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Some Results on Vertex Equitable Labeling

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ABSTRACT

Let G be a graph with p vertices and q edges and let $A=\left\{0,1,2,\cdots,\left\lceil\frac{q}{2}\right\rceil\right\}$. A vertex labeling $f:V(G)\to A$ is said to be a vertex equitable labeling of G if it induces an edge labeling f^* given by $f^*(uv)=f(u)+f(v)$ such that $\left|v_f(a)-v_f(b)\right|\le 1$ and $f^*(E)=\{1,2,3,\cdots,q\}$, where $v_f(a)$ is the number of vertices v with $f(v)=a$ for $a\in A$. A graph G is said to be a vertex equitable graph if it admits vertex equitable labeling. In this paper, we establish the vertex equitable labeling of a T_p-tree, $T\odot\overline{K_n}$ where T is a T_p-tree with even number of vertices, bistar $B(n,n+1)$, the caterpillar $S(x_1,x_2,\cdots,x_n)$ and crown $C_n\odot K_1, P_n^2$.

Keywords: Vertex Equitable Labeling; Vertex Equitable Graph

1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [1]. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G. Let $G(p,q)$ be a graph with $p=|V(G)|$ vertices and $q=|E(G)|$ edges. A labeling f of a graph G is a mapping that assigns elements of a graph to the set of numbers (usually to positive or non-negative integers). If the domain of the mapping is the set of vertices (the set of edges) then we call the labeling *vertex labeling* (*edge labeling*). The labels of the vertices induce labels of the edges. There are several types of labeling. A detailed survey of graph labeling can be found in [2]. A vertex labeling f is said to be difference labeling if it induces the label $|f(x)-f(y)|$ for each edge xy which is called as weight of the edge xy.

A difference labeling f of a graph G is said to be k-equitable if for each weight induced by f on the edges of G appears exactly k times. If a graph G has a k-equitable labeling then G is said to be k-equitable. Equitable labeling of graphs was introduced by Bloom and Ruiz in [3]. A brief summary of definitions which are useful for the present study is given below.

Definition 1.1 [4] *Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let u and v be two pendant vertices of T such that the length of the path u_0-u is equal to the length of the path v_0-v. If the edge u_0v_0 is deleted from T and u and v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation* (*or an ept, for short*) *and the edge u_0v_0 is called transformable edge.*

If by a sequence of ept's, T can be reduced to a path, then T is called a T_p tree (transformed tree) and such sequence is regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$.

A T_p tree and a sequence of two ept's reducing it to a path are illustrated in **Figure 1**.

Definition 1.2 *The corona $G_1\odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.*

Definition 1.3 *Caterpillar is a tree with the property that the removal of its pendant vertices leaves a path.*

Definition 1.4 *The square graph G^2 of a graph G has the vertex set $V(G^2)=V(G)$ with u,v adjacent in G^2 whenever $d(u,v)\le 2$ in G.*

$\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

The concept of mean labeling was introduced by S. Somasundaram and R. Ponraj in [5] and further studied in [6-8]. A. Lourdusamy and M. Seenivasan introduced a vertex equitable labeling in [9]. In a vertex equitable labeling we use the labels $0,1,2,\cdots,\lceil q/2\rceil$ for the vertices,

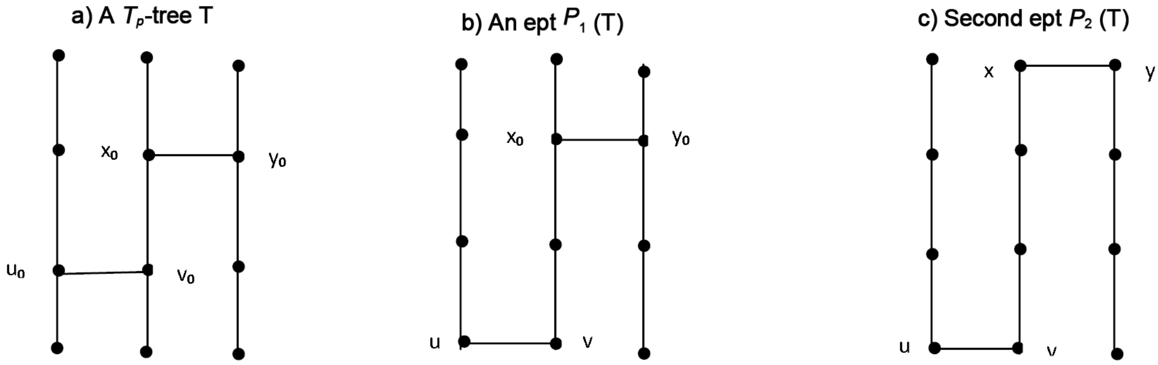

Figure 1. A T_p-tree and a sequence of two ept's reducing it to a path.

the number of times the different vertex labels appear cannot differ by more than one. The induced edge labels are defined as the sum of the incident vertex labels. They proved that the graphs like path, bistar $B(n,n)$, combs $P_n \odot K_1$, bipartite complete $K_{2,n}$, friendship graph $C_3^{(t)}$ for $t \geq 2$, quadrilateral snake,

$$K_2 + mK_1,\ K_{1,n} \cup K_{1,n+k}$$

if and only if $1 \leq k \leq 3$, ladder graph $L_n = P_n \times K_2$, arbitrary super division of a path and cycle C_n with $n \equiv 0$ or $3(mod\ 4)$ are vertex equitable. Also they proved that the graph $K_{1,n}$ if $n \geq 4$, Eulerian graph with n edges where $n \equiv 1$ or $2(mod\ 4)$, the wheel W_n, the complete graph K_n if $n > 3$ and triangular cactus with q edges where $q \equiv 0$ or 6 or 9 $(mod\ 12)$ are not vertex equitable. Moreover they proved that if G is a graph with p vertices and q edges, q is even and $p < \lceil q/2 \rceil + 2$ then G is not vertex equitable.

Definition 1.5 [9] *Suppose G is a graph with p vertices and q edges. Let* $A = \{0,1,2,\cdots,\lceil q/2 \rceil\}$. *A vertex labeling* $f: V \to A$ *induces an edge labeling* f^* *defined by* $f^*(uv) = f(u) + f(v)$ *for all edges uv. For* $a \in A$, *let* $v_f(a)$ *be the number of vertices v with* $f(v) = a$. *A graph G is vertex equitable if there exists a vertex labeling f such that for all a and b in A,*

$$|v_f(a) - v_f(b)| \leq 1$$

and the induced edge labels are $1,2,3,\cdots,q$.

P. Jeyanthi and A. Maheswari proved in [10,11] that tadpoles, $C_m \oplus C_n$, armed crowns, $[P_m;\ C_n^2]$ and, $\langle P_m \hat{o} K_{1,n} \rangle$, the graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle C_n, total graph of P_n, splitting graph of P_n and fusion of two edges of a cycle C_n are vertex equitable graphs. In this paper, we establish the vertex equitable labeling of a T_p-tree, $T \odot \overline{K_n}$ where T is a T_p-tree with even number of vertices, the bistar $B(n,n+1)$, the caterpillar $S(x_1,x_2,\cdots,x_n)$ and the crown $C_n \odot K_1, P_n^2$.

2. Main Results

Theorem 2.1 *Let* $G_1(p_1,2n)$ *and* $G_2(p_2,q_2)$ *be any two vertex equitable graphs with equitable labeling f and g respectively. Let u and v be the vertices of* G_1 *and* G_2 *respectively such that* $f(u) = n$ *and* $g(v) = 0$. *Then the graph* $(G_1)_f * (G_2)_g$ *obtained from* G_1 *and* G_2 *by identifying the vertices u and v is a vertex equitable graph.*

Proof. Clearly $(G_1)_f * (G_2)_g$ has $2n + q_2$ edges and $p_1 + p_2 - 1$ vertices. Let

$$V(G_1) = \{u, u_i : 1 \leq i \leq p_1 - 1\},$$
$$V(G_2) = \{v, v_i : 1 \leq i \leq p_2 - 1\}.$$

Define $h: V((G_1)_f * (G_2)_g) \to \left\{0,1,2,\cdots,\left\lceil \frac{2n+q_2}{2} \right\rceil\right\}$

by $h(u_i) = f(u_i)$ for $1 \leq i \leq p_1 - 1$, $h(v) = f(u)$ and $h(v_i) = g(u_i) + n$ for $1 \leq i \leq p_2 - 1$. Clearly,

$$v_h(a) = \begin{cases} v_f(a) & if\ 0 \leq a \leq n \\ v_g(a) & if\ n+1 \leq a \leq \left\lceil \frac{q_2}{2} \right\rceil + n. \end{cases}$$

Therefore, $|v_h(a) - v_h(b)| \leq 1$ and the labels of the edges of the copy of G_1 are $1,2,\cdots,2n$ and the labels of the edges of the copy of G_2 are $2n+1, 2n+2, \cdots, 2n+q_2$. Hence, $(G_1)_f * (G_2)_g$ is a vertex equitable graph.

Theorem 2.2 *Let* $G_1(p_1,2n+1)$ *and* $G_2(p_2,q_2)$ *be any two vertex equitable graphs with equitable labeling f and g respectively. Let u and v be the vertices of* G_1 *and* G_2 *respectively such that* $f(u) = n+1$ *and* $g(v) = 0$. *Then the graph G obtained by joining u and v by an edge is vertex equitable.*

Proof. Clearly G has $2n + 2 + q_2$ edges and $p_1 + p_2$ vertices. Let

$$V(G_1) = \{u, u_i : 1 \leq i \leq p_1 - 1\}, V(G_2) = \{v, v_i : 1 \leq i \leq p_2 - 1\}.$$

Define $h: V(G) \to \left\{0,1,2,\cdots,\left\lceil \frac{2n+2+q_2}{2} \right\rceil\right\}$

by $h(w) = f(w)$, if $w \in V(G_1)$, $h(w) = g(w) + n + 1$ if $w \in V(G_2)$. The labels of the edges of the copy of G_1 are $1, 2, \cdots, 2n+1$ and the labels of the edges of the copy of G_2 are $2n+3, 2n+4, \cdots, 2n+2+q_2$ and

$$h^*(uv) = h(u) + h(v) = 2n+2.$$

Hence, G is a vertex equitable graph.

Theorem 2.3 *Every T_p-tree is a vertex equitable graph.*

Proof. Let T be a T_p-tree with n vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$ we have 1) $V(P(T)) = V(T)$, 2) $E(P(T)) = (E(T) - E_d) \cup E_p$ where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the epts P used to arrive the path $P(T)$. Clearly, E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_n$ starting from one pendant vertex of $P(T)$ right up to the other.

For $1 \le i \le n$, define the labeling f as

$$f(v_i) = \begin{cases} \dfrac{i-1}{2} & \text{if } i \text{ is odd} \\ \dfrac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

Then f is a vertex equitable labeling of the path $P(T)$

Let $v_i v_j$ be any edge of T with $1 \le i < j \le n$ and P_1 be the ept that deletes this edge and add the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.

Since $v_{i+t} v_{j-t}$ is an edge of the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore i and j are of opposite parity.

The induced label of the edge $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) \\ &= f(v_i) + f(v_{i+2t+1}) \\ &= \begin{cases} \dfrac{i}{2} + \dfrac{i+2t}{2} & \text{if } i \text{ is even} \\ \dfrac{i-1}{2} + \dfrac{i+2t+1}{2} & \text{if } i \text{ is odd} \end{cases} \\ &= i + t,\ 1 \le i \le n \end{aligned}$$

Now

$$\begin{aligned} f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) \\ &= f(v_{i+t}) + f(v_{i+t+1}) \\ &= i + t,\ 1 \le i \le n. \end{aligned}$$

Therefore, we have $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$ and hence f is a vertex equitable labeling of the T_p-tree T.

An example for the vertex equitable labeling of a T_p-tree with 12 vertices is given in **Figure 2**.

Theorem 2.4 *Let T be a T_p-tee with even number of vertices. Then the graph $T \odot \overline{K}_n$ is a vertex equitable graph for all $n \ge 1$.*

Proof. Let T be a T_p-tree of even order m and the vertex set $V(T) = \{v_1, v_2, v_3, \cdots, v_m\}$. Let $u_1^j, u_2^j, \cdots, u_n^j$ be the pendant vertices joined with v_j $(1 \le j \le m)$ by an edge. Then

$$V(T \odot \overline{K_n}) = \{v_j, u_i^j : 1 \le i \le n, 1 \le j \le m\}.$$

By the definition of a T_p-tree, there exists a parallel transformation P of T such that for the path $P(T)$ we have 1) $V(P(T)) = V(T)$, 2) $E(P(T)) = (E(T) - E_d) \cup E_p$ where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \cdots, P_k)$ of the epts P used to arrive the path $P(T)$. Clearly, E_d and E_p have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, \cdots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other. The labeling f defined by

$$f(v_j) = \begin{cases} \dfrac{(n+1)(j-1)}{2} & \text{if } j \text{ is odd} \\ \dfrac{(n+1)j}{2} & \text{if } j \text{ is even,} \end{cases}$$

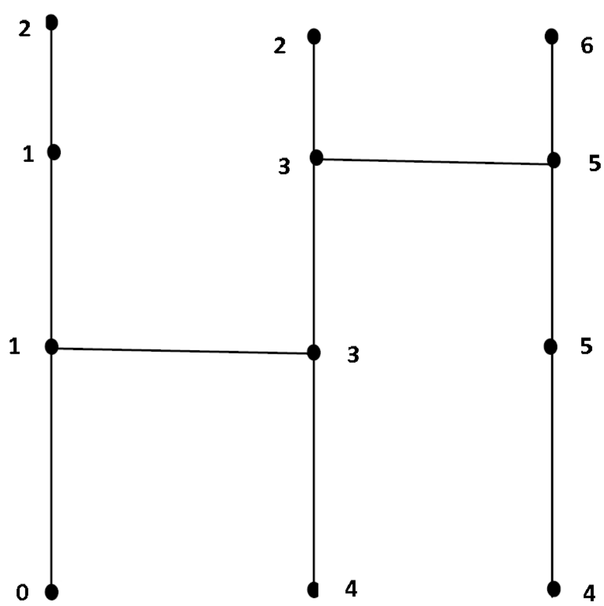

Figure 2. Vertex equitable labeling of a T_p-tree with 12 vertices.

$$f\left(u_i^j\right)=\begin{cases}\dfrac{(n+1)(j-1)}{2}+i & \text{if } j \text{ is odd}\\[2ex] \dfrac{(n+1)(j-2)}{2}+i & \text{if } j \text{ is even.}\end{cases},\ 1\le i\le n$$

is a vertex equitable labeling graph.

Let v_iv_j be any edge of T with $1\le i<j\le m$ let P_1 be the ept that deletes this edge and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1=j-t$ which implies $j=i+2t+1$. Therefore i and j are of opposite parity.

The induced label of the edge v_iv_j is given by

$$\begin{aligned}f^*\left(v_iv_j\right)&=f^*\left(v_iv_{i+2t+1}\right)=f\left(v_i\right)+f\left(v_{i+2t+1}\right)\\ &=\begin{cases}\dfrac{(n+1)i}{2}+\dfrac{(n+1)(i+2t+1-1)}{2} & \text{if } i \text{ is even}\\[2ex] \dfrac{(n+1)(i-1)}{2}+\dfrac{(n+1)(i+2t+1)}{2} & \text{if } i \text{ is odd}\end{cases}\\ &=(n+1)(i+t),\ 1\le i\le n\end{aligned}$$

$$\begin{aligned}f^*\left(v_{i+t}v_{j-t}\right)&=f^*\left(v_{i+t}v_{i+t+1}\right)=f\left(v_{i+t}\right)+f\left(v_{i+t+1}\right)\\ &=(n+1)(i+t),\ 1\le i\le n.\end{aligned}$$

Therefore, we have $f^*\left(v_iv_j\right)=f\left(v_{i+t}v_{j-t}\right)$ and thus f is a vertex equitable labeling of $T\odot\overline{K_n}$.

An example for the vertex equitable labeling of $T\odot\overline{K_5}$, where T is a T_p-tree with 12 vertices is shown in **Figure 3**.

Let $B(n,n+1)$ be a graph obtained from K_2 by attaching n pendant edges at one vertex and $n+1$ pendant edges at the other vertex.

Theorem 2.5 *The bistar* $B(n,n+1)$ *is a vertex equitable graph.*

Proof. Let $V(K_2)=\{u,v\}$ and $u_i\ (1\le i\le n)$ and $v_j\ (1\le j\le n+1)$ be the vertices adjacent to u and v respectively. Now, $B(n,n+1)$ has $2n+2$ edges and $2n+3$ vertices. Define

$$f:V\left(B(n,n+1)\right)\to\left\{0,1,2,\cdots,\left\lceil\frac{2n+2}{2}\right\rceil\right\}$$

by $f(u)=0$, $f(v)=n+1$, $f(u_i)=i$ if $1\le i\le n$ and $f(v_j)=j$ if $1\le j\le n+1$. Then f is a vertex equitable labeling of $B(n,n+1)$.

Theorem 2.6 *Let* $x_1<x_2\le x_3\le\cdots\le x_n$ *and*

$$k=\begin{cases}\left(x_2+x_4+\cdots+x_n\right)-\left(x_1+x_3+\cdots+x_{n-1}\right)-1 & \text{if } n \text{ is even}\\ \left(x_1+x_3+\cdots+x_n\right)-\left(x_2+x_4+\cdots+x_{n-1}\right) & \text{if } n \text{ is odd}\end{cases}.$$

Then $S\left(x_1,x_2,\cdots,x_n,k+1\right)$ is a vertex equitable graph.

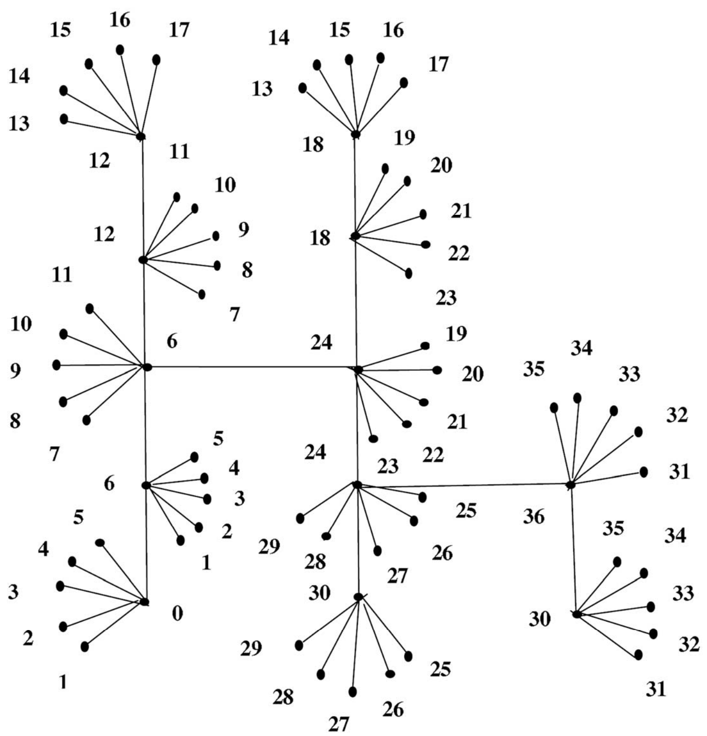

Figure 3. Vertex equitable labeling of $T\odot\overline{K_5}$ **.**

Proof. By Theorem 2.5, $S(x_1, x_1+1)$ is a vertex equitable graph. Let f_1 be the corresponding vertex equitable labeling of $S(x_1, x_1+1)$. Let $y_1 = x_2 - (x_1+1)$. Since $x_1 < x_2$, $y_1 \geq 0$. Consider the graphs $S(x_1, x_1+1)$ and $S(y_1, y_1+1)$. The number of edges of the graph $S(x_1, x_1+1)$ is $2x_1+2$.

Now,
$$\left(S(x_1,x_1+1)\right)_{f_1} * \left(S(y_1,y_1+1)\right)_{f_1}$$
$$= S(x_1, x_1+y_1+1, y_1+1) = S(x_1, x_2, y_1+1).$$

Therefore, by Theorem 2.1, $S(x_1, x_1+y_1+1, y_1+1)$ is a vertex equitable graph. Let f_2 be the corresponding vertex equitable labeling of $S(x_1, x_2, y_1+1)$. Again the number of edges of $S(x_1, x_2, y_1+1)$ is even.

Now take $y_2 = x_3 - (y_1+1) = x_3 - x_2 + x_1 + 1 - 1$. Hence $y_2 \geq 0$. Also
$$\left(S(x_1,x_2,y_1+1)\right)_{f_2} * \left(S(y_2,y_2+1)\right)_{f_1}$$
$$= S(x_1, x_2, y_1+y_2+1, y_2+1)$$
$$= S(x_1, x_2, x_3, y_2+1).$$

Therefore, by Theorem 2.1, $S(x_1, x_2, x_3, y_2+1)$ is a vertex equitable graph and the number of edges is even. Proceeding like this, at the $(n-2)^{th}$ step we get $S(x_1, x_2, x_3, \cdots, x_{n-1}, y_{n-2}+1)$ is a vertex equitable graph where

$$y_{n-2} = \begin{cases} (x_2+x_4+\cdots+x_{n-1}) - (x_1+x_3+\cdots+x_{n-2}) - 1 & \text{if } n \text{ is even} \\ (x_1+x_3+\cdots+x_{n-1}) - (x_2+x_4+\cdots+x_{n-2}) & \text{if } n \text{ is odd} \end{cases}.$$

Let f_{n-1} be the corresponding vertex equitable labeling of $S(x_1, x_2, x_3, \cdots, x_{n-1}, y_{n-2}+1)$. Take

$$y_{n-1} = \begin{cases} (x_2+x_4+\cdots+x_n) - (x_1+x_3+\cdots+x_{n-1}) - 1 & \text{if } n \text{ is even} \\ (x_1+x_3+\cdots+x_n) - (x_2+x_4+\cdots+x_{n-1}) & \text{if } n \text{ is odd} \end{cases}.$$

Clearly $y_{n-2} + y_{n-1} + 1 = x_n$. Now,
$$\left(S(x_1,x_2,x_3,\cdots,x_{n-1},y_{n-2}+1)\right)_{f_{n-1}} * \left(S(y_{n-1},y_{n-1}+1)\right)_{f_1}$$
$$= S(x_1, x_2, x_3, \cdots, x_{n-1}, y_{n-2}+1+y_{n-1}, y_{n-1}+1)$$
$$= S(x_1, x_2, x_3, \cdots, x_n, k+1).$$

Therefore, $S(x_1, x_2, x_3, \ldots, x_{n-1}, x_n, k+1)$ is a vertex equitable graph.

An example for the vertex equitable labeling of $S(4,6,9,7+1)$ if n is odd is given in **Figure 4**.

An example for the vertex equitable labeling of $S(5,7,9,10,2+1)$ if n is even is given in **Figure 5**.

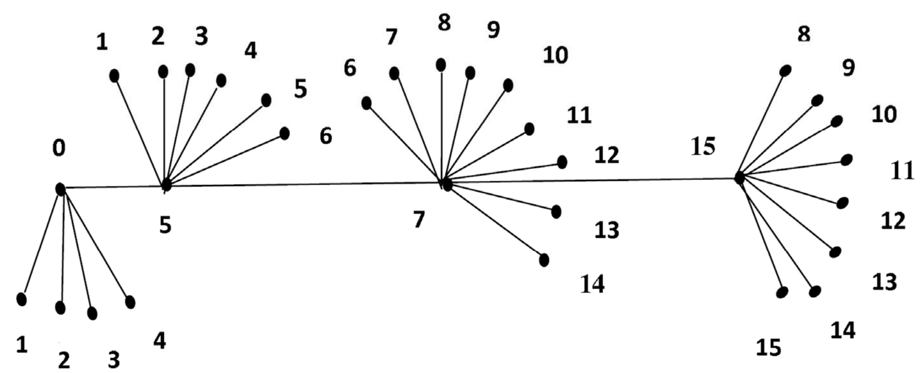

Figure 4. Vertex equitable labeling of *S* (4, 6, 9, 7 + 1).

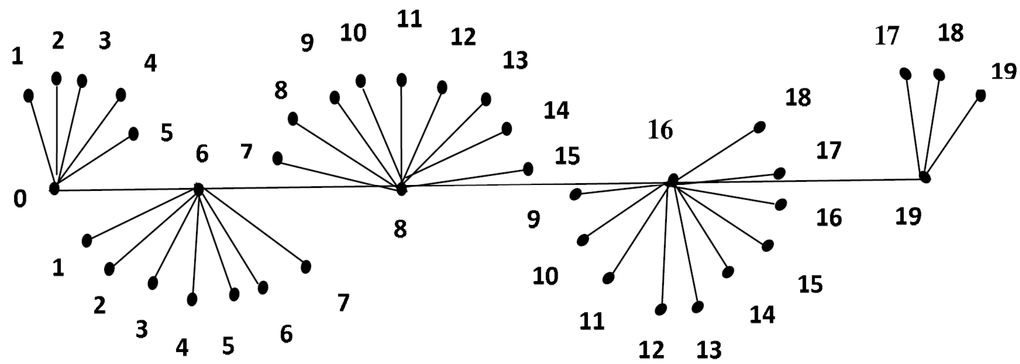

Figure 5. Vertex equitable labeling of S (5, 7, 9, 10, 2 + 1).

Theorem 2.7 *The crown* $C_n \odot K_1$ *is a vertex equitable graph.*

Proof: Let $u_1, u_2, \cdots u_n$ be the vertices of the cycle C_n and let v_i be the vertex adjacent to u_i for $1 \le i \le n$. Then the vertex set $V(C_n \odot K_1) = \{u_i, v_i : 1 \le i \le n\}$ and the edge set $E(C_n \odot K_1) = \{u_i u_{i+1},\ u_i v_i,\ u_n u_1\ u_n v_n, : 1 \le i \le n-1\}$. Define $f : V(C_n \odot K_1) \to \{0, 1, 2 \cdots, n\}$ for the following cases:

Case 1. $n \equiv 0 (mod\ 4)$.

$$f(u_i) = \begin{cases} (i-1) & \text{for } i = 1,3,5,\cdots,\frac{n}{2}-1 \\ i & \text{for } i = 2,4,6,\cdots,\frac{n}{2} \\ i & \text{for } \frac{n}{2}+1 \le i \le n, \end{cases}$$

$$f(v_i) = \begin{cases} i & \text{for } i = 1,3,5,\cdots,\frac{n}{2}-1 \\ (i-1) & \text{for } i = 2,4,6,\cdots,\frac{n}{2} \\ i & \text{for } \frac{n}{2}+1 \le i \le n. \end{cases}$$

Case 2. $n \equiv 1 (mod\ 4)$.

$$f(u_i) = \begin{cases} i & \text{for } i = 1,3,5,\cdots,\left\lceil \frac{n}{2} \right\rceil \\ i-1 & \text{for } i = 2,4,6,\cdots,\left\lfloor \frac{n}{2} \right\rfloor \\ i & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases}$$

$$f(v_i) = \begin{cases} i-1 & \text{for } i = 1,3,5,\cdots,\left\lceil \frac{n}{2} \right\rceil \\ i & \text{for } i = 2,4,6,\cdots,\left\lfloor \frac{n}{2} \right\rfloor \\ i & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases}$$

Case 3. $n \equiv 2 (mod\ 4)$.

$$f(u_i) = \begin{cases} (i-1) & \text{for } i = 1,3,5,\cdots,\frac{n}{2} \\ i & \text{for } i = 2,4,6,\cdots,\frac{n}{2}-1 \\ i & \text{for } \frac{n}{2}+2 \le i \le n, \end{cases}$$

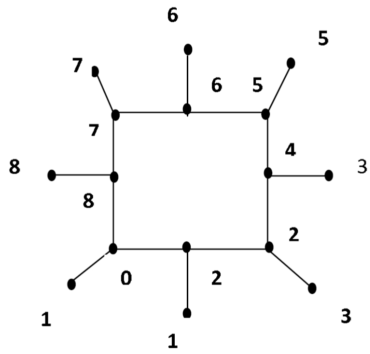

Figure 6. Vertex equitable labeling of $C_8 \odot K_1$.

$$f(v_i) = \begin{cases} i & \text{for } i = 1,3,5,\cdots,\frac{n}{2} \\ (i-1) & \text{for } i = 2,4,6,\cdots,\frac{n}{2}-1 \\ i & \text{for } \frac{n}{2}+3 \le i \le n. \end{cases}$$

$$f\left(v_{(n/2)+1}\right) = \frac{n}{2},\quad f\left(v_{(n/2)+2}\right) = \frac{n}{2}+1,$$

$$f\left(u_{(n/2)+1}\right) = \frac{n}{2}+2.$$

Case 4. $n \equiv 3 (mod\ 4)$.

$$f(u_i) = \begin{cases} (i-1) & \text{for } i = 1,3,5,\cdots,\left\lfloor \frac{n}{2} \right\rfloor \\ i & \text{for } i = 2,4,6,\cdots,\left\lceil \frac{n}{2} \right\rceil \\ i & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases}$$

$$f(v_i) = \begin{cases} i & \text{for } i = 1,3,5,\cdots,\left\lfloor \frac{n}{2} \right\rfloor \\ (i-1) & \text{for } i = 2,4,6,\cdots,\left\lceil \frac{n}{2} \right\rceil - 1 \\ i & \text{for } \left\lceil \frac{n}{2} \right\rceil \le i \le n. \end{cases}$$

In all the above cases, f is a vertex equitable labeling. Hence $C_n \odot K_1$ is a vertex equitable graph.

An example for the vertex equitable labeling of $C_8 \odot K_1$ is shown in **Figure 6**.

Theorem 2.8 *The graph* P_n^2 *is a vertex equitable graph.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the path P_n. Clearly, P_n^2 has n vertices and $2n-3$ edges. Define

$$f : V\left(P_n^2\right) \to \left\{0,1,2,\cdots,\left\lceil \frac{2n-3}{2} \right\rceil\right\}$$

by $f(u_i) = i - 1,\ 1 \le i \le n$. Evidently, P_n^2 is a vertex equitable graph.

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