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An Application of Eulerian Graph to PI on $M_n(C)$

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ABSTRACT

We obtain a new class of polynomial identities on the ring of $n \times n$ matrices over any commutative ring with 1 by using the Swan's graph theoretic method [1] in the proof of Amitsur-Levitzki theorem. Let Γ be an Eulerian graph with k vertices and d edges. Further let $n \geq 1$ be an integer and assume that $d \geq 2kn$. We proof that $\sum_{\pi \in \Pi(\Gamma)} \operatorname{sgn}(\pi) x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(d)} = 0$ is an PI on $M_n(C)$. Standard and Chang [2] Giambruno-Sehgal [3] polynomial identities are the special examples of our conclusions.

Keywords: Eulerian Graph; Eulerian Path; Admissible; Polynomial Identity

1. Introduction

Let Γ be a finite directed graph with multiple edges allowed, and let $V(\Gamma) = \{1, \cdots, k\}$ denote the vertex set of Γ and $E(\Gamma) = \{e_1, \cdots, e_d\}$ the edges set of Γ. Let σ and τ be the functions from $E(\Gamma)$ to $V(\Gamma)$ defined by $(\sigma(e_s), \tau(e_s)) = (i, j)$ where e_s is an edge from vertex i to vertex j. For a vertex $i \in V(\Gamma)$ we put

$$\phi_+(i) = \left|\{e_s \mid \sigma(e_s) = i\}\right|, \phi_-(i) = \left|\{e_s \mid \tau(e_s) = i\}\right|$$

and

$$\gamma(i) = \max\{\phi_+(i), \phi_-(i)\}$$

We say that $e_{\pi(1)} e_{\pi(2)} \cdots e_{\pi(N)}$ is an Eulerian path of Γ if π is an element of Sym(d) (the symmetric group acting on the set $\{1, \cdots, d\}$ and $\tau\left(e_{\pi(i)}\right) = \sigma\left(e_{\pi(i+1)}\right)$ for $i = 1, \cdots, d-1$.

It is well known that a connected graph Γ has an Eulerin path starting at vertex p and ending at vertex q if and only if one of the following two conditions applies:

1) $p = q$ and $\phi_+(i) = \phi_-(i)$ for each $i = 1, \cdots, k$;

2) $p \neq q$ and $\phi_+(p) = \phi_-(q) + 1$, $\phi_-(q) = \phi_+(q) + 1$ and $\phi_+(i) = \phi_-(i)$ for each $i \in \{1, \cdots, k\} \setminus \{p, q\}$.

A directed connected graph $\Gamma_{p,q}$ with fixed vertices p and q is called Eulerian if either condition 1) or 2) is satisfied. We note that if $\Gamma_{p,q}$ is an Eulerian graph of type (b), then the vertices p, q are uniquely determined, but in the other case we may choose any vertex $p = q$. For an Eulerian graph $\Gamma_{p,q}$ denote by

$\Pi(\Gamma_{p,q}) = \{\pi \in Sym(d) \mid e_{\pi(1)} \cdots e_{\pi(d)}$ is an Eulerian path of $\Gamma_{p,q}$ starting at vertex p and ending at vertex $q\}$.

2. Main Results

Let Γ be an Eulerian graph with d edges $e_1, e_2, \cdots, e_d$ and distinguished points p and q. the polynomial $f_\Gamma(X)$ associated with Γ is defined as follows:

$$f_\Gamma(X) = \sum_{\pi \in \Pi(\Gamma)} \operatorname{sgn}(\pi) x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(d)}$$

Thus $f_\Gamma(X)$ is a multilinear polynomial in the set $X = \{x_1, \cdots, x_d\}$ of non-commuting indeterminates.

Let $n \geq 1$ be an integer, C a commutative ring with 1 and $T : X \to \{E_{uv} \mid 1 \leq u, v \leq n\}$ a set map where the E_{uv}'s are the standard matrix units over C. It is clear that T can be viewed as a substitution. we shall define a directed graph $\overline{\Gamma}_T$ induced from Γ by T. First consider the directed graph on the vertex set $V \times \{1, 2, \cdots, n\}$ with edge set $\overline{e}_1, \cdots, \overline{e}_d$ where $\sigma(\overline{e}_r) = (\sigma(e_r), u)$, $\tau(\overline{e}_r) = (\tau(e_r), v)$ and $x_r^T = E_{uv}$. Now we define $\overline{\Gamma}_S$ by restricting the vertex set to $\bigcup_{r=1}^{d} \{\sigma(\overline{e}_r), \tau(\overline{e}_r)\}$. We note that the graph so obtained need by no means be connected let alone Eulerian. If it is Eulerian however, by construction $\overline{\Gamma}_T$ has at most $\Sigma_{i=1}^{k}$ min $\{n, \gamma(i)\}$ vertices, where $\gamma(i) = \max\{\phi_+(i), \phi_-(i)\}$, *i.e.*, $\gamma(i) = \phi_+(i) = \phi_-(i)$ for all $i \in V \setminus \{p, q\}$ and $\gamma(p) = \phi_+(p)$, $\gamma(q) = \phi_-(q)$. Those elements of $\Pi(\Gamma)$ which do lift to an Eulerian path of $\overline{\Gamma}_T$ will be called admissible (with respect to T). It is clear that $\pi \in \Pi(\Gamma)$ is admissible if and only if $\overline{e}_{\pi(1)} \cdots \overline{e}_{\pi(d)}$ is an Eulerian path of $\bar{\Gamma}_T$. For the remainder of this section, we introduce Swan's theorem and our main results.

Swan [1]. Let Γ be an Eulerian graph with *d* edges and *k* vertices satisfying $d \geq 2k$. Then $\Pi(\Gamma)$ has the same number of odd and even permutations (with respect to the fixed order)

Theorem 1. Let Γ be an Eulerian graph with vertex set $V = \{1, 2, \cdots, k\}$ and *d* edges. Further let $n \geq 1$ be an integer such that

$$d \geq 2\left(\sum_{i=1}^{i} \min\{n, \gamma(i)\}\right).$$

Then $f_\Gamma(X) = 0$ is a polynomial identity on the ring $M_n(C)$ of $n \times n$ matrices over a commutative ring *C* with 1

Corollary 2. Let Γ be an Eulerian graph with *k* vertices and *d* edges. Further let $n \geq 1$ be an integer and assume that $d \geq 2kn$. Then $f_\Gamma(X) = 0$ is a polynomial identity on $M_n(C)$.

3. Proof of Theorem 1

Since $f_\Gamma(X)$ is multilinear, it suffices to show that $f_\Gamma(X^T) = 0$ for any substitution *T* of $n \times n$ matrix units over *C*. Fix such an *T* and put $x_r^T = E_{u(r)v(r)}$, $1 \leq r \leq d$. Then

$$f_\Gamma(X^T) = \sum_{\pi \in \Pi(\Gamma)} \operatorname{sgn}(\pi) E_{u(\pi(1))v(\pi(1))} \cdots E_{u(\pi(d))v(\pi(d))} \quad (*)$$

Now consider $\bar{\Gamma}_T$. Clearly, and summand in (*) vanishes unless, for the given $\pi \in \Pi(\Gamma)$,

$$v(\pi(r)) = u(\pi(r+1))$$

for all $1 \leq r \leq N-1$, *i.e.*, if π is admissible. If so, on multiplying the matrix units, we obtain

$$\operatorname{sgn}(\pi) E_{u(\pi(1))v(\pi(d))}.$$

It follows that

$$f_\Gamma(X^T) = \sum_{u,v} \left(\sum \operatorname{sgn}(\pi)\right) E_{uv},$$

where the inner sum is taken over all admissible permutations with $u(\pi(1)) = u$ and $v(\pi(d)) = v$. If no such admissible π exists, the inner sum is 0 by definition. We want to prove that this inner sum is 0 anyway. It is readily seen that for any choice of *u* and *b*, *a* sum and $\operatorname{sgn}(\pi)$ in the inner sum arises precisely of π lifts to an Eulerian path of $\bar{\Gamma}_T$ from (*p*, (*u*) to (*q*, *v*). Thus, on applying Swan's theorem to $\bar{\Gamma}_T$ with $\left|E(\bar{\Gamma}_T)\right| = d$ and $\left|V(\bar{\Gamma}_T)\right| \leq \sum_{i=1}^{k} \min\{n, \gamma(i)\}$, we find that the number of even and odd admissible permutations π with $u(\pi(1)) = u$ and $v(\pi(d)) = v$ coincide whence the inner sum is 0 for any choice of *u* and *v*. This completes the proof.

4. Applications

1) Let Γ be the Eulerian graph on one vertex with *d* loops. Then $\Pi(\Gamma) = \operatorname{Sym}(d)$ and

$$f_\Gamma(X) = \sum_{\pi \in \operatorname{Sym}(d)} \operatorname{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(d)}$$

the standard polynomial [2] in *d* indeterminates.

More generally, let Γ be the Eulerian graph on *k* vertices with distinguished points $p = q = 1$ and the number $\alpha(i, j)$ of edges from vertex *i* to *j*:

$$\alpha(i, j) = \begin{cases} m & \text{if } j = i+1 \text{ and } 1 \leq i \leq k-1 \\ m & \text{if } i = k \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now clearly $d = km$ and

$$\Pi(\Gamma) = \operatorname{sym}(m) \times \cdots \times \operatorname{sym}(m),$$

k times. On putting $\pi = \pi_1 \times \cdots \times \pi_k$ and labelling the indeterminates, corresponding to the edges from *i* to $i+1$ by $x_1^{(i)}, \cdots, x_m^{(i)}$, from the corollary 2 it follows that

$$f_\Gamma(X) = \sum_{\pi \in \Pi(\Gamma)} \operatorname{sgn}(\pi) \left(\prod_{r=1}^{m} x_{\pi_1(r)}^{(1)} \cdots x_{\pi_k(r)}^{(k)} \right) = 0$$

is *a* polynomial identity on $M_n(C)$ [3] if $km = d \geq 2kn$, *i.e.*, if $m \geq 2n$.

2) For $\pi \in \Pi(\Gamma)$ we define a sequence $g(1), g(2), \cdots, g(d+1)$ of staircase steps, and the staircase height $g(\pi) = \max\{g(1), g(2), \cdots, g(d+1)\}$. We will construct a substitution *T*, such that π lifts to the unique converging directed path of $\bar{\Gamma}_S$ (*i.e.*, $\Pi(\bar{\Gamma}_S) = \{\pi\}$). First define a function $\pi^*: \{1, 2, \cdots, d, d+1\} \to A$ by

$$\pi^*(1) = \sigma(e_{\pi(1)}); \ \pi^*(r) = \sigma(e_{\pi(r)}) = \tau(e_{\pi(r-1)}),$$

$$2 \leq r \leq d, \ \pi^*(d+1) = \tau(e_{\pi(d)}).$$

Next we define by recursion the sequence of pair $(g(r), w_r)$, $1 \leq r \leq d+1$, where $g(r)$ is a natural number and w_r is a subset of $\{1, 2, \cdots, d, d+1\}$. We put $g(1) = 1$ and $w_1 = \phi$. Having $(g(1), w_1), \cdots, (g(r), w_r)$ in hand $(1 \leq r \leq d)$. There are three cases to consider:

a) $\pi^*(r+1) \neq \pi^*(t), \forall 1 \leq t \leq r$,

b) $\pi^*(r+1) = \pi^*(t), \pi^*(r+1) \neq \pi^*(s), \ \forall t+1 \leq s \leq r$, $t \in w_r$,

c) $\pi^*(r+1) = \pi^*(t), \pi^*(r+1) \neq \pi^*(s), \ \forall t+1 \leq s \leq r$, $t \notin w_r$.

We now put

$$g(r+1)=\begin{cases}1 & \text{in case (1)}\\ g(t)+1 & \text{in case (2)}\\ g(t) & \text{in case (3),}\end{cases}$$

and

$$w_{r+1}=\begin{cases}w_r & \text{in case (1) and (2)}\\ w_r\cup\{t,t+1,\cdots,r,r+1\} & \text{in case (3).}\end{cases}$$

Let $n\ge g(r)$ for all $1\le r\le d+1$, it is clear that $x_r^T=E_{g(r)g(r+1)}(1\le r\le d)$ gives a substitution of $n\times n$ matrix units over C. Now

$$\left(\pi^*(1),g(1)\right)\xrightarrow{\bar{e}_{\pi(1)}}\left(\pi^*(2),g(2)\right)\to\cdots\to$$
$$\left(\pi^*(d),g(d)\right)\xrightarrow{\bar{e}_{\pi(d)}}\left(\pi^*(d+1),g(d+1)\right)$$

is the unique covering directed path of $\bar{\Gamma}_S$ from $\left(\pi^*(1),g(1)\right)$ to $\left(\pi^*(d+1),g(d+1)\right)$ [4-7]. Since the $\left(g(1),g(d+1)\right)$ entry of the $n\times n$ matrix $f_\Gamma\left(X^T\right)$ is $\operatorname{sgn}(\pi)$, we have

Theorem 3. Let Γ be an Eulerian graph and $\pi\in\Pi(\Gamma)$. If $n\ge g(\pi)$, then $f_\Gamma(X)=0$ is not a polynomial identity on the ring $M_n(C)$ of $n\times n$ matrices over a commutative ring C with 1.

Remark. It is an obvious consequence of the above theorem that if $n\ge\min\{g(\pi)|\pi\in\Pi(\Gamma)\}$ is not the least integer $n\ge1$ for which $f_\Gamma(X)=0$ is not a polynomial identity on $M_n(C)$.

We note that, in general $\min\{g(\pi)|\pi\in\Pi(\Gamma)\}$ is not the least integer $n\ge1$ for which $f_\Gamma(X)=0$ is not a polynomial identity on $M_n(C)$.

Let Γ be the Eulerian graph on one vertex d loops. It is easily see that

$$g(1)=g(2)=1,\ g(3)=g(4)=2,\cdots,$$
$$g(2s+1)=g(2s+2)=s+1,\cdots.$$

Thus $g(\pi)=[d/2]+1$ for all $\pi\in sym(d)$ and the minimality assertion of the Amitsur-Levitzki theorem follows; the main part is an immediate consequence of the corollary.

Let Γ be the Eulerian graph on k vertices with distinguished points $p=q=1$ and the number $\alpha(i,j)$ of edges from vertex i to j:

$$\alpha(i,j)=\begin{cases}m & \text{if } j=i+1 \text{ and } 1\le i\le k-1\\ m & \text{if } i=k \text{ and } j=1\\ 0 & \text{otherwise}\end{cases}$$

Analogously, for any $\pi\in\Pi(\Gamma)$ we have

$$g(1)=g(2)=\cdots=g(k+1)=1,$$
$$g(k+2)=g(k+3)=\cdots=g(2k+2)=2,$$
$$\cdots g\left(s(k+1)+1\right)=g\left(s(k+1)+2\right)=$$
$$\cdots=g\left((s+1)(k+1)\right)=s+1.$$

In consequence $g(\pi)=m-\left[(m-1)/(k+1)\right]$ for all $\pi\in\Pi(\Gamma)$.

For $k=2$ we get the double Capelli polynomial; it is known, however, that in this case $m-\left[(m-1)/3\right]$ is not the smallest n for which $f_\Gamma(X)=0$ is not a polynomial identity on $M_n(C)$.

When $k=3$ we use x, y and z instead of the symbols $x^{(1)}$, $x^{(2)}$, and $x^{(3)}$ respectively to denote the indeterminates of the triple Capelli polynomial and continue to write m for the number of edges from vertex i to $i+1$. Thus the triple Capelli polynomial is

$$C_M(X,Y,Z)$$
$$=\sum_{\pi\in\Pi(\Gamma)}\operatorname{sgn}(\pi)x_{\pi_1(1)}y_{\pi_2(1)}z_{\pi_3(1)}\cdots x_{\pi_1(m)}y_{\pi_2(m)}z_{\pi_3(m)}.$$

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