$$x^2 - 2\alpha_0 x = -\alpha_0^2 - (\alpha_i \alpha_i)$$

and we see that $t(x) = \alpha_0$ and $q(x) = \alpha_0^2 + (\alpha_i \alpha_i)$.

A commutative quadratic algebra will be a Jordan algebra. $\square$

Since the algebra is commutative the trace is associative; the norm is symmetric.

Lemma 3. *The norm of a Abbas algebra is nondegenerate.*

Proof. Let H denote a Abbas algebra. Then if $x \in H$, $x = \alpha_0 e + \alpha_i b^i$ is arbitrary and $d = \delta_0 + \delta_i b^i$ is fixed, then

$$\begin{aligned} q(d,x) &= q(d+x) - q(d) - q(x) \\ &= (\delta_0 + \alpha_0)^2 + \sum_{i=1}^{n} (\delta_i + \alpha_i)^2 \\ &\quad - (\delta_0^2 + \delta_i \delta_i) - (\alpha_0^2 + \alpha_i \alpha_i) \end{aligned}$$

$$q(d,x) = 2\delta_0 \alpha_0 + 2\delta_i \alpha_i$$

$\square$

3. Special $G\mathbb{C}$ Algebras

Hamieh and Abbas [1] pass to a representation of the point $q = ae + bI + cJ$ of the algebra $G\mathbb{C}$ in spherical coordinates, $a = r\cos(\theta), b = r\sin(\theta)\cos(\varphi)$ and $c = r\sin(\theta)\sin(\varphi)$. The subalgebras, called special $G\mathbb{C}$ algebras and denoted by $SG\mathbb{C}$ are the subalgebras spanned by all elements in which the "azimutal phase angle φ is constant". Each of these subalgebras is (isomorphic to) the complex numbers.

Lemma 4. *The algebra* $G\mathbb{C}$ *is isomorphic to an algebra of two by two matrices*

$$\begin{bmatrix} a & bI + cJ \\ bI + J & a \end{bmatrix}$$

under the usual matrix operation of addition and multiplication.

Proof. The straight forward verification that the mapping $\theta(ae+bI+cJ)=\begin{bmatrix} a & bI+cJ \\ bI+J & a \end{bmatrix}$ is an isomorphism is left to the reader. □

Lemma 5. *Each of the algebras $SG\mathbb{C}$ is isomorphic to the complex numbers.*

Proof. We note that if $b=r\sin(\theta)\cos(\varphi)$ then $c=b\left(\frac{\sin(\varphi)}{\cos(\varphi)}\right)$ if $\cos(\varphi)\neq 0$. If $\varphi=\frac{\pi}{2}$ or $\frac{3\pi}{2}$, then $q=a+cJ$ and the subalgebra $SG\mathbb{C}$ is (isomorphic to) the complex numbers. Otherwise, $q=ae+bI+bsJU$ or $q=ae+b(I+sJ)$ for some $s\in\mathbb{R}$. Let $X=\frac{I+sJ}{\sqrt{1+s^2}}$, then

$$X^2=\left(\frac{I+sJ}{\sqrt{1+s^2}}\right)^2=\frac{-(1+s^2)}{1+s^2}e=-e.$$

The multiplication, using the basis $\{e,X\}$ will be given by

$$(ae+bX)(ce+dX)=(ac-bd)e+(ad+bc)X.$$

□

4. The Spinor Matrices

The classical reference on spinors and wave equations is the book by Corson [16].

The associator spinor matrices are

$$\boldsymbol{C}_t=\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix},\ \boldsymbol{C}_x=\begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix},$$

$$\boldsymbol{C}_y=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},\ \boldsymbol{C}_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $1,I,J\in G\mathbb{C}$. Denoting the 2 by 2 identity matrix by I_2, these matrices satisfy

$$C_t^2=-I_2 \text{ and } \{C_\mu,C_\nu\}=C_\mu C_\nu+C_\nu C_\mu=2\delta_{\mu\nu}I_2$$

for $\mu,\nu=x,y,z.$

The spinor matrices generate a 6-dimensional real algebra with elements

$$\left\{\begin{pmatrix} a & bI+cJ \\ eI+fJ & d \end{pmatrix} a,b,c,d,e,f\in\mathbb{R},I,J\in GC\right\}$$

that contains the matrix representation of the $G\mathbb{C}$ algebra. Denote this algebra by $SP(6)$.

Lemma 6. *The algebra $SP(6)$ is a quadratic algebra.*

Proof. If $x=\begin{pmatrix} a & bI+cJ \\ eI+fJ & d \end{pmatrix}$ is an element of $SP(6)$, then

$$x^2=\begin{pmatrix} a^2-be-cf & (a+d)(bI+cJ) \\ (a+d)(fI+cJ) & d^2-be-cf \end{pmatrix}$$

$$-2\left(\frac{1}{2}\right)(a+d)x=-\begin{pmatrix} a^2+ad & (a+d)(bI+cJ) \\ (a+d)(eI+fJ) & ad+d^2 \end{pmatrix}$$

Adding the left and right sides gives

$$x^2-2\left(\frac{1}{2}\right)(a+d)x=\begin{pmatrix} -ad-be-cf & 0 \\ 0 & -ad-cb-cf \end{pmatrix}$$

□

Lemma 7. *The algebra $SP(6)$ is flexible.*

Proof. Because of the trilinearlty of the associator, we can write the elements x and y of the associator (x,y,x) as $x=\begin{pmatrix} 0 & bI+cJ \\ eI+fJ & d \end{pmatrix}$ and $y=\begin{pmatrix} 0 & yI+zJ \\ sI+tJ & v \end{pmatrix}$. Then

$$xy=\begin{pmatrix} -bs-ct & bvI+cvJ \\ \mathrm{d}sI+\mathrm{d}tJ & \mathrm{d}v-ey-fz \end{pmatrix}$$

$$(xy)x=\begin{pmatrix} -ebv+fcv & (-bs-ct)(bI+cJ)+d\left[(bvI+cvJ\right] \\ (\mathrm{d}v-ey-fz)(eI+fJ) & b\mathrm{d}s+c\mathrm{d}t+d(\mathrm{d}v-ey-fz) \end{pmatrix}$$

$$yx=\begin{pmatrix} -ey-fz & (xb+\mathrm{d}y)I+(cx+\mathrm{d}z)J \\ evI+fvJ & \mathrm{d}v-bs-ct \end{pmatrix}$$

$$x(yx)=\begin{pmatrix} -bev-cfv & (\mathrm{d}v-bs-ct)(bI+cJ) \\ (-ey-fz)(eI+fJ)+(\mathrm{d}evI+\mathrm{d}fvJ) & d(\mathrm{d}v-bs-ct)-e\mathrm{d}y-f\mathrm{d}z \end{pmatrix}$$

□

Theorem 1. *The algebra $SP(6)$ is a quadratic noncommutative Jordan algebra.*

5. The Dirac Equation

We proceed as in Hamieh and Abbas [1]. The Dirac equation over the complex numbers is often written as

$$\left(i\gamma^\mu\partial_\mu-m\right)\Psi=0$$

utilizing the Einstein summation convention for $\mu=x,y,z,t$. A more general form is, setting

$$\gamma^{\mu} = C_{\mu}, H\Psi = \left(C_{\mu}\partial_{\mu}\right)\Psi = m\Psi$$

$$\left(C_{\mu}\partial_{\mu} - m\right)\Psi = 0,$$

where $\mu = x, y, z, t$.

Upon substituting the matrices for C_{μ} and simplifying we get

$$\begin{pmatrix} \partial_z - m & \partial_z \\ J\partial_x + I\partial_y + J\partial_t & -\partial_z - m \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0.$$

In dimensions *x* and *t*, the solution is given by

$$\Psi\left(x,t\right) = N\begin{pmatrix} \dfrac{E+p}{m} \\ 1 \end{pmatrix} e^{J\left(px-ct\right)}$$

p and $E = \pm\sqrt{p^2 + m^2}$ are respect the momentum and energy. N is a normalization factor.

6. Conclusion

We have shown that the $G\mathbb{C}$ algebra belongs to a large class of Jordan algebras and have examined a few of the algebraic properties of these algebras and, like the Jordan algebra and the algebra of color, there is a very rich mathematical structure to further explore.

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