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The Best *m*-Term One-Sided Approximation of Besov Classes by the Trigonometric Polynomials*

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ABSTRACT

In this paper, we continue studying the so called best *m*-term one-sided approximation and Greedy-liked one-sided approximation by the trigonometric polynomials. The asymptotic estimations of the best *m*-terms one-sided approximation by the trigonometric polynomials on some classes of Besov spaces in the metric $L_p\left(T^d\right)\left(1 \le p \le \infty\right)$ are given.

Keywords: Besov Classes; *m*-Term Approximation; One-Sided Approximation; Trigonometric Polynomial; Greedy Algorithm

1. Introduction

In [1,2], R. A. Devore and V. N. Temlyakov gave the asymptotic estimations of the best *m*-term approximation and the *m*-term Greedy approximation in the Besov spaces, respectively. In [3,4], by combining Ganelius' ideas on the one-sided approximation [5] and Schmidt's ideas on *m*-term approximation [6], we introduced two new concepts of the best *m*-term one-sided approximation (Definition 2.2) and the *m*-term Greedy-liked one-sided approximation (Definition 2.3) and studied the problems on classes of some periodic functions defined by some multipliers. We know that the best *m*-term approximation has many applications in adaptive *PDE* solvers, compression of images and signal, statistical classification, and so on, and the one-sided approximation has wide applications in conformal algorithm and operational research, etc. Hence, we are interested in the problems of the best *m*-term one-sided approximation and corresponding *m*-term Greedy-liked one-sided approximation. As a continuity of works in [3,4], we will study the same kinds of problems on some Besov classes in the paper.

There are a lot of papers on the best *m* term approximation problem and the best onee-sided approximation problem, we may see the papers [7-10] on the best *m* term approximation problem and see [11,12] on the best one-sided approximation problem.

Let $T^d := [0,2\pi)^d \left(T^1 = [0,2\pi)\right)$ be the *d* dimensional torus. For any two elements $x = \left(x_1, x_2, \cdots, x_d\right)$, $y = \left(y_1, y_2, \cdots, y_d\right) \in R^d$, set $e_k(x) := e^{ikx}$, $k = \left(k_1, k_2, \cdots, k_d\right) \in Z^d$, where *xy* denotes the inner product of *x* and *y*, *i.e.*, $xy = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$.

Denote by $L_p\left(T^d\right)\left(1 \le p \le \infty\right)$ the space of all 2π-periodic and measurable functions *f* on R^d for which the following quantity

$$\|f\|_p := \left(\int_{T^d} \left|f(x)\right|^p \mathrm{d}x\right)^{1/p}, 1 \le p < \infty,$$

$$\|f\|_\infty = ess\sup_{x \in T^d} \left|f(x)\right|, p = \infty,$$

is finite. $L_p\left(T^d\right)$ is a Banach space with the norm $\|\cdot\|_p$.

For any $f \in L_p\left(T^d\right)$, we denote by

$$\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{T^d} f(x) e_k(x) \mathrm{d}x, \left(k \in Z^d\right),$$

the Fourier coefficients of *f* (see [13]).

For any positive integer *m*, set $n = n(m) =: \left[m^{1/d}\right]$. For any $f \in L_1\left(T^d\right)$, as Popov in [11,12], by using the multivariate Fejér kernels,

$$\Phi_n(x) := (\pi/2)^{2d} \prod_{i=1}^{d} \left(\frac{\sin n x_i/2}{n \sin x_i/2}\right)^2,$$

$$x = \left(x_1, x_2, \cdots, x_d\right) \in T^d,$$

we defined

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$$T_m^+(f,x) := T_m(f,x) + \sum_{|l|=0}^{n-1} \Phi_n(x - 2\pi l/n) \sup_{|y-2\pi l/n| \le \pi/n} |f(y) - T_m(f,y)|, \quad (1)$$

and called it to be the best m-term one-sided trigonometric approximation operators, where and in the sequel the operator $T_m(f,x)$ is the best m-term trigonometric approximation operators and $\sum_{|l|=0}^{n-1}$ denotes $\sum_{l_1=0}^{n-1}\sum_{l_2=0}^{n-1}\cdots\sum_{l_d=0}^{n-1}$. It is easy to see that $f(x) \le T_m^+(f,x)$.

Meantime, for any $f \in L_1(T^d)$, we also defined

$$g_m^+(f,x) := g_m(f,x) + \sum_{|l|=0}^{n-1} \Phi_n(x - 2\pi l/n) \sup_{|y-2\pi l/n| \le \pi/n} |f(y) - g_m(f,y)|, \quad (2)$$

where $g_m(f,x) = \sum_{i=1}^m \hat{f}(k(i)) e_{k(i)}$ and $\{\hat{f}(k(i))\}_{i=1}^{\infty}$ is a sequence determined by the Fourier coefficients $\{\hat{f}(k)\}_{k \in Z^d}$ of f in the decreasing rearrangement, *i.e.*, $|f(k(1))| \ge |f(k(2))| \ge \cdots$.

It is easy to see that two operators T_m^+ and g_m^+ are non-linear. We will see that for any $x \in T^d$, $g_m^+(f,x) \ge f(x)$ (see Lemma 3.1 2)).

The main results of this paper are Theorems 2.5 and 2.6. In Theorem 2.5, by using the properties of the operator $T_m^+(f,x)$, we give the asymptotic estimations of the best m-term one-sided approximations of some Besov classes under the trigonometric function system. From this it can be seen easily that the approximation operator $T_m^+(f,x)$ is the ideal one. In Theorem 2.6, by using the properties of the approximation operator $g_m^+(f,x)$, the asymptotic estimations of the one-sided Greedy-liked algorithm of the best m-term one-sided approximation of Besov spaces under the trigonometric function system are given.

2. Preliminaries

For each positive integer m, denote by Σ_m the non-linear manifold consists of complex trigonometric polynomials T, where each trigonometric polynomial T can be written as a linear combination of at most m exponentials $e_k(x)$, $k \in Z^d$. Thus $T \in \Sigma_m$ if and only if there exits $\Lambda \subset Z^d$ such that $|\Lambda| \le m$ and

$$T(x) = \sum_{k \in \Lambda} c_k e_k(x),$$

where $|\Lambda|$ is the cardinality of the set Λ.

Let D be a finite or infinite denumerable set. Denote by $l_p(D)(1 \le p \le \infty)$ the space of all subset of some complex numbers $X = \{x_j\}_{j \in D}$ with the following finite l_p norm

$$\|X\|_{l_p(D)} := \left(\sum_{j \in D} |x_j|^p\right)^{1/p}, 1 \le p < \infty; \ \|X\|_{l_\infty} := \sup_{j \in D} |x_j|.$$

For any $f \in L_1(T^d)$, let $\{\hat{f}(k)\}_{k \in Z^d}$ be the set of Fourier coefficients of f. As in the page 19 of [14], denote by

$$\|f\|_{l_p} = \left\|\{\hat{f}(k)\}_{\mathbb{Z}^d}\right\|_{l_p(\mathbb{Z}^d)}$$

the l_p norm of the set of Fourier coefficients of f.

Throughout this paper, let $\mathcal{T}_n$ denote the set of the trigonometric polynomials of d variables and degree n with the form $T = \sum_{|k| \le n} \hat{T}(k) e_k$ and $\mathcal{A}_q(\mathcal{T}_n)$ denote the set of all trigonometric polynomials T in $\mathcal{T}_n$ such that

$$\|T\|_{\mathcal{A}_q(\mathcal{T}_n)} := \left\|\{\hat{T}(k)\}_{k \in Z^d}\right\|_{l_q(Z^d)} \le 1.$$

Here we take as $\hat{T}(k) = 0$ if $|k| > n$, $|k| := \max\{|k_1|, |k_2|, \cdots, |k_d|\}$.

Definition 2.1. (*see cf.* [1]) *For a given function f, we call*

$$\sigma_m(f)_p := \inf_{T \in \Sigma_m} \|f - T\|_p$$

the best m-term approximation error of f with trigonometric polynomials under the norm L_p. *For the function set* $A \subset L_p(T^d)$, *we call*

$$\sigma_m(A)_p := \sup_{f \in A} \sigma_m(f)_p$$

the best m-term approximation error of the function class A with trigonometric polynomials under the norm L_p.

Definition 2.2. (*see cf.* [3,4]) *For given function f, set* $\sum_m^+ := \{T \mid T \in \sum_{2m}, T \ge f\}$. *The quantity*

$$\sigma_m^+(f)_p := \inf_{T \in \Sigma_m^+} \|f - T\|_p$$

is called to be the best m-term one-sided approximation error of f with trigonometric polynomials under the norm L_p. *For given function set* $A \subset L_p(T^d)$, *the quantity*

$$\sigma_m^+(A)_p := \sup_{f \in A} \sigma_m^+(f)_p$$

is called to be the best m-term one-sided approximation error of the function class A with trigonometric polynomials under the norm L_p.

Definition 2.3. (*see cf.* [3,4]) *For given function f, we call* $g_m^+(f,x)$ (*given by relation* (2)) *the Greedy-liked algorithm of the best m-term one-sided approximation of f under trigonometric function system. For given function set* $A \subset L_p(T^d)$, *we call*

$$\alpha_m^+(A)_p := \sup_{f \in A} \|f - g_m^+(f,x)\|_p$$

the Greedy-liked one-sided approximation error of the best m-term one-sided approximation of function class A given by trigonometric polynomials with norm L_p.

As in [1,15], denote by $B_s^\alpha(L_q)$, $\alpha > 0$, $0 < q, s \le \infty$, the Besov space. The definition of the Besov space is given by using the following equivalent characterization. A function *f* is in the unit ball $U(B_s^\alpha(L_q))$ of the Besov space $B_s^\alpha(L_q)$, if and only if there exist trigonometric polynomials $R_j(x) := \sum_{|k|\le 2^j} c_{jk} e_k(x)$, such that $f(x) := \sum_{j=0}^\infty R_j(x)$ and

$$\left\| \left(2^{j\alpha} \left\| R_j \right\|_q \right)_{j=0}^\infty \right\|_{l_s(\mathbb{Z}_+)} \le 1. \tag{3}$$

Here $\mathbb{Z}_+ = \{0,1,2,\cdots\}$. In the case $1 < q < \infty$, we can take $R_j = f_j := \sum_{2^{j-1}\le|k|<2^j} \hat{f}(k) e_k$, $j \ge 1$, $f_0 := \hat{f}(0) e_0$, $k = (k_1, k_2, \cdots, k_d) \in Z^d$, $|k| = \max\{|k_1|, |k_2|, \cdots, |k_d|\}$.

We define the seminorm $|f|_{B_s^\alpha(L_q)}$ as the infimum over all decompositions (3) and denote by $U(B_s^\alpha(L_q))$ the unit ball with respect to this seminorm.

Throughout this paper, for any two given sequences of non-negative numbers $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ if there is a non-negative constant *c* independent of all *n*, such that $\alpha_n \le c\beta_n$, then we write $\alpha_n \ll \beta_n$. If both $\alpha_n \ll \beta_n$ and $\beta_n \ll \alpha_n$ hold, then we write $\alpha_n \asymp \beta_n$.

For any $1 \le p \le \infty$, $0 < q, s \le \infty$, set

$$\alpha(p,q) := \begin{cases} d\left(\dfrac{1}{q} - \dfrac{1}{p}\right)_+, & 0 < q \le p \le 2 \text{ and } 1 \le p \le q \le \infty, \\ \max\left\{\dfrac{d}{q}, \dfrac{d}{2}\right\}, & \text{otherwise}, \end{cases} \tag{4}$$

and

$$\beta(p,q) := \begin{cases} d + \alpha(p,q), & 0 < q \le p \le 2 \text{ and } 1 \le p \le q \le \infty, \\ \alpha(p,q), & \text{otherwise}. \end{cases} \tag{5}$$

For the unit ball $U(B_s^\alpha(L_q))$ of the Besov spaces $B_s^\alpha(L_q)$, Devore and Temlyakov in [1] gave the following result:

Theorem 2.4. (*c.f.* [1]) *For any* $1 \le p \le \infty$, $0 < q$, $s \le \infty$, *let* $\alpha(p,q)$ *be defined as in* (4). *Then, for* $\alpha > \alpha(p,q)$ *the estimate*

$$\sigma_m\left(U\left(B_s^\alpha(L_q)\right)\right)_p \asymp m^{-\alpha/d + \left(\frac{1}{q} - \max\left\{\frac{1}{p}, \frac{1}{2}\right\}\right)_+}$$

is valid.

In this paper, we give the following results about the best *m*-term one-sided approximation and corresponding Greedy-liked one-sided algorithm of some Besov classes by taking the *m*-term trigonometric polynomials as the approximation tools. Our results is the following theorems.

Theorem 2.5. *For any* $1 \le p \le \infty$, $0 < q$, $s \le \infty$, *let* $\beta(p,q)$ *be defined as in* (5). *Then, for* $\alpha > \beta(p,q)$, *we have*

$$\sigma_m^+\left(U\left(B_s^\alpha(L_q)\right)\right)_p \asymp m^{-\alpha/d + \left(\frac{1}{q} - \max\left\{\frac{1}{p}, \frac{1}{2}\right\}\right)_+}.$$

Theorem 2.6. *For* $1 \le p \le \infty$, $1 \le q \le \infty$, $0 < s \le \infty$ *and for* $\alpha > \beta(p,q)$, *we have*

$$m^{-\alpha/d + (1/q - 1/p)_+} \ll \alpha_m^+\left(U\left(B_s^\alpha(L_q)\right)\right)_p \ll m^{-\alpha/d + (1/q - 1/2)_+},$$

when $1 \le p \le 2$, and

$$m^{-\alpha/d + (1/q - 1/2)_+} \ll \alpha_m^+\left(U\left(B_s^\alpha(L_q)\right)\right) \ll m^{-\alpha/d + \max\{1/q, 1/2\}},$$

when $2 \le p \le \infty$.

3. The Proofs of the Main Results

In order to prove Theorem 2.5 and Theorem 2.6, we need following lemmas for $\Phi_n(x)$.

Lemma 3.1. *For the d variable trigonometric polynomial* $\Phi_n(x)$ *of degree n above, we have*

1) If $x \in T^d$ then $\Phi_n(x) \ge 0$;

2) If $|x| \le \pi/n$ then $\Phi_n(x) \ge 1$;

3) $\sum_{|l|=0}^{n-1} \Phi_n(x - 2\pi l/n) \le C_1$, where C_1 is a constant independent of *n*;

4) $\int_{T^d} \Phi_n(x)\mathrm{d}x \le C_2/n^d$, where C_2 is a constant independent of *n*.

Proof. We only prove 4).

If $0 \le t \le \pi$, then from $t/\pi \le \sin t/2 \le t/2$, we have

$$\int_{T^d} \Phi_n(x)\mathrm{d}x = (\pi/2)^{2d} \prod_{i=1}^d \int_0^{2\pi} \left(\frac{\sin nx_i/2}{n \sin x_i/2}\right)^2 \mathrm{d}x_i \asymp (1/n)^{2d} \prod_{i=1}^d \int_0^{\pi} \left(\frac{\sin nx_i/2}{x_i}\right)^2 \mathrm{d}x_i \asymp n^{-2d} \prod_{i=1}^d n \int_0^{n\pi/2} \left(\frac{\sin y_i}{y_i}\right)^2 \mathrm{d}y_i \asymp n^{-d}$$

4) follows from above equalities.

Similarly, we have

Lemma 3.2. *For* $1 \le p \le \infty$, $a_j \ge 0$, $l \in Z^d$ *there is positive constant C independent of n, such that*

$$\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)a_l\right\|_p \le C\left\{\left(2\pi/n\right)^d\sum_{|l|=0}^{n-1}a_l^p\right\}^{1/p}. \quad (6)$$

Proof. For the integral properties of $\Phi_n(x)$ mainly determined by the properties of free variables in the neighborhood of zero, we have

$$\begin{aligned}\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)a_l\right\|_p &= \left\{\left(\frac{1}{2\pi}\right)^d\int_{T^d}\left(\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)a_l\right)^p \mathrm{d}x\right\}^{1/p}\\ &= \left\{\left(\frac{1}{2\pi}\right)^d\int_{T^d}\left(\sum_{|l|=0}^{n-1}\left(\pi/2\right)^{2d}a_l\prod_{i=1}^{d}\frac{\sin^2\left(n\left(x_i-2\pi l_i/n\right)/2\right)}{\left(n\sin\left(x_i-2\pi l_i/n\right)/2\right)^2}\right)^p \mathrm{d}x\right\}^{1/p}\\ &\ll \left\{\sum_{|l|=0}^{n-1}a_l^p\prod_{i=1}^{d}\int_0^{2\pi}\left(\frac{\sin^2\left(n\left(x_i-2\pi l_i/n\right)/2\right)}{\left(n\left(x_i-2\pi l_i/n\right)/2\right)^2}\right)^p \mathrm{d}x_i\right\}^{1/p}\\ &\ll \left\{\sum_{|l|=0}^{n-1}a_l^p\prod_{i=1}^{d}n^{-1}\int_{-\pi l_i}^{n\pi-\pi l_i}\left(\frac{\sin^2 y_i}{y_i^2}\right)^p \mathrm{d}y_i\right\}^{1/p} \ll \left\{\left(2\pi/n\right)^d\sum_{|l|=0}^{n-1}a_l^p\right\}^{1/p}.\end{aligned}$$

The proof of Lemma 3.2 is finished.

Proof of Theorem 2.5. First, we consider the upper estimation. For a given function $f\in L_p\left(T^d\right)$, $T_m\in\sum_m$ set

$$T_m^+\left(f,x\right) := T_m+\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)\sup_{|y-2\pi l/n|\le\pi/n}\left|f\left(y\right)-T_m\left(y\right)\right|.$$

By Lemma 3.1 2) and Remark 1.1, we have $f\left(x\right)\le T_m^+\left(f,x\right)$ and $T_m^+\left(f,x\right)$ is a linear combination of at most $2m$ exponentials $e_k\left(x\right)$, $k\in Z^d$.

When $p=\infty$, $q=2$, $s=\infty$, by Definition 2.2, we have

$$\begin{aligned}\sigma_{2^{md}}^+\left(U\left(B_\infty^\alpha\left(L_2\right)\right)\right)_\infty &\le \sup_{f\in U\left(B_\infty^\alpha\left(L_2\right)\right)}\left\{\inf_{T\in\Sigma_{2^{md}}}\left(\left\|f-T\right\|_\infty+\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)\sup_{|y-2\pi l/n|\le\pi/n}\left|f-T\right|\right\|_\infty\right)\right\}\\ &\le \sigma_{2^{md}}\left(U\left(B_\infty^\alpha\left(L_2\right)\right)\right)_\infty+\sup_{f\in U\left(B_\infty^\alpha\left(L_2\right)\right)}\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)\sup_{|y-2\pi l/n|\le\pi/n}\left|f-T_{2^{md}}\left(f\right)\right|\right\|_\infty =: S_1+S_2,\end{aligned} \quad (7)$$

where we have written $n=2^m$ in (7).

By the conditions of Theorem 2.5, for any given natural number m, we have $\alpha>\alpha\left(p,q\right)=d/2$. Notice that $1/q-\max\left\{1/p,1/2\right\}=0$ in Theorem 2.4. Thus,

$$S_1 =: \sigma_{2^{md}}\left(U\left(B_\infty^\alpha\left(L_2\right)\right)\right)_\infty \ll 2^{-m\alpha}. \quad (8)$$

For any $f\in U\left(B_s^\alpha\left(L_q\right)\right)$, by Lemma 3.2, under the condition of Theorem 2.5, we have

$$\begin{aligned}S_2 &:= \sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)}\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)\sup_{|y-2\pi l/n|\le\pi/n}\left|f\left(y\right)-T_{2^{md}}\left(f,y\right)\right|\right\|_\infty = \sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)}\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x-2\pi l/n\right)a_l\right\|_\infty\\ &\ll \left(2\pi l/n\right)^d n^d\left\|f\left(y\right)-T_{2^{md}}\left(f,y\right)\right\|_\infty \ll \left\|f-T_{2^{md}}\right\|_\infty \ll 2^{-m\alpha},\end{aligned} \quad (9)$$

where $a_l := \sup_{|y-2\pi l/n|\le\pi/n}\left|f\left(y\right)-T_{2^{md}}\left(f,y\right)\right|$.

By the monotonicity of σ_m^+ and (8), (9), we have

$$\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_\infty \ll m^{-\alpha/d}. \quad (10)$$

When $p=q$, $s=\infty$, for any $f\in U\left(B_s^\alpha\left(L_q\right)\right)$, then, by the definition of Besov classes, there exists a sequence $\left\{R_j\left(x\right)\right\}_{j=0}^\infty$ of the trigonometric polynomials of coordinate degree 2^j such that $f\left(x\right):=\sum_{j=0}^\infty R_j\left(x\right)$, and $\left\|\left(2^{j\alpha}\left\|R_j\right\|_p\right)_{j=1}^\infty\right\|_{l_\infty}\le 1$.

In particular, take $R_0\left(x\right)=T_1\left(f,x\right)$, $R_j\left(x\right)=T_{2^j}\left(f,x\right)-T_{2^{j-1}}\left(f,x\right)$, $j=1,2,3,\cdots$.

Here the operator $T_m(f,x)$ are the best m-term trigonometric approximation operators in (1). From the relation between linear approximation and non-linear approximation and Lemma 3.2, we have

$$\begin{aligned}\sigma_{2^{md}}^{+}\left(U\left(B_\infty^\alpha\left(L_p\right)\right)\right)_p &= \sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)}\left(\inf_{g\in\Sigma_{2^{md}}^{+}}\|f-g\|_p\right)\le E_{2^{md}}\left(U\left(B_\infty^\alpha\left(L_p\right)\right)\right)_p\\ &+\sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)}\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(\cdot-2\pi l/n\right)\sup_{|y-2\pi l/n|\le\pi/n}\left|\sum_{j=m+1}^{\infty}R_j\left(y\right)\right|\right\|_p\\ &\ll \sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)}\sum_{j=m+1}^{\infty}2^{-j\alpha}\left\|2^{j\alpha}R_j\right\|_p+\sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)}\left\|\sum_{|l|=0}^{n-1}\Phi_n\left(\cdot-2\pi l/n\right)a_l\right\|_p\\ &\ll \sum_{j=m+1}^{\infty}2^{-j\alpha}+\left\{\left(2\pi/n\right)^d\sum_{|l|=0}^{n-1}a_l^p\right\}^{1/p}:=S_1'+S_2'.\end{aligned}\tag{11}$$

Here $a_l=\sup_{|y-2\pi l/n|\le\pi/n}\left|\sum_{j=m+1}^{\infty}R_j\left(y\right)\right|$. Under the condition of Theorem 2.5, it is easy to see that

$$S_1'=\sum_{j=m+1}^{\infty}2^{-j\alpha}\ll 2^{-m\alpha}.\tag{12}$$

Next, we will estimate S_2'. Set $h(y)=\sum_{j=m+1}^{\infty}R_j\left(y\right)$, and

$$\tau_1\left(h,1/n\right)_p=\left\{\int_{T^d}\left(\sup_{y\in U\left(x,2\pi/n\right)}\left|h\left(y\right)-h\left(x\right)\right|\right)^p\mathrm{d}x\right\}^{1/p}.$$

Since the measure of the neighborhood $U\left(2\pi j/n,\pi/n\right)=\prod_{i=1}^{d}\left[\frac{2j_i\pi}{n}-\frac{\pi}{n},\frac{2j_i\pi}{n}+\frac{\pi}{n}\right]$ is $\left(2\pi/n\right)^d$, so, by the definition of Besov classes and Minkowskii inequality, we have

$$\begin{aligned}S_2'(f)&:=\left(\sum_{|l|=0}^{n-1}\int_{U\left(2\pi l/n,\pi/n\right)}a_l^p\mathrm{d}x\right)^{1/p}:=\left(\sum_{|l|=0}^{n-1}\int_{U\left(2\pi l/n,\pi/n\right)}\sup_{y\in U\left(2\pi l/n,\pi/n\right)}\left|h\left(y\right)\right|^p\mathrm{d}x\right)^{1/p}\\ &\le\left(\sum_{|l|=0}^{n-1}\int_{U\left(2\pi l/n,\pi/n\right)}\sup_{y\in U\left(2\pi l/n,\pi/n\right)}\left(\left|h\left(y\right)-h\left(x\right)\right|+\left|h\left(x\right)\right|\right)^p\mathrm{d}x\right)^{1/p}\\ &\le\left\{\sum_{|l|=0}^{n-1}\int_{U\left(2\pi l/n,\pi/n\right)}\left(\sup_{y\in U\left(2\pi l/n,\pi/n\right)}\left|h\left(y\right)-h\left(x\right)\right|\right)^p\mathrm{d}x\right\}^{1/p}+\left\{\sum_{|l|=0}^{n-1}\int_{U\left(2\pi l/n,\pi/n\right)}\left|h\left(x\right)\right|^p\mathrm{d}x\right\}^{1/p}\\ &\ll\left\{\int_{T^d}\left(\sup_{y\in U\left(x,2\pi/n\right)}\left|h\left(y\right)-h\left(x\right)\right|\right)^p\mathrm{d}x\right\}^{1/p}+\left\{\int_{T^d}\left|h\left(x\right)\right|^p\mathrm{d}x\right\}^{1/p}=\tau_1\left(h,1/n\right)_p+\|h\|_p.\end{aligned}$$

For a fixed $\mu=\left(\mu_1,\mu_2,\cdots\mu_d\right)\in\mathbb{Z}_+^d$, set $D^\mu=\frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}\partial x_2^{\mu_2}\cdots\partial x_d^{\mu_d}}$. By the mathematical induction on d, it is easy to see that

$$\tau_1\left(R_s,1/n\right)_p\ll\sum_{|\mu|=1}^{d}n^{-|\mu|}\left\|D^\mu R_s\right\|_p.$$

Notice that $n=2^m$. From the properties of smooth modulus [12] and Bernstein inequality, we have

$$\begin{aligned}\tau_1\left(h,1/n\right)_p&=\tau_1\left(\sum_{s=m+1}^{\infty}R_s,1/n\right)_p\ll\sum_{s=m+1}^{\infty}\sum_{|\mu|=1}^{d}n^{-|\mu|}\left\|D^\mu R_s\right\|_p\ll\sum_{s=m+1}^{\infty}\sum_{|\mu|=1}^{d}n^{-|\mu|}2^{s|\mu|}\left\|R_s\right\|_p\\ &\le\sum_{s=m+1}^{\infty}\sum_{|\mu|=1}^{d}n^{-|\mu|}2^{s|\mu|}2^{-s\alpha}\left(2^{s\alpha}\left\|R_s\right\|_p\right)\ll 2^{-m\alpha}\sum_{j=1}^{\infty}2^{-j(\alpha-d)}.\end{aligned}$$

By the conditions of Theorem 2.5, and $\alpha>d$, we have

$$\tau_1\left(h,1/n\right)_p\ll 2^{-m\alpha}$$

From (11), (12) and (3), we have

$$\|h\|_p\ll 2^{-m\alpha}.$$

So

$$S_2' := \sup_{f\in U\left(B_\infty^\alpha\left(L_q\right)\right)} S_2'\left(f\right) \ll \tau_1\left(h,1/n\right)_p + \|h\|_p \ll 2^{-m\alpha}. \quad (13)$$

For sufficiently large m, by (12), (13) and the monotonicity of σ_m^+, we have

$$\sigma_m^+\left(U\left(B_\infty^\alpha\left(L_q\right)\right)\right)_p \ll m^{-\alpha/d}. \quad (14)$$

The upper estimations for the other cases can be obtained by the embedding Theorem. In detail, we may show them in the following.

If $2 \le q \le p \le \infty$, then $\|\cdot\|_p \le \|\cdot\|_\infty$. So for any $f \in U\left(B_\infty^\alpha\left(L_q\right)\right)$, by $\left\|\left(2^{j\alpha}\|f_j\|_q\right)_{j=0}^\infty\right\|_{l_s} \le 1$, we have $2^{j\alpha}\|f_j\|_q \le 1$, for all $j \in \mathbb{N}$. Thus, $2^{j\alpha}\|f_j\|_2 \le 2^{j\alpha}\|f_j\|_q \le 1$. Hence we have $\left\|\left(2^{j\alpha}\|f_j\|_2\right)_{j=0}^\infty\right\|_{l_\infty} \le 1$, *i.e.*, $f \in U\left(B_\infty^\alpha\left(L_2\right)\right)$. So, we have following embedding relation

$$U\left(B_\infty^\alpha\left(L_q\right)\right) \subset U\left(B_\infty^\alpha\left(L_2\right)\right).$$

By (10), we have

$$\sigma_m^+\left(U\left(B_\infty^\alpha\left(L_q\right)\right)\right)_p \le \sigma_m^+\left(U\left(B_\infty^\alpha\left(L_2\right)\right)\right)_\infty \ll m^{-\alpha/d}.$$

If $0 < q \le 2 \le p \le \infty$, then for any j and $f \in U\left(B_\infty^\alpha\left(L_q\right)\right)$, by (3), we have $2^{j\alpha}\|f_j\|_q \le 1$ (if q takes different values, replacing f_j by T_j, does not influence the proof). So by Nikol'skii inequality (see [1], p. 102) for the inequality), we have

$$2^{j\alpha}2^{-jd(1/q-1/2)}\|f_j\|_2 \le 2^{j\alpha}\|f_j\|_q \le 1.$$

Hence $f \in U\left(B_s^{\alpha-d(1/q-1/2)}\left(L_q\right)\right)$ and we have following embedding formula

$$U\left(B_s^\alpha\left(L_q\right)\right) \subset U\left(B_s^{\alpha-d(1/q-1/2)}\left(L_2\right)\right).$$

By (10) we can get

$$\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \le \sigma_m^+\left(U\left(B_s^{\alpha-d(1/q-1/2)}\left(L_2\right)\right)\right)_\infty \ll m^{-\alpha/d+(1/q-1/2)}.$$

If $0 < q \le p \le 2$, then for any j and $f \in U\left(B_s^\alpha\left(L_q\right)\right)$, we have $\left\|\left(2^{j\alpha}\|f_j\|_q\right)_{j=0}^\infty\right\|_{l_s} \le 1$. By Nikol'skii inequality we have

$$\left\|\left(2^{j\alpha}2^{-jd(1/q-1/p)}\|f_j\|_p\right)_{n=0}^\infty\right\|_{l_s} \le \left\|\left(2^{j\alpha}\|f_j\|_q\right)_{j=0}^\infty\right\|_{l_s}.$$

Thus we have following embedding formula

$$U\left(B_s^\alpha\left(L_q\right)\right) \subset U\left(B_s^{\alpha-d(1/q-1/p)_+}\left(L_p\right)\right).$$

By (14), we have

$$\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \le \sigma_m^+\left(U\left(B_s^{\alpha-d(1/q-1/p)_+}\left(L_p\right)\right)\right)_p \ll m^{-\alpha/d+(1/q-1/p)_+}.$$

If $1 \le p \le q \le \infty$, then, for any $f \in U\left(B_s^\alpha\left(L_q\right)\right)$, by $\left\|\left(2^{j\alpha}\|f_j\|_q\right)_{j=0}^\infty\right\|_{l_s} \le 1$, we have $2^{j\alpha}\|f_n\|_q \le 1$, for any $j \in N$. So there hold $2^{j\alpha}\|f_j\|_p \le 2^{j\alpha}\|f_j\|_q \le 1$ and $\left\|\left(2^{j\alpha}\|f_j\|_p\right)_{j=0}^\infty\right\|_{l_\infty} \le 1$. Therefore, we have

$$U\left(B_s^\alpha\left(L_q\right)\right) \subset U\left(B_s^\alpha\left(L_p\right)\right).$$

By (14) we have

$$\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \ll \sigma_m^+\left(U\left(B_s^\alpha\left(L_p\right)\right)\right)_p \ll m^{-\alpha/d}.$$

The upper estimation is finished.

By the definition of σ_m^+ and σ_m, the lower estimation can be gotten from Theorem 2.4, and the following relation

$$\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \ge \sigma_{2m}\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p.$$

Proof of Theorem 2.5 is finished.

Proof of Theorem 2.6. First, we consider the case $1 \le p \le 2 \le q \le \infty$. By Definition 2.2 and 2.3, we have

$$\begin{aligned}\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p &\le \alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \\ &\le \alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_2 \\ &= \sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_2.\end{aligned} \quad (15)$$

By Theorem 2.5, we have

$$\alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \asymp m^{-\alpha/d+(1/q-1/p)_+}.$$

When $1 \le p \le 2$, for $1 \le q \le 2$, by Theorem 2.5, the upper estimation is

$$\sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_2 \ll m^{-\alpha/d+(1/q-1/2)_+}.$$

From (15) we can get

$$\alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \le \sigma_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_2 \ll m^{-\alpha/d+(1/q-1/2)_+}. \tag{16}$$

When $2 \le p \le \infty$, $1 \le q \le 2$, by the relation between best m-term approximation and Greedy algorithm [7], we have

$$\left\|f-g_m(f)\right\|_p \ll m^{|1/2-1/p|}\sigma_m(f)_p. \tag{17}$$

For $f \in U\left(B_s^\alpha\left(L_q\right)\right)$, and any l, by (16), (17) and Theorem 2.4, we have

$$\begin{aligned} a_l &:= \sup_{|y-2\pi l/n|\le\pi/n}\left|f(y)-g_m(f,y)\right| \\ &\le \left\|f-g_m\right\|_\infty \ll m^{1/2}m^{-\alpha/d}m^{1/q-1/2}. \end{aligned} \tag{18}$$

From Lemma 3.2 and relation (17), (18), we have

$$\begin{aligned} \alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p &\le \sup_{f\in U\left(B_s^\alpha\left(L_q\right)\right)}\left(\left\|f-g_m\right\|_p + \left\|\sum_{|l|=0}^{n-1}\Phi_n\left(x\,y-2\pi l/n\right)\sup_{|y-2\pi l/n|\le\pi/n}\left|f-g_m\right|\right\|_p\right) \\ &\ll m^{-\alpha/d}m^{1/q-1/p} + m^{1/2}m^{-\alpha/d}m^{1/q-1/2} \ll m^{-\alpha/d+1/q}. \end{aligned}$$

When $2 \le p \le \infty$, we consider the case $2 \le q \le \infty$. By the $\left\|f_j\right\|_2 \le \left\|f_j\right\|_q$, we have

$$U\left(B_s^\alpha\left(L_2\right)\right) \supset U\left(B_s^\alpha\left(L_q\right)\right). \tag{20}$$

By (19) and (20), we can get

$$\alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \le \alpha_m^+\left(U\left(B_s^\alpha\left(L_2\right)\right)\right)_p \ll m^{-\alpha/d+1/2}. \tag{21}$$

In the following we will give the lower estimation. By Definition 2.3, we have

$$\alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \ge \sigma_{2m}\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p.$$

And by Theorem 2.4, we have

$$\alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \ll m^{-\alpha/d+(1/q-1/p)_+}$$

when $1 \le p \le 2$, and

$$\alpha_m^+\left(U\left(B_s^\alpha\left(L_q\right)\right)\right)_p \gg m^{-\alpha/d+(1/q-1/2)_+}$$

when $2 \le p \le \infty$.

This finishes the proof of Theorem 2.6.

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