

Semiclassical Husimi Function of Simple and Chaotic Systems

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Received April 10, 2012; revised May 5, 2012; accepted May 31, 2012

ABSTRACT

We review the semiclassical method proposed in [1], a generalization of this method for n-dimensional system is presented. Using the cited method, we present an analytical method of obtain the semiclassical Husimi Function. The validity of the method is tested using Harmonic Oscillator, Morse Potential and Dikie’s Model as example, we found a good accuracy in the classical limit.

Keywords: Classical Limit; Husimi Function; Quantum Chaos

1. Introduction

Since early times of quantum theory, some quantization difficulties of non integrable systems were pointed by Einstein [2,3]. Recently, due to the pioneer discoveries of classically chaotic systems, the subject has yielded many interesting and important results both from the point of view of numerical models and (not as many) analytical proofs [4-6]. Also in this direction, the phenomena of scar [7-11] drew much attention. They showed that the Hamiltonians eigenfunctions of chaotic systems exhibit “scars” around unstable periodic orbit. An question that appears from those analyses is related to chaotic manifestation of classical chaos over the eigenfunctions in terms of quantities that are base independent [12-15]. In opposition, it has been reported that scars can exist in regions where there are no periodic orbits [16]. The search for classical “imprints” than the celebrated phenomena of scars on eigenfunctions of quantum systems with classical analog has also gained a lot of attention.

In the present contribution we begin by generalizing the semiclassical expansion [1] for n-dimensional system. The semiclassical expansion is built in a way that the first order wave function contains the classical dynamics for the system in question as completely as possible, in the sense that the dominant term is given only in terms of classical trajectories. Higher order contributions contain essentially quantum effects and make possible a precise identification of a classical behavior in the quantum dynamics for short times. We use this expansion to obtain the semiclassical Husimi distributions of simple systems.

This paper is organized as follows. In Sections 2 and 3 we present the method, we closely follow ref. [1]. In

Section 4 we present a method of determining a semiclassical Husimi function. Section 7 contains conclusions.

2. The Semiclassical Expansion

Let us consider a classical one degree of freedom Hamiltonian of the form

$$H_{\text{cls}} = \frac{p^2}{2m} + V(q), \quad (1)$$

where p stands for the particle momentum and q for its position. We make a change of variables

$$p = \frac{\alpha - \alpha^*}{i\sqrt{\frac{2}{m\omega\hbar}}}; \quad q = \frac{\alpha + \alpha^*}{\sqrt{\frac{2m\omega}{\hbar}}}, \quad (2)$$

where

$$\omega = \sqrt{k/m}, \quad k = \left. \frac{\partial^2 V(q)}{\partial q^2} \right|_{q=0}. \quad (3)$$

The Hamiltonian can then be rewritten as

$$H_{\text{cls}} = \hbar\omega\alpha^*\alpha + U(\alpha^*\alpha) \quad (4)$$

with $U(\alpha^*\alpha) = V(q) - \left[k(\alpha + \alpha^*) / \sqrt{2m\omega/\hbar} \right]^2$.

We can write H_{cls} as a Taylor expansion,

$$H_{\text{cls}} = \hbar\omega\alpha^*\alpha + \sum_{m,n} A_{m,n} (\alpha^*)^m \alpha^n,$$

where $A_{1,1} = 0$.

The classical equations of motions are

$$\frac{d}{dt}\alpha = \frac{1}{i\hbar} \frac{\partial H_{\text{cls}}}{\partial \alpha^*} = -i\omega\alpha - \frac{i}{\hbar} \sum_{m,n} m A_{m,n} (\alpha^*)^{m-1} \alpha^n, \quad (5)$$

$$\frac{d}{dt}\alpha^* = -\frac{1}{i\hbar} \frac{\partial H_{\text{cls}}}{\partial \alpha} = i\omega\alpha^* + \frac{i}{\hbar} \sum_{m,n} n A_{m,n} (\alpha^*)^m \alpha^{n-1}. \quad (6)$$

We choose the quantum Hamiltonian \hat{H}_q in order to have $\langle \alpha | \hat{H}_q | \alpha \rangle = H_{\text{cls}}$, if $|\alpha\rangle$ is a coherent field state.

We make our semiclassical expansion around a quantum operator $\hat{H}_{sc}(\alpha(t))$. The difference

$$\hat{H}_q - \hat{H}_{sc}(\alpha(t)) = \hat{\delta}(\alpha, \beta, t)$$

will be considered as a perturbation. We choose the semiclassical Hamiltonian, $\hat{H}_{sc}(\alpha(t))$, in a way that for a coherent initial state, all expectation values of point classical observables will be precisely reproduced.

The semiclassical Hamiltonian which satisfies this condition is [1]:

$$\begin{aligned} & \hat{H}_{sc}(\alpha(t)) \\ &= \hbar\omega\hat{a}^\dagger\hat{a} + \sum_{m \neq 0} m A_{m,m} (\alpha^*)^{m-1} \alpha^{m-1} (\hat{a}^\dagger\hat{a} - \alpha^*\alpha) \\ &+ \sum_{m,n} A_{m,n} (\alpha^*)^m \alpha^n + \sum_{m \neq n} m A_{m,n} (\alpha^*)^{m-1} \alpha^n (\hat{a}^\dagger - \alpha^*) \quad (7) \\ &+ \sum_{m,n} n A_{m,n} (\alpha^*)^m \alpha^{n-1} (\hat{a} - \alpha). \end{aligned}$$

We can write the semiclassical evolution operator for an one degree of freedom, observing that

$$\begin{aligned} \hat{U}_{sc}(\alpha(t))|\alpha_0\rangle &= e^{i\phi(t)}|\alpha(t)\rangle \\ \Rightarrow \hat{U}_{sc}(\alpha(t)) &= e^{i\phi(t)} e^{i\Omega(t)\hat{a}^\dagger\hat{a}} \hat{D}(\alpha(t)) \hat{D}(\alpha_0)^{-1} \end{aligned} \quad (8)$$

where \hat{D} is the well known displacement operator

$$\hat{D}(\beta) = e^{\beta\hat{a}^\dagger - \beta^*\hat{a}},$$

and $\alpha(t)$ is given by 5, 6 and $\alpha_0 = \alpha(0)$.

Thus, for the N dimensional case we have

$$\begin{aligned} \hat{U}_{sc}(\overline{\alpha(t)}) &= \prod_j^N e^{i\phi(t)_j} e^{\alpha_j(t)\hat{a}_j^\dagger - \alpha_j^*(t)\hat{a}_j} e^{-\alpha_j\hat{a}_j^\dagger + \alpha_j^*\hat{a}_j} \quad (9) \\ &= \prod_j^N \hat{U}_{sc}^j(\alpha_j(t)). \end{aligned}$$

where, $\hat{U}_{sc}(\overline{\alpha(t)}) = \hat{U}_{sc}(\alpha_1(t), \alpha_2(t) \dots \alpha_N(t))$, and \hat{U}_{sc}^j is the semiclassical evolution operator related to the k-th degree of freedom, note that it depends solely on $\alpha_j(t)$ but in general we have

$$\alpha_j(t) = f(\alpha_1(t), \alpha_2(t) \dots \alpha_N(t)).$$

The phase $\phi(\alpha(t))_j$ is given by

$$\phi(\alpha(t))_j = -\frac{i}{\hbar} \int_0^t \mathcal{L}[\alpha, \alpha^*] dt' \quad (10)$$

where $\mathcal{L}[\alpha, \alpha^*]$ is the classical Lagrangian of the (in-

dependent) systems. In equation (9) we chose $\omega_j(t) = 0^1$, what can be done choosing a specific form of the semiclassical Hamiltonian, see [1]. A generalization for SU(2) algebra or for any subspace where coherent states can be included, is immediate. The action of the semiclassical evolution operator over a coherent state can always be written as [17]

$$\hat{U}_{sc}^j(\alpha(t))|\beta_j\rangle = e^{i\rho_j(t)}|\alpha_j(t) + \beta_j - \alpha_j(0)\rangle \quad (11)$$

where

$$\begin{aligned} \rho_j(\alpha_j(t), \beta_j) &= \phi_j(\alpha(t)) - \text{Im}(\alpha_j(t)\beta_j^*) \\ &+ \text{Im}[(\alpha_j(0) - \beta_j)^* \alpha(t)]. \end{aligned}$$

In general $\alpha_k(t)$ is a function of all $\alpha_j(t)$. In the next sections we use the fact that the labels of coherent states follow the classical trajectories.

3. Time Evolution

We consider a two degrees of freedom system, which the complete Hamiltonian is given by

$$\hat{H}_q = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12} \quad (12)$$

where $\hat{H}_{1(2)}$ represent the autonomous dynamics of the degree of freedom 1 (2) and \hat{H}_{12} is their interaction. The semiclassical Hamiltonian has the following form²

$$\begin{aligned} & \hat{H}_{sc}(\alpha(t), \beta(t), t) \\ &= \hat{H}_{sc}^1(\alpha(t), \beta(t), t) + \hat{H}_{sc}^2(\alpha(t), \beta(t), t) + H_{cl}, \end{aligned} \quad (13)$$

and by definition we have

$$\left[\hat{H}_{sc}^1(\alpha(t), \beta(t), t), \hat{H}_{sc}^2(\alpha(t), \beta(t), t) \right] = 0.$$

As discussed in section II we rewrite the Hamiltonian (12) in the following form $\hat{H} = \hat{H}_{sc}(\alpha, \beta, t) + \hat{\delta}(\alpha, \beta, t)$, where $\hat{\delta}(\alpha, \beta, t) = \hat{H} - \hat{H}_{sc}(\alpha, \beta, t)$. We make a perturbation expansion about $\hat{\delta}(\alpha, \beta, t)$. Using Schrödinger's equation, where we will always use as initial state $|\Psi(0)\rangle = |\alpha(0)\rangle|\beta(0)\rangle$, $|\alpha(0)\rangle$ and $|\beta(0)\rangle$ are coherent states. Thus, after some straightforward algebraic manipulations [1] we get

$$\begin{aligned} |\Psi\rangle(t) &= \hat{U}_{sc}(\alpha, \beta, t) \left\{ 1 + \frac{1}{i\hbar} \int_0^t \hat{\Delta}_s(t_1) dt_1 \right. \\ &\left. + \left(\frac{1}{i\hbar} \right)^2 \int_0^t \int_0^{t_1} \hat{\Delta}_s(t_1) \hat{\Delta}_s(t_2) dt_1 dt_2 + \dots \right\} |\Psi\rangle(0) \end{aligned} \quad (14)$$

¹ ω_j refers to j-th system.

² $\hat{H}_{sc}^1(\alpha, \beta, t)$ represents the semiclassical Hamiltonian in the subspace 1, and $\hat{H}_{sc}^2(\alpha, \beta, t)$ represents the semiclassical Hamiltonian in the subspace 2.

where $\hat{\Delta}_s(t, \alpha, \beta) = \hat{U}_{sc}^\dagger(\alpha, \beta, t) \hat{\delta}(\alpha, \beta, t) \hat{U}_{sc}(\alpha, \beta, t)$. The general problem of convergence of the serie (14) is an open problem. The convergence of the method was demonstrated for the quartic oscilator [1] and there is strong evidence of it for chaotic Dikie model [18], then it is reasonable to assume that the method is convergent at least to a group of non-integrable systems.

4. Husimi's Quantum Phase Space Distribution

The Q-function or Husimi's function is, see refs. [4,19], defined by:

$$\mathbf{H}_\Psi(Q, P) = \langle \alpha | \rho_\Psi | \alpha \rangle, \tag{15}$$

ρ_Ψ is a density operator, and $|\alpha\rangle$ is the harmonic coherent state according to the definitions:

$$Q = \frac{\langle \hat{q} \rangle}{\beta}, P = \frac{\langle \hat{p} \rangle}{\hbar} \beta \tag{16}$$

$$\beta = \sqrt{\frac{\hbar}{m\omega}}, \alpha = \frac{Q + iP}{\sqrt{2}}. \tag{17}$$

q and p are position and momentum operator respectively, and the mean is calculated in the coherent state $|\alpha\rangle$. From this definition, we are able to write the Husimi function as

$$\mathbf{H}_\Phi(Q, P) = \left| \iiint \Phi(x, y, z) \chi(x, y, z) dx dy dz \right|^2. \tag{18}$$

$\Phi(x, y, z)$ is the system Hamiltonian eigenfunction, and $\chi(x, y, z)$ is the harmonic coherent state in three dimensions, it is given by

$$\chi_\alpha(x, y, z) = \prod_{i=1}^3 \mu_{\theta_i} \exp \left[- \left(\frac{x_i - \langle x_i \rangle_{\alpha_i}}{2\Delta x} \right)^2 + i \langle p_{x_i} \rangle_{\alpha_i} \frac{x_i}{\hbar} \right], \tag{19}$$

where $x_1 = x, x_2 = y, x_3 = z, p_{x_i}$ is the momentum related with x_i , and

$$\mu_{\theta_i} = e^{\frac{\alpha_i^2 - \alpha_i'^2}{4}} \left(\frac{\mu\omega}{\pi\hbar} \right)^{\frac{1}{4}}, \tag{20}$$

$$\Delta x = \Delta y = \Delta z = \sqrt{\frac{\hbar}{2\mu\omega}}. \tag{21}$$

For the simplest case of the Harmonic Oscillator, using equation [15], the Husimi Function for an eigenstate, n , can be written as:

$$\mathbf{H}_n(\alpha) = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{(2\pi\hbar)n!} \tag{22}$$

In terms of Q and P , we have

$$\mathbf{H}_n(Q, P) = \frac{e^{-(Q^2+P^2)/2} (Q^2 + P^2)^n}{(4\pi\hbar)n!} \tag{23}$$

4.1. Husimi Function for the Morse Potential

The Morse potential is used to model diatomic molecules, it is defined as :

$$U(x) = D(e^{-2\eta x} - 2e^{-\eta x}), \tag{24}$$

where
$$x \equiv \frac{r - r_0}{r_0}. \tag{25}$$

The r_0 values are the equilibrium position of the center of mass, μ is the reduced mass of the two atoms and r is the distance between the atoms. The constant D defines the minimum value o the potential wich is $-D$. The constant η determines the potential range. The Hamiltonian that describes the center of mass can be written as:

$$H = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} \frac{L^2}{\mu r^2} + U(r). \tag{26}$$

where L is the angular momentum. The time independent Schrödinger equation is:

$$\left[-\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{L^2}{2\mu r^2} + V(r) \right] \Phi(r, \theta, \varphi) = E \Phi(r, \theta, \varphi). \tag{27}$$

We can write the wavefunction as

$$\Phi(r, \theta, \varphi) = \frac{\Psi(r)}{r} Y_{lm}(\theta, \varphi), \tag{28}$$

where Y_{lm} is the spherical harmonics:

$$Y_l^m(\theta, \varphi) = A_{lm} P_l^m[\cos(\theta)] e^{im\varphi}. \tag{29}$$

For $L = 0$ case we find the eigenvalues:

$$E_\nu = -D + \hbar\omega \left(\nu + \frac{1}{2} \right) - \frac{1}{\zeta} \left(\nu + \frac{1}{2} \right)^2 \tag{30}$$

where $\nu = 0, 1, 2, 3, \dots, \nu < \frac{1}{2}(\zeta - 1)$, and

$$\beta^2 = -\frac{2\mu E_\nu r_0^2}{\hbar^2}, \gamma^2 = \frac{2\mu D r_0^2}{\hbar^2}, \tag{31}$$

$$\zeta = \frac{2\beta}{\eta} + 1, \tilde{a} = \frac{1}{2}\zeta - \frac{\gamma}{\eta}, \zeta = \frac{2\gamma}{\eta}.$$

and for the eigenfunctions:

$$\begin{aligned} \frac{\Psi(x)}{r} &= \frac{A_1}{r_0} \frac{\Gamma(\zeta)}{(x+1)\Gamma(\tilde{a})} \exp\left(-\beta x - \frac{1}{2}\zeta e^{-\eta x}\right) \\ &\times \sum_{n=0}^{\infty} \left[\frac{\Gamma(\tilde{a} + n)}{\Gamma(\zeta + n)} \frac{(\zeta e^{-\eta x})^n}{n!} \right]. \end{aligned} \tag{32}$$

A_1 is fixed by normalization, $\Gamma(x)$ is the gamma function.

Following the definition (20), we obtain the Husimi the Function [20] as

$$\mathbf{H}_\Phi(Q, P) = \left| 2\pi \int_{-1}^{\infty} I_{\theta 0}(x) \chi(x) \Psi(x) r_0^2 (x+1) dx \right|^2, \quad (33)$$

where $I_{\theta 0}(r) = \frac{1}{\sqrt{\pi}} \frac{\sinh(\alpha_0 r)}{\alpha_0 r}$, $l = 0$. The exact Husimi function is obtained by numerically integration of (33).

4.2. Semiclassical Husimi's Function

The semiclassical expansion, as defined above, gives us the time evolution of a quantum state as a perturbative expansion. An eigenstate has only a time dependent phase as its dynamics. The nearest semiclassical scenario we can build is to choose a coherent state with the same energy. The time dependence can be eliminated by a time integration, *i.e.*, a mean in time. This integration can be justified noting that as we are dealing with eigenstates we have not time precision. Under this considerations we may write the semiclassical Husimi function as

$$\mathbf{H}_{sc}(\beta) = \overline{\left| \langle \beta | \alpha(t) \rangle \right|^2}. \quad (34)$$

The states $|\alpha(t)\rangle$ and $|\beta\rangle$ are coherent states of the harmonic oscillator. $\alpha(t)$ is defined as

$$\alpha(t) = \frac{Q(t) + iP(t)}{2} \quad \text{and} \quad \beta = x + ip_x,$$

where x and p_x are parameters of the Husimi Function, $Q(t)$ and $P(t)$ correspond to the classical canonical conjugate pairs redefined as in Equations (16) and (17). In case of classical mixed dynamics we must perform a mean considering all possible initial condition with the same specific energy. We should also use adequate coherent state base for each algebra. This semiclassical Husimi function is calculated by taking a long time mean, formally we write it as

$$\mathbf{H}_{sc}(\beta) = \lim_{\tau \rightarrow \infty} \frac{\int_0^\tau \left| \langle \beta | \alpha(t) \rangle \right|^2 dt}{\tau}. \quad (35)$$

As we known, classically chaotic systems stay longer times [4] near a stable or unstable periodic orbit. In our numerical calculation it means that this region has a huge contribution in the mean (35) or (34). Easily we can show that

$$\mathbf{H}_{sc}(\beta) = e^{\overline{|Q(t)+iP(t)-\beta|^2}}. \quad (36)$$

The semiclassical Husimi distributions is determined by the classical trajectory solely. In general the semiclassical Husimi function is obtained by numerical

methods, it is the case of all chaotic model.

4.2.1. Morse Potential

For the Morse potential, with $L = 0$, we obtain the classical trajectory [20] as:

$$Q(t) = \frac{1}{\alpha} \ln \left\{ \frac{D}{|E|} + \left[\sin \left(t \sqrt{\frac{2|E|\alpha^2}{\mu r_0^2}} \right) \sqrt{\frac{D^2 + ED}{|E|^2}} \right] \right\} \quad E < 0 \quad (37)$$

We also have $P(t) = \mu \frac{dQ(t)}{dt}$, and we can choose

$p(0) = 0$ and choosing the energy as $E = E_v$ into (36) to obtain the semiclassical Husimi function. The mean (35) is obtained by a numerical integration. In **Figure 1** we show the approximated semiclassical Husimi for the Morse potential with the parameters of the H_2 molecule, for $v = 0$. In **Figure 2** we have the exact result, **Figure 3** shows the semiclassical Husimi function for $n = 1$ and **Figure 4** the exact result, details about exact calculation can be found in ref. [20]. As we can observe in this **Figures 1-4**, the semiclassical Husimi function does not reproduce exactly the Husimi function, but it regards some similarities. Notice that the main region is located in the same phase space area for the exact and semiclassical Husimi function.

4.2.2. Harmonic Oscillator

Now consider the Harmonic potential with a natural frequency ω , its classical dynamics is given by

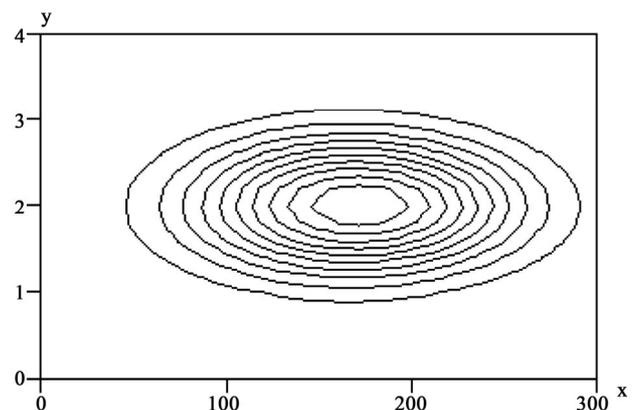


Figure 1. Semiclassical Husimi function for the Morse Potential, $U(x) = D(e^{-2\alpha x} - 2e^{-\alpha x})$, where $x = \frac{r-r_0}{r_0}$. We have used the experimental hydrogen molecule^o values, $\alpha = 1.440$ and $D = 4.75(\text{eV})$. The principal quantum number $n = 0$ and the angular momentum $L = 0$. The x axis is related to the position, r , as $r = r_0 \frac{(x+10)}{250}$, the momentum p is related to the y axis variable as $p = m\omega r_0 \frac{(y-2)}{1.2}$. We have used $r_0 = 7.42 \times 10^{-11} \text{ m}$ and $\omega = 8.3 \times 10^{14} \text{ rad/s}$.

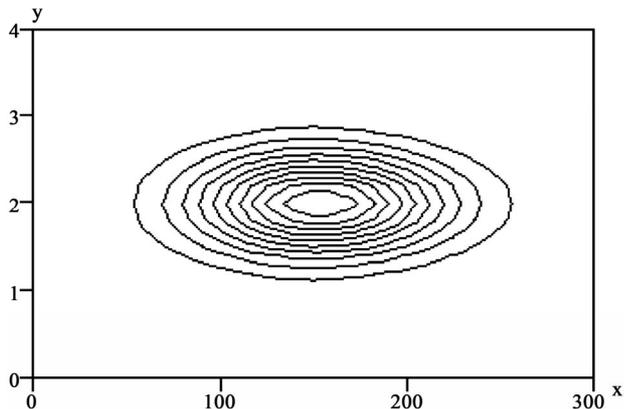


Figure 2. Husimi function for the Morse potential with the same parameters of Figure 1.

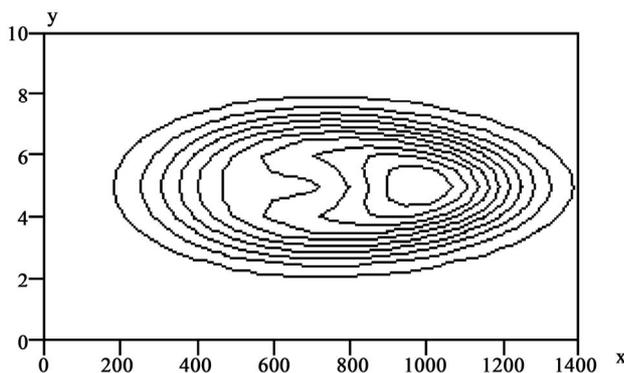


Figure 3. Semiclassical Husimi function for the Morse Potential, $U(x) = D(e^{-2ax} - 2e^{-ax})$, where $x = \frac{r-r_0}{r_0}$. We have used the experimental hydrogen molecule values, $\alpha = 1.440$ and $D = 4.75(\text{eV})$. The principal quantum number $n=1$ and the angular momentum $L = 0$. The x axis is related to the position, r , as $r = r_0 \frac{(x + 300)}{1000}$, the momentum p is related to the y axis variable as $p = m\omega r_0 \frac{(y-5)}{5}$. We have used $r_0 = 7.42 \times 10^{-11} \text{ m}$ and $\omega = 8.3 \times 10^{14} \text{ rad/s}$.

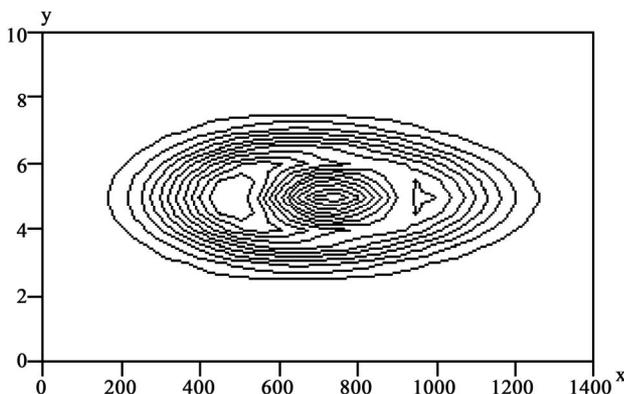


Figure 4. Husimi function for the Morse potential with the same parameters of Figure 3.

$$Q(t) = Q(0)\cos(\omega t) + \frac{P(0)}{\omega}\sin(\omega t) \quad (38)$$

and

$$P(t) = P(0)\cos(\omega t) - Q(0)\omega\sin(\omega t) \quad (39)$$

We redefined Q and P in a such way that the Hamiltonian can be written as

$$H = \hbar\omega \frac{(Q^2 + P^2)}{2}. \quad (40)$$

Substituting (38) and (39) into (36) we obtain a semiclassical Husimi Function for the Harmonic oscillator with an energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. Without any loss of generality we can use $Q(0) = (2E_n)^{1/2}$, and $P(0) = 0$. In Figure 5 we show the approximated and exact Husimi function for the Harmonic potential for $n = 5$. Figure 6 shows the semiclassical and exact Husimi function for $n = 100$. Again the mean (35) is obtained numerically.

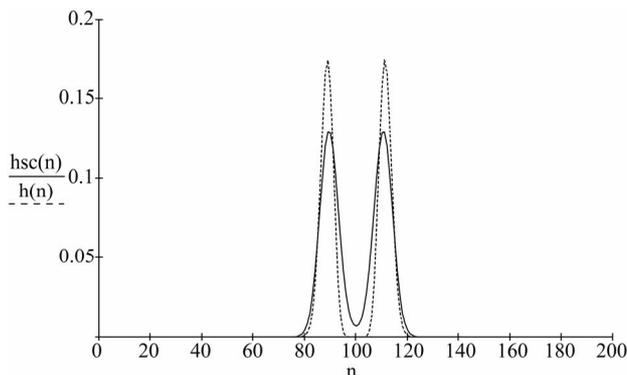


Figure 5. Full line shows a cross section of Husimi function for the harmonic potential for $p = 0$. The principal quantum number $n = 5$. The x axis is related to the position. Dotted line, correspondent semiclassical Husimi function.

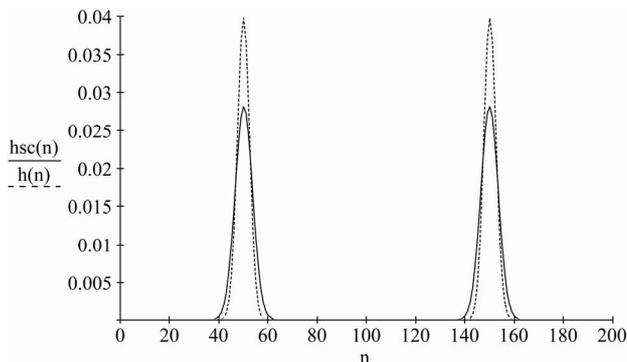


Figure 6. Full line shows a cross section of Husimi function for the harmonic potential for $p = 0$. The principal quantum number $n = 100$. The x axis is related to the position. Dotted line, correspondent semiclassical Husimi function.

4.2.3. Dickie Model

In the **Figures 1 to 7** we have used classical integrable models, although that our approach is also useful for non-integrable thus let us take a look in Dickie model [21], his quantum Hamiltonian is

$$H = H_O + H_{G'} + H_G, \tag{41}$$

where

$$H_O = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\varepsilon\hat{J}_z, \tag{42}$$

$$H_{G'} = \frac{G'}{\sqrt{2J}}\left(\hat{a}\hat{J}_- + \hat{a}^\dagger\hat{J}_+\right), \tag{43}$$

$$H_G = \frac{G}{\sqrt{2J}}\left(\hat{a}\hat{J}_+ + \hat{a}^\dagger\hat{J}_-\right), \tag{44}$$

ω is the harmonic oscillator natural frequency, ε is precession frequency, G and G' are coupling constants. In the harmonic term of (42) \hat{a} and \hat{a}^\dagger are bosonic annihilation and creation operators of harmonic oscillator, \hat{J}_k is the angular momentum operator in k direction and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. In order to obtain the Semiclassical Husimi function for the Dickie model we have to integrate numerically the corresponding classical equations of motion and calculate the mean (35).

In **Figure 7** we show the semiclassical Wigner function of ground state of Dickie model [21] in an integrable regime. In **Figure 8** we have the semiclassical Wigner function for Dickie model in a non-integrable regime. Comparing with exact Wigner³ function that can be found in ref. [21], we see that semiclassical Husimi function contains the main information about the exact one. A detailed semiclassical analysis of Dickie's model can be found in [18,22]. In [18] they show a numerical evidence of the expansion convergence for the Dickie's model.

4.2.4. The Quality of the Approximation

Now we quantify the quality of the approximation using the function $\Delta Q_{\psi_n}(q)$, which is defined as

$$\Delta Q_{\psi_n}(q) = \mathbf{H}(\mathbf{q}, \mathbf{p} = \mathbf{0}) - \mathbf{H}_{sc}(\mathbf{q}, \mathbf{p} = \mathbf{0}), \tag{45}$$

where \mathbf{H} is the exact Husimi function and \mathbf{H}_{sc} is the semiclassical Husimi function. Due to the symmetry we have chosen $p = 0$. As we increase the principal quantum number (n), in the classical limit, we hope we have $\Delta Q_{\psi_n}(q) \rightarrow 0$. In order to see the classical limit, let us define the function SQ_{ψ_n} , which is

$$SQ_{\psi_n} = \int_{-\infty}^{\infty} |\Delta Q_{\psi_n}(q)| dq \tag{46}$$

³For stationary states semiclassical Wigner function and semiclassical Husimi function are identical.

⁴For example, the Wigner function can be negative at some point while the semiclassical Husimi function and the exact one are almost identical.

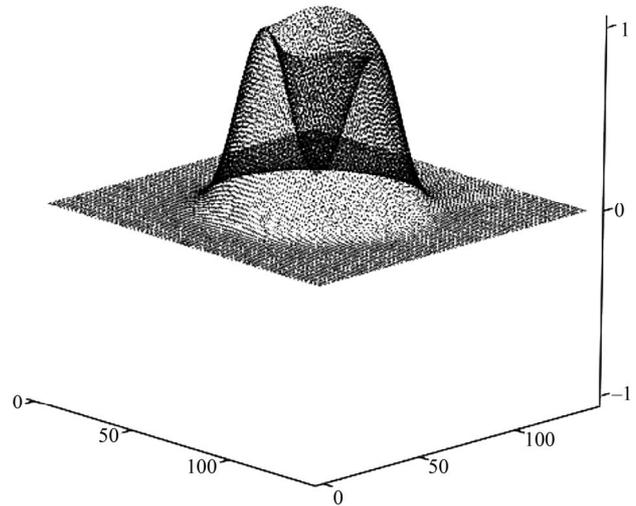


Figure 7. Semiclassical Wigner function for the fundamental state of Dickie model where $G' = 0$, $\lambda = 1.5$, $\omega = 1$.

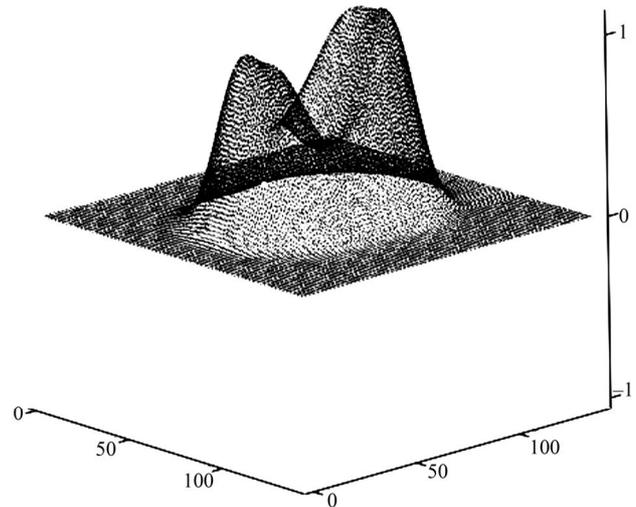


Figure 8. Semiclassical Wigner function for the fundamental state of Dickie model $G = G' = 0.75$.

Suppose we have $SQ_{\psi_n} \approx 0$. It means that quantum description of the state ψ , in the Husimi's representation, is almost contained in the semiclassical one. In spite of that we can say that the quantum classical difference becomes smaller, as expected. Of course it does not mean that we have no quantum features⁴, it only means that Husimi is not a good observable for this situation [23]. **Figure 9** shows SQ_{ψ_n} say that the approximation works better as we increase the principal quantum number, as expected. From these figures we may conclude that the classical ingredient is very strong on the state formation of regular systems. In other words, those figures suggest that scars are essentially a classical manifestation in the quantum system since the building blocks of the semiclassical Husimi are classical trajectories.

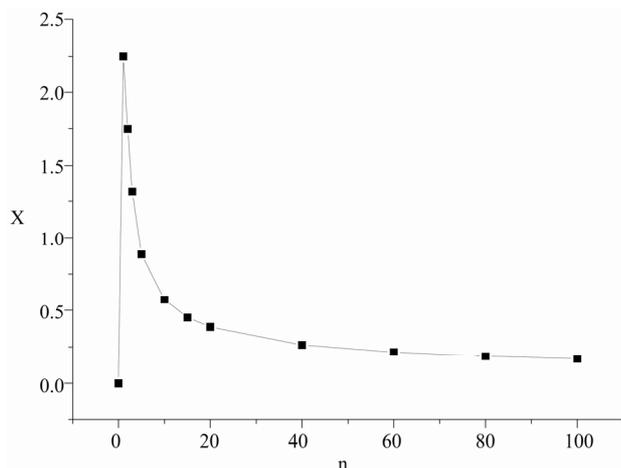


Figure 9. SQ function for the harmonic oscillator. X axis corresponds to the principal quantum number n .

5. Conclusion

We show that the semiclassical Husimi function reproduces the major features of the quantum one, in the particular harmonic case, we show that the first semiclassical term is able to reproduce the Husimi function with a increasing accuracy as we increase the principal quantum number n . We must remark that there is no demonstration that would suggest an existence of the limit procedure which turns quantum corrections less important in terms of the proposed semiclassical expansion. The building blocks of semiclassical Husimi Function are the classical trajectories, then we can conclude that classical periodic orbits (stable or unstable) contribute with higher weight. The fidelity decay rate has a gaussian regime that is only perturbation potential dependent, although its validity is determined by $t < \tau_{sc}$.

6. Acknowledgements

The author is grateful to Fapesb for partial financial support. The author also acknowledge, A. R. Bosco de Magalhaes, M. C. Nemes, Fernanda Alves de Oliveira for helpful comments.

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