

Minimum Rank of Graphs Powers Family

Alimohammad Nazari¹, Marzieh Karimi Radpoor²

¹Department of Mathematics, University of Arak, Arak, Iran

²Institute Scientific and Research Bahar, Hamedan, Iran

Email: a-nazari@araku.ac.ir, karimiradpoor@yahoo.com

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ABSTRACT

In this paper we study the relationship between minimum rank of graph G and the minimum rank of graph G^j for some families of special graph G , where G^j is the j th power of graph G .

Keywords: Minimum Rank; Power of Graphs; Zero Forcing Set

1. Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices (usually $\{1, \dots, n\}$ or a subset thereof) and E is the set of edges (an edge is a two-element subset of vertices); what we call a graph is sometimes called a simple undirected graph. In this paper each graph is finite and has nonempty vertex set. The order of a graph G , denoted $|G|$, is the number of vertices of G . A path is a graph $P_n = (\{v_1, \dots, v_n\}, E)$ such that

$$E = \{\{v_i, v_{i+1}\} : i = 1, \dots, n-1\}.$$

A cycle is a graph $C_n = (\{v_1, \dots, v_n\}, E)$ such that

$$E = \{\{v_i, v_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{v_n, v_1\}\}.$$

The length of a path or cycle is the number of its edges. A complete graph is a graph $K_n = (\{v_1, \dots, v_n\}, E)$ such that

$$E = \{\{v_i, v_j\} : 1 \leq i < j \leq n\}.$$

A graph (V, E) is bipartite if the vertex set V can be partitioned into two nonempty subsets U and W , such that every edge of E has one endpoint in U and one in W . A complete bipartite graph is a bipartite graph $K_{p,q} = (U \cup W, E)$ such that $|U| = p$, $|W| = q$ and $E = \{\{u, w\} : u \in U, w \in W\}$.

The line graph of a graph $G = (V, E)$, denoted $L(G)$, is the graph having vertex set E , with two vertices in $L(G)$ adjacent if and only if the corresponding edges share an endpoint in G . Since we require a graph to have a nonempty set of vertices, the line graph $L(G)$ is defined only for a graph G that has at least one edge.

The corona of G with H , denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by taking one copy of G and $|G|$ copies of H , and joining all the vertices in the i th copy of H to the i th vertex of G . See **Figures 6, 7** for

a picture of $K_6 \circ K_1$. Note that $G \circ H$ and $H \circ G$ are usually not isomorphic (in fact, if $|G| \neq |H|$, then $|G \circ H| \neq |H \circ G|$).

Definition 1.1 The j th power of a graph G is a graph with the same set of vertices as G and an edge between two vertices if there is a path of length at most j between them.

Definition 1.2 For such a matrix, the graph of A , denoted $G(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $G(A)$. The set of symmetric matrices of graph G (over R) is defined to be

$$S(G) = \{A \in S_n(R) : G(A) = G\}$$

The minimum rank of a graph G (over R) is defined to be

$$mr(G) = \min\{\text{rank}(A) : A \in S(G)\}$$

For $A \in R^{n \times n}$ the corank of A is the nullity of A and the maximum nullity (or maximum corank) of a graph G (over R) is defined to be

$$M(G) = \max\{\text{corank}(A) : A \in S(G)\}$$

Clearly

$$mr(G) + M(G) = |G|.$$

More generally, the minimum rank of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose ij th entry (for $i \leq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise [1,2].

The solution to the minimum rank problem is equivalent to the determination of the maximum multiplicity of an eigenvalue among the same family of matrices [3].

2. Zero Forcing Sets and the Graph Parameter $Z(G)$

Here we introduce the graph parameter $Z(G)$ as the minimum size of a zero forcing set from [1]. The zero forcing number is a useful tool for determining the minimum rank of structured families of graphs and small graphs [4].

Definition 2.1 *Color-change rule:*

- If G is a graph with each vertex colored either white or black, u is a black vertex of G , and exactly one neighbor v of u is white, then change the color of v to black.
- Given a coloring of G , the derived coloring is the result of applying the color-change rule until no more changes are possible.
- A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.
- $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

For example, an endpoint of a path is a zero forcing set for the path. In a cycle, any set of two adjacent vertices is a zero forcing set.

Corollary 2.2 [1,5] *Let $G = (V, E)$ be a graph and let $Z \subseteq V$ be a zero forcing set. Then $M(G) \leq |Z|$, and thus $M(G) \leq Z(G)$.*

The Colin deVerdiere-type parameter ξ can be useful in computing minimum rank or maximum nullity (over the real numbers). A symmetric real matrix M is said to satisfy the Strong Arnold Hypothesis provided there does not exist a nonzero symmetric matrix X satisfying:

- $MX = 0$.
- $M \circ X = 0$.
- $I \circ X = 0$.

where \circ denotes the Hadamard (entrywise) product and I is the identity matrix. For a graph G , $\xi(G)$ is the maximum nullity among matrices $A \in S(G)$ that satisfy the Strong Arnold Hypothesis.

It follows that $\xi(G) \leq M(G)$.

A *contraction* of G is obtained by identifying two adjacent vertices of G , and suppressing any loops or multiple edges that arise in this process. A minor of G arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contractions of edges. A graph parameter ζ is minor monotone if for any minor G' of G , $\zeta(G') \leq \zeta(G)$. The parameter ξ was introduced in [6], where it was shown that ξ is minor monotone. It was also established that $\xi(K_n) = n-1$ and $\xi(K_{p,q}) = p+1$ (under the assumptions that $p \leq q$, $3 \leq q$) (see [5,7]).

Corollary 2.3 [2,6] *Let G be a graph.*

- 1) If K_p is a minor of G , then

$$M(G) \geq \xi(K_p) = p-1.$$

- 2) If $p \leq q$, $3 \leq q$ and $K_{p,q}$ is a minor of G , then

$$M(G) \geq \xi(K_{p,q}) = p+1.$$

Other possible bounds for minimum rank derived from certain easy to compute parameters of the graph were considered, leading to an investigation of the connection between minimum degree of a vertex, $\delta(G)$, and minimum rank [8].

Corollary 2.4 *For any graph G and infinite field F , $mr^F(G) \leq |G| - \delta(G)$.*

3. Main Results

In here we calculate the minimum rank of graph G^j . For this purpose we obtained Zero forcing set and the graph parameter $Z(G)$ and the parameter $\xi(G)$ and determined the upper bound and lower bound for maximum nullity $M(G)$. Since we have

$$\xi(G) \leq M(G) \leq Z(G).$$

Then we can achieve minimum rank of graphs (see **Table 1**).

Theorem 3.1 *For all $j \geq 1$ and for all graph such that $Z(G^j) = M(G^j)$, we have $mr(G^{j+1}) \leq mr(G^j)$.*

Proof. It is clear that

$$mr(G^j) = |G^j| - M(G^j) = |G| - Z(G^j)$$

and $Z(G) \leq Z(G^2) \leq \dots \leq Z(G^j)$,

then

$$|G| - Z(G^{j+1}) \leq |G| - Z(G^j) \Rightarrow mr(G^{j+1}) \leq mr(G^j).$$

Proposition 3.2 [1] *For each of the following families of graphs, $Z(G) = M(G)$:*

- 1) Any graph G such that $|G| \leq 6$. (The minimum ranks of all graphs of order at most 7 are available in the spreadsheet [9]).
- 2) K_n, C_n, P_n .
- 3) Any tree T .
- 4) Some families of special graphs.

By this Proposition we have following Theorem.

Theorem 3.3 *For each of the following families of graphs $Z(G^j) = M(G^j)$:*

- 1) Any graph G such that $|G| \leq 6$.
- 2) Complete graph and complete multipartite graph.
- 3) Petersen graph.
- 4) Wheel graph.
- 5) Clebsch Graph for $j > 1$.
- 6) Complement of cycle graph $\overline{C_n}$, where $n \geq 5$.

Proof. The proof is trivial.

Proposition 3.4 *If P_n be a path graph, $(P_n)^{n-1}$ is homomorphic to the complete graph K_n .*

Table 1. Summary of minimum rank results established in this paper.

G	$ G^j $	$Z(G^j)$	$mr(G^j)$	$j: G^j \sim$ Complete Graph	preamble
K_n	n	$n-1$	1	1	
K_{n_1, \dots, n_p}	$n_1 + \dots + n_p$	$n_1 + \dots + n_p - 1$	1	2	$j \neq 1$
W_n	n	$n-1$	1	2	$j \neq 1$
Petersen Graph	10	9	1	2	$j \neq 1$
$\overline{C_n}$	n	$n-1$	1	2	$j \neq 1$
Clebsch Graph	16	15	1	2	$j \neq 1$
C_n	n	$2j$	$n-2j$	$\lfloor n/2 \rfloor$	$j \leq n/2$
P_n	n	j	$n-j$	$n-1$	$n > j$
$K_n \circ K_1$	$2n$	n	n	3	$j = 2$
$C_n \circ K_1$	$2n$	$n+2(j-2)$	$n+2(j-2)$	$\lfloor n/2 \rfloor + 2$	$\lfloor n/2 \rfloor \leq j < \lfloor n/2 \rfloor + 2$ & $n > 5$
$L(G^j)$	n	-	$n-2$	-	$j \geq 3$ & $n \geq 2$

Theorem 3.5 For all $j < n$, we have $mr((P_n^j)^j) = n - j$.

Proof. (Figure 1) In graph P_n^j , we have $Z(P_n^j) = j = M(P_n^j)$, because if we start coloring from the end point vertex u , this vertex at least is adjacent with j vertices. The vertex u with its $j-1$ adjacent vertices are coloring. The other vertices also are coloring since they are adjacent to coloring vertices, and the number of coloring vertices is j , therefore we have $Z(P_n^j) = j$. From Corollary 2.2 we have $M(P_n^j) \leq Z(P_n^j) = j$.

On the other hand with $n-j-1$ contraction of the vertices of G^j , we reach to the complete graph K_{j+1} , and we know that K_{j+1} is a minor for P_n^j . Then we have

$$j = \xi(K_{j+1}) \leq \xi(P_n^j) \leq M(P_n^j).$$

Consequently

$$Z(P_n^j) = j = M(P_n^j),$$

thus,

$$mr(P_n^j) = n - j.$$

(see also [10]).

Proposition 3.6 $(C_n)_{\lfloor n/2 \rfloor}$ is homomorphic to K_n .

Theorem 3.7 $mr(C_n^j) = n - 2j$, for all $j \leq \lfloor n/2 \rfloor$.

Proof. (Figure 2) In C_n^j any vertex u is adjacent to $2j$ vertices, then

$$M(C_n^j) \leq Z(C_n^j) = 2j.$$

On the other hand, $\delta(C_n^j) = 2j$. Then

$$2j = \delta(C_n^j) \leq M(C_n^j),$$

and finally

$$mr(C_n^j) = n - 2j.$$

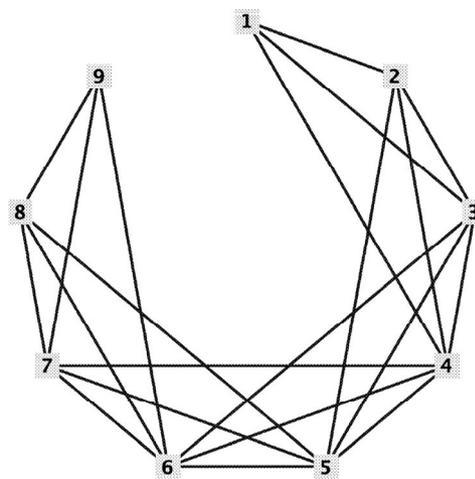


Figure 1. Graph P_9^3 .

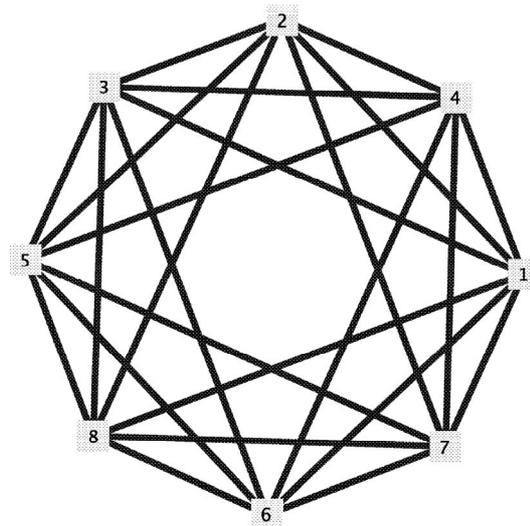


Figure 2. Graph C_8^3 .

Proposition 3.8 For all $j = \lfloor \frac{n}{2} \rfloor + 2$, we have

$$(C_n \circ K_1)^j = K_{2n}.$$

Theorem 3.9 For all $j \leq \lfloor n/2 \rfloor + 2$, we have

$$Z[(C_n \circ K_1)^j] = n + 2(j - 2).$$

Proof. (Figures 3, 4, 5) In the j th power of graph $C_n \circ K_1$ an external vertex u_1 is adjacent to $3 + 4(j - 2)$ vertices. If we start to coloring of external vertex, then $3 + 4(j - 2)$ vertices are coloring, which $2(j - 2)$ colored vertices are in external cycle and the remaining vertices are located in the inner cycle. For more coloring we use the nearest adjacent vertex to u_1 on the set of external vertices. We call this vertex by u_2 . $4(j - 2) + 1$ adjacent vertices to u_2 are same adjacent vertices to u_1 , and only two of them is different. One of these, which is located on the inner cycle has colored and the another vertex that is located on the external cycle colored from “color-change rule”. We continue the process until all vertices are colored on the internal cycle. Finally

$$Z[(C_n \circ K_1)^j] = n + 2(j - 2).$$

Theorem 3.10 For all $\lfloor n/2 \rfloor \leq j < \lfloor n/2 \rfloor + 2$, and $n \geq 5$ we have $mr(C_n \circ K_1)^j = n - 2(j - 2)$.

Proof. (Figures 3, 4, 5) when we make $\lfloor n/2 \rfloor$ power of $(C_n \circ K_1)$, then its internal cycle reach to complete graph. With contraction of $2j - 1$ vertices to $2j - 3$ vertices of this graph we reach to complete graph of order $n + 2(j - 2) + 1$, then K_{n+2j-1} is a minor of $(C_n \circ K_1)$. On the other hand we have

$$\begin{aligned} & n + 2(j - 2) \\ &= \xi(K_{n+2j-1}) \leq \xi((C_n \circ K_1)^j) \leq M((C_n \circ K_1)^j). \end{aligned}$$

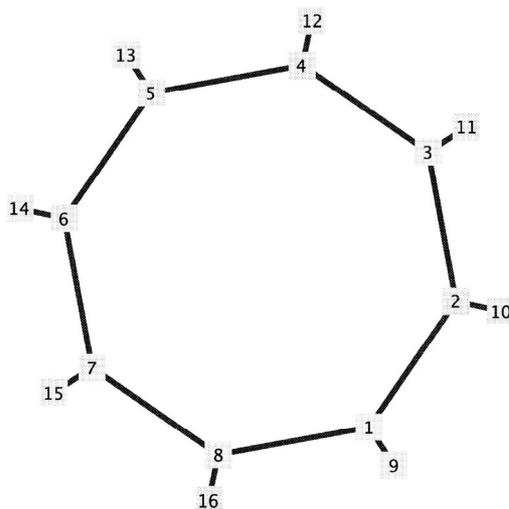


Figure 3. $C_8 \circ K_1$.

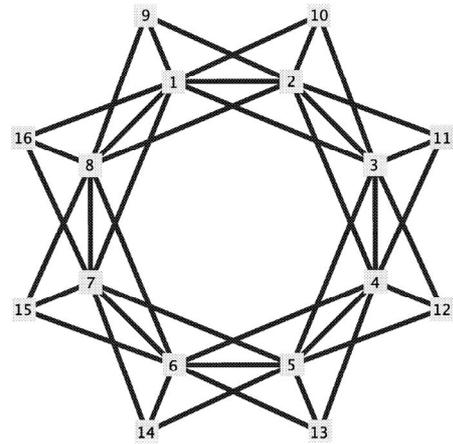


Figure 4. $(C_8 \circ K_1)^2$.

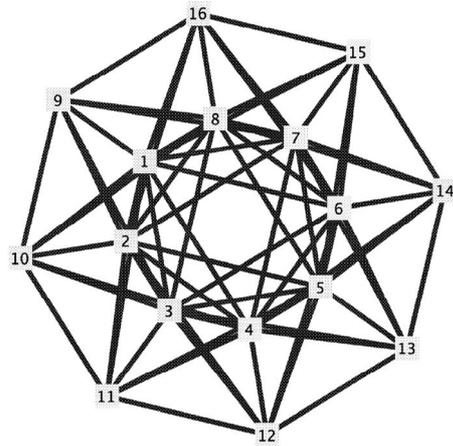


Figure 5. $(C_8 \circ K_1)^3$.

Also according to the previous Theorem, we have $Z((C_n \circ K_1)^j) = n + 2(j - 2)$, then

$$M((C_n \circ K_1)^j) = Z((C_n \circ K_1)^j) = n + 2(j - 2)$$

and hence $mr((C_n \circ K_1)^j) = n - 2(j - 2)$.

Proposition 3.11 $(K_n \circ K_1)^3$ is homomorphic to K_{2n} .

Theorem 3.12 $mr((K_n \circ K_1)^2) = n$.

Proof. (Figures 6, 7) With the contraction of $n - 1$ external vertices of $(K_n \circ K_1)^2$ on the vertices which is located in internal cycle, we have the complete graph K_{n+1} , then the minor of $(K_n \circ K_1)^2$, is K_{n+1} . So we have

$$n = \xi(K_{n+1}) \leq \xi((K_n \circ K_1)^2) \leq M((K_n \circ K_1)^2).$$

On the other hand it is trivial that $Z((K_n \circ K_1)^2) = n$. And we have $Z((K_n \circ K_1)^2) = n$ then

$$mr((K_n \circ K_1)^2) = n.$$

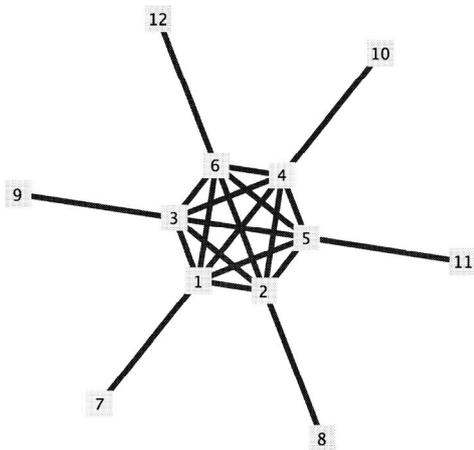


Figure 6. $C_6 \circ K_1$.

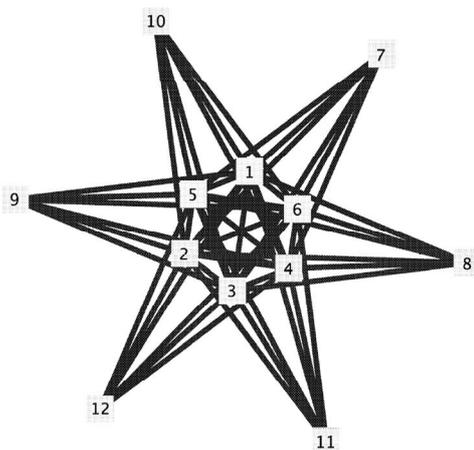


Figure 7. $(K_6 \circ K_1)^2$.

Proposition 3.13 [1] *If G has $n \geq 2$ vertices and contains a Hamiltonian path, then $mr(L(G)) = n - 2$.*

Theorem 3.14 *For all $j \geq 3$ and $n \geq 2$ vertices we have, $mr(L(G^j)) = n - 2$.*

Proof. By using the induction on the $|G|$ it can be shown that G^3 contains a Hamiltonian path. So we have, $mr(L(G^j)) = n - 2$.

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