

# Periodicity and Solution of Rational Recurrence Relation of Order Six

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## ABSTRACT

Difference equations or discrete dynamical systems is a diverse field whose impact almost every branch of pure and applied mathematics. Every dynamical system  $a_{n+1} = f(a_n)$  determines a difference equation and vice versa. We obtain in this paper the solution and periodicity of the following difference equation.

$$x_{n+1} = (x_n x_{n-2} x_{n-4}) / (x_{n-1} x_{n-3} x_{n-5}), \quad (1)$$

$n = 0, 1, \dots$  where the initial conditions  $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$  and  $x_0$  are arbitrary real numbers with  $x_{-1}, x_{-3}$  and  $x_{-5}$  not equal to be zero. On the other hand, we will study the local stability of the solutions of Equation (1). Moreover, we give graphically the behavior of some numerical examples for this difference equation with some initial conditions.

**Keywords:** Difference Equation; Solutions; Periodicity; Local Stability

## 1. Introduction

Difference equations or discrete dynamical systems is a diverse field whose impact almost every branch of pure and applied mathematics. Every dynamical system  $a_{n+1} = f(a_n)$  determines a difference equation and vice versa. Recently, there has been great interest in studying difference equations. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology, ...etc. Difference equations usually describe the evolution of certain phenomena over the course of time. Recently there are a lot of interest in studying the global attractivity, boundedness character the periodic nature, and giving the solution of nonlinear difference equations. Recently there has been a lot of interest in studying the boundedness character and the periodic nature of nonlinear difference equations. Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincaré and Perron in the end of nineteenth and the beginning of twentieth centuries. For some results in this area, see for example [1-13].

Although difference equations are sometimes very simple in their forms, they are extremely difficult to un-

derstand throughly the behavior of their solutions.

Many researchers have investigated the behavior of the solution of difference equations for examples.

Cinar [1,2] investigated the solutions of the following difference equations

$$x_{n+1} = x_{n-1} / (-1 + x_n x_{n-1})$$

$$x_{n+1} = ax_{n-1} / (1 + bx_n x_{n-1})$$

Karatas *et al.* [4] gave that the solution of the difference equation

$$x_{n+1} = x_{n-5} / (1 + x_{n-2} x_{n-5})$$

G. Ladas, M. Kulenovic *et al.* [12] have studied period two solutions of the difference equation

$$x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1}) / (A + Bx_n + Cx_{n-1})$$

Simsek *et al.* [13] obtained the solution of the difference equation

$$x_{n+1} = x_{n-3} / (1 + x_{n-1})$$

Ibrahim [5] studied the third order rational difference Equation

$$x_{n+1} = (x_n x_{n-2}) / (x_{n-1} (\alpha + \beta x_n x_{n-2}))$$

In this paper we obtain the solution and study the periodicity of the following difference equation

$$x_{n+1} = (x_n x_{n-2} x_{n-4}) / (x_{n-1} x_{n-3} x_{n-5}), \tag{1}$$

$n = 0, 1, \dots$  where the initial conditions  $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$ , and  $x_0$  are arbitrary real numbers with  $x_{-1}, x_{-3}$  and  $x_{-5}$  not equal to be zero. On the other hand, we will study the local stability of the solutions of Equation (1). Moreover, we give graphically the behavior of some numerical examples for this difference equation with some initial conditions.

Here, we recall some notations and results which will be useful in our investigation.

Let  $I$  be some interval of real numbers and Let  $F : I^{k+1} \rightarrow I$  be a continuously differentiable function. Then for every set of initial conditions  $x_{-k}, x_{-k+1}, \dots, x_0 \in I$ , the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{2}$$

has a unique solution  $\{x_n\}_{n=-k}^\infty$  [11].

**Definition (1.1)** A point  $\bar{x} \in I$  is called an equilibrium point of Equation (2) if

$$\bar{x} = F(\bar{x}, \bar{x}, \dots, \bar{x}).$$

That is,  $x_n = \bar{x}$  for  $n \geq 0$ , is a solution of Equation (2), or equivalently,  $\bar{x}$  is a fixed point of  $F$ .

**Definition (1.2)** The difference Equation (2) is said to be persistence if there exist numbers  $m$  and  $M$  with  $0 < m \leq M < \infty$  such that for any initial  $x_{-k}, x_{-k+1}, \dots, x_0 \in (0, \infty)$  there exists a positive integer  $N$  which depends on the initial conditions such that  $m \leq x_n \leq M$  for all  $n \geq N$ .

**Definition (1.3)** (Stability)

Let  $I$  be some interval of real numbers.

1) The equilibrium point  $\bar{x}$  of Equation (2) is locally stable if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for  $x_{-k}, x_{-k+1}, \dots, x_0 \in I$  with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

we have  $|x_n - \bar{x}| < \varepsilon$  for all  $n \geq -k$ .

2) The equilibrium point  $\bar{x}$  of Equation (2) is locally asymptotically stable if  $\bar{x}$  is locally stable solution of Equation (2) and there exists  $\gamma > 0$ , such that for all  $x_{-k}, x_{-k+1}, \dots, x_0 \in I$  with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

we have  $\lim_{n \rightarrow \infty} x_n = \bar{x}$ .

3) The equilibrium point  $\bar{x}$  of Equation (2) is global attractor if for all  $x_{-k}, x_{-k+1}, \dots, x_0 \in I$ , we have  $\lim_{n \rightarrow \infty} x_n = \bar{x}$ .

4) The equilibrium point  $\bar{x}$  of Equation (2) is globally asymptotically stable if  $\bar{x}$  is locally stable, and  $\bar{x}$  is also a global attractor of Equation (2).

5) The equilibrium point  $\bar{x}$  of Equation (2) is unstable if  $\bar{x}$  not locally stable.

The linearized equation of Equation (2) about the equi-

librium  $\bar{x}$  is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k (\partial F(\bar{x}, \bar{x}, \dots, \bar{x}) / (\partial x_{n-i})) y_{n-i}$$

**Theorem (1.4)** [10] Assume that  $p, q \in \mathfrak{R}$  (real numbers) and  $k \in \{0, 1, 2, \dots\}$ . Then

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \dots.$$

**Remark (1.5)** Theorem (1.4) can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \quad n = 0, 1, \dots,$$

where  $p_1, p_2, \dots, p_k \in \mathfrak{R}$  (real numbers) and  $k \in \{1, 2, \dots\}$ . Then Equation (4) is asymptotically stable provided that

$$\sum_{i=1}^k |p_i| < 1.$$

**Definition (1.6)** (Periodicity)

A sequence  $\{x_n\}_{n=-k}^\infty$  is said to be periodic with period  $p$  if  $x_{n+p} = x_n$  for all  $n \geq -k$ .

## 2. Solution and Periodicity

In this section we give a specific form of the solutions of the difference Equation (1).

**Theorem (2.1)**

Let  $\{x_n\}_{n=-k}^\infty$  be a solution of Equation (1). Then Equation (1) have all solutions and the solutions are

$$x_{14n-5} = k, x_{14n-4} = h, x_{14n-3} = L, x_{14n-2} = \alpha,$$

$$x_{14n-1} = \beta, x_{14n} = \gamma, x_{14n+1} = (\gamma\alpha h) / (\beta L k),$$

$$x_{14n+2} = 1/k, x_{14n+3} = 1/h, x_{14n+4} = 1/L, x_{14n+5} = 1/\alpha,$$

$$x_{14n+6} = 1/\beta, x_{14n+7} = 1/\gamma, x_{14n+8} = \beta L k / \gamma \alpha h$$

where  $x_{-5} = k, x_{-4} = h, x_{-3} = L, x_{-2} = \alpha, x_{-1} = \beta, x_0 = \gamma$ .

**Proof:**

For  $n = 0$  the result holds. Now suppose that  $n > 0$  and that our assumption holds for  $n - 1$ . We shall show that the result holds for  $n$ . By using our assumption for  $n - 1$ , we have the following:

$$x_{14n-19} = k, x_{14n-18} = h, x_{14n-17} = L, x_{14n-16} = \alpha,$$

$$x_{14n-15} = \beta, x_{14n-14} = \gamma, x_{14n-13} = (\gamma\alpha h) / (\beta L k),$$

$$x_{14n-12} = 1/k, x_{14n-11} = 1/h, x_{14n-10} = 1/L, x_{14n-9} = 1/\alpha$$

$$x_{14n-8} = 1/\beta, x_{14n-7} = 1/\gamma, x_{14n-6} = \beta L k / \gamma \alpha h$$

Now, it follows from Equation (1) that

$$x_{14n-5} = x_{14n-6} x_{14n-8} x_{14n-10} / (x_{14n-7} x_{14n-9} x_{14n-11}) = (\beta L k / \gamma \alpha h) (1/\beta) (1/L) / ((1/\gamma) (1/\alpha) (1/h)) = k$$

$$\begin{aligned}
 x_{14n-4} &= x_{14n-5}x_{14n-7}x_{14n-9}/(x_{14n-6}x_{14n-8}x_{14n-10}) = h \\
 x_{14n-3} &= x_{14n-4}x_{14n-6}x_{14n-8}/(x_{14n-5}x_{14n-7}x_{14n-9}) = L \\
 x_{14n-2} &= x_{14n-3}x_{14n-5}x_{14n-7}/(x_{14n-4}x_{14n-6}x_{14n-8}) = \alpha
 \end{aligned}$$

similarly we can derive,

$$\begin{aligned}
 x_{14n-2} &= \alpha, \quad x_{14n} = \gamma, \\
 x_{14n+1} &= (\gamma\alpha h)/(\beta Lk), \quad x_{14n+2} = 1/k, \\
 x_{14n+3} &= 1/h, \quad x_{14n+4} = 1/L, \quad x_{14n+5} = 1/\alpha, \\
 x_{14n+6} &= 1/\beta, \quad x_{14n+7} = 1/\gamma, \quad x_{14n+8} = \beta Lk/\gamma\alpha h.
 \end{aligned}$$

Thus, the proof is completed.

**Theorem (2.2)**

Suppose that  $\{x_n\}_{n=-k}^\infty$  be a solution of Equation (1).

Then all solutions of Equation (1) are periodic with period fourteen.

**Proof:**

From Equation (1), we see that

$$\begin{aligned}
 x_{n+1} &= (x_n x_{n-2} x_{n-4}) / (x_{n-1} x_{n-3} x_{n-5}) \\
 x_{n+2} &= (x_{n+1} x_{n-1} x_{n-3}) / (x_n x_{n-2} x_{n-4}) = 1/x_{n-5} \\
 x_{n+3} &= (x_{n+2} x_{n-2} x_{n-4}) / (x_{n+1} x_{n-1} x_{n-3}) = 1/x_{n-4} \\
 x_{n+4} &= (x_{n+3} x_{n+1} x_{n-1}) / (x_{n+2} x_n x_{n-2}) = 1/x_{n-3} \\
 x_{n+5} &= 1/x_{n-2} \ \& \ x_{n+6} = 1/x_{n-1} \ \& \ x_{n+7} = 1/x_n \\
 x_{n+8} &= 1/x_{n+1} \ \& \ x_{n+9} = 1/x_{n+2} = x_{n-5} \ \& \ x_{n+10} = x_{n-4} \\
 x_{n+11} &= x_{n-3} \ \& \ x_{n+12} = x_{n-2} \ \& \ x_{n+13} = x_{n-1} \ \& \ x_{n+14} = x_n
 \end{aligned}$$

which completes the proof.

**3. Stability of Solutions**

In this section we study the local stability of the solutions of Equation (1).

**Lemma (3.1)**

Equation (1) have two equilibrium points which are 0 and 1.

**Proof:**

For the equilibrium points of Equation (1), we can write

$$\bar{x} = \bar{x} \cdot \bar{x} \cdot \bar{x} / (\bar{x} \cdot \bar{x} \cdot \bar{x})$$

Then  $\bar{x}^4 = \bar{x}^3$ , *i.e.*  $\bar{x}^4 - \bar{x}^3 = 0$

Thus the equilibrium points of Equation (1) is are 0 and 1.

**Theorem (3.2)**

The equilibrium points  $\bar{x} = 0$  and  $\bar{x} = 1$  are unstable.

**Proof:**

We will prove the theorem at the equilibrium point  $\bar{x} = 1$  and the proof at the equilibrium point  $\bar{x} = 0$  by the same way.

Let  $f : (0, \infty)^6 \rightarrow (0, \infty)$  be a continuous function defined by

$$f(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1 u_3 u_5) / (u_2 u_4 u_6)$$

Therefore it follows that

$$\begin{aligned}
 \partial f / \partial u_1 &= (u_3 u_5) / (u_2 u_4 u_6) \\
 \partial f / \partial u_2 &= 0 - (u_1 u_3 u_5) (u_4 u_6) / (u_2 u_4 u_6)^2 \\
 &= -(u_1 u_3 u_5) / (u_2^2 u_4 u_6) \\
 \partial f / \partial u_3 &= (u_1 u_5) / (u_2 u_4 u_6) \\
 \partial f / \partial u_4 &= -(u_1 u_3 u_5) / (u_2 u_4^2 u_6) \\
 \partial f / \partial u_5 &= (u_1 u_3) / (u_2 u_4 u_6) \\
 \partial f / \partial u_6 &= -(u_1 u_3 u_5) / (u_2 u_4 u_6^2)
 \end{aligned}$$

At the equilibrium point  $\bar{x} = 1$  we have

$$\begin{aligned}
 \partial f / \partial u_1 = 1 = p_1, \quad \partial f / \partial u_2 = -1 = p_2, \\
 \partial f / \partial u_3 = 1 = p_3, \quad \partial f / \partial u_4 = -1 = p_4, \\
 \partial f / \partial u_5 = 1 = p_5, \quad \partial f / \partial u_6 = -1 = p_6
 \end{aligned}$$

Then the linearized equation of Equation (1) about  $\bar{x} = 1$  is

$$\begin{aligned}
 y_{n+1} - p_1 y_n - p_2 y_{n-1} - p_3 y_{n-2} - p_4 y_{n-3} \\
 - p_5 y_{n-4} - p_6 y_{n-5} = 0
 \end{aligned}$$

*i.e.*

$$y_{n+1} - y_n + y_{n-1} - y_{n-2} + y_{n-3} - y_{n-4} + y_{n-5} = 0$$

Whose characteristic equation is

$$\lambda^6 - \lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1 = 0$$

By the generalization of theorem (1.4) we have

$$|1| + |-1| + |1| + |1| + |-1| + |1| < 1$$

which is impossible. This means that the equilibrium point  $\bar{x} = 1$  is unstable. Similarly, we can see that the equilibrium point  $\bar{x} = 0$  is unstable.

**4. Numerical Examples**

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Equation (1).

**Example 4.1**

Consider  $x_{-5} = 1, x_{-4} = 2, x_{-3} = 4, x_{-2} = -1, x_{-1} = -2,$  and  $x_0 = 7$ . See **Figure 1**.

**Example 4.2**

Consider  $x_{-5} = 1, x_{-4} = -1, x_{-3} = 2, x_{-2} = -6, x_{-1} = 5,$  and  $x_0 = -4$ . See **Figure 2**.

**Example 4.3**

Consider  $x_{-5} = 3, x_{-4} = 5, x_{-3} = -7, x_{-2} = -3, x_{-1} = 2,$  and  $x_0 = -1$ . See **Figure 3**.

**Example 4.4**

Consider  $x_{-5} = -4, x_{-4} = 3, x_{-3} = -2, x_{-2} = 9, x_{-1} = 17,$

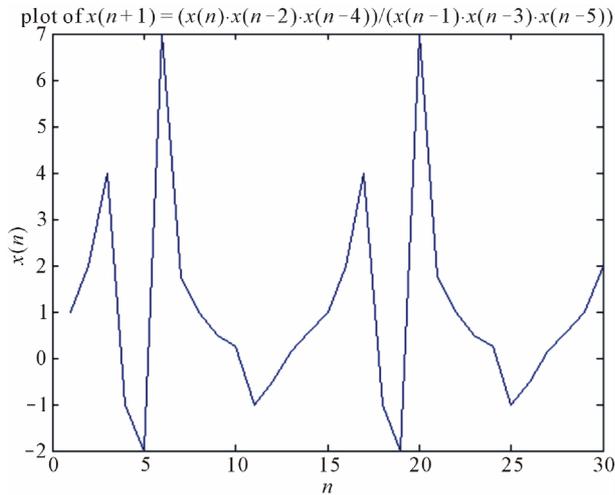


Figure 1. The periodicity of solutions with period 14 with unstable equilibrium points  $\bar{x} = 1$  and  $\bar{x} = 0$ .

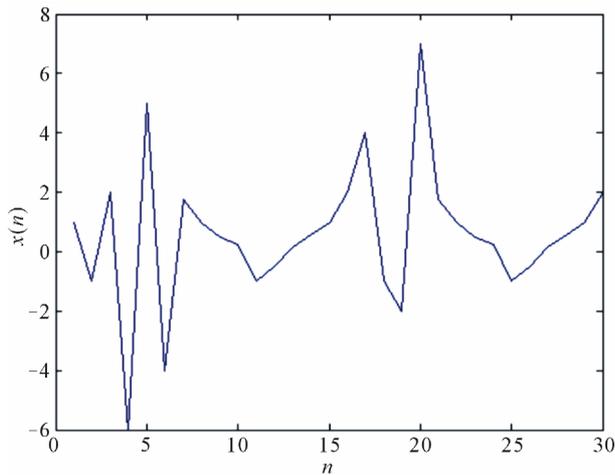


Figure 2. Periodicity of solutions with period 14 with unstable equilibrium points  $\bar{x} = 1$ ,  $\bar{x} = 0$ .

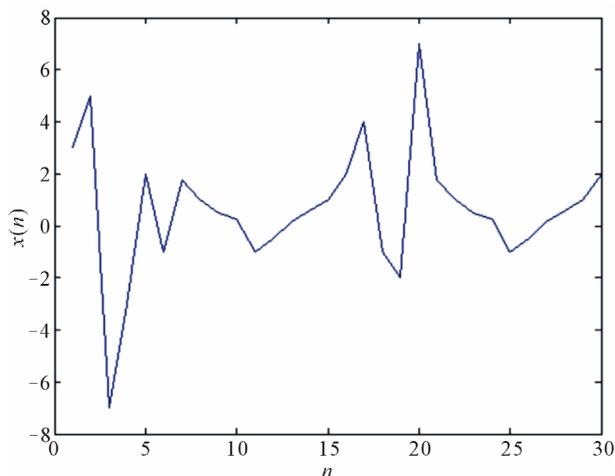


Figure 3. Periodicity of solutions with period 14 with unstable equilibrium points  $\bar{x} = 1$  and  $\bar{x} = 0$ .

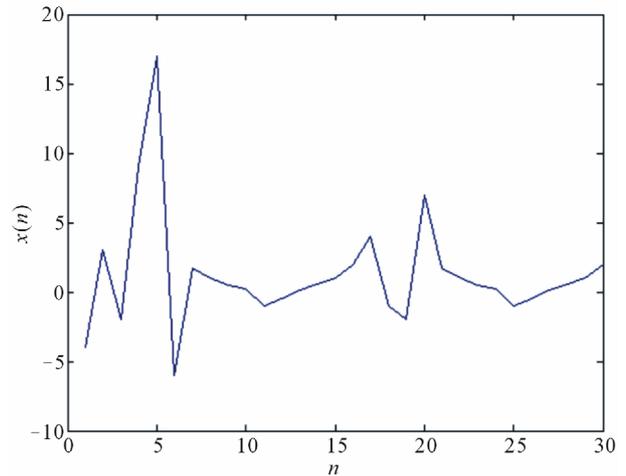


Figure 4. The periodicity of solutions with period 14 with unstable equilibrium points  $\bar{x} = 1$ ,  $\bar{x} = 0$ .

and  $x_0 = -6$ . See Figure 4.

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