

# Anisotropic Scattering for a Magnetized Cold Plasma Sphere

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## Abstract

The transformation of parameter tensors for anisotropic medium in different coordinate systems is derived. The electric field for a magnetized cold plasma sphere and the general expression of scattering field from anisotropic target are obtained. The functional relations of differential scattering cross section and the RCS for the magnetized plasma sphere are presented. Simulation results are in agree with that in the literatures, which shows the method used and results obtained are correct and the results provide a theoretical base for anisotropic target identification etc.

**Keywords:** Anisotropy, Scattering, Plasma

## 1. Introduction

Plasma is an anisotropic medium in the outside magnetic field. It has widely applied fields such as modern Radar system, antenna system and target concealing and thus has been of increasing interest [1-4]. In [5], the E.M. scattering features for a conductor sphere coated with plasma are researched by expanding the electromagnetic field into a series of vector spherical functions. The propagation and absorbability of a circular polarizing E.M. wave in asymmetric plasma are also studied [6,7], some results are tested with experiments. The interaction of an E.M. wave and an inhomogeneous plasma slab with electron distribution in the form of partially linear and sinusoidal profiles [8] is researched in which it has been found that inhomogeneous plasma slab can be used as a broad band radar absorbing layer. The scattering characteristics of target coated with plasma are researched [9] by using physical optical and input impedance methods. Other targets coated with plasma are studied [10] based on the medium laminar modeling and the effect induced by the plasma parameters on scattering features is analyzed. In recent years, the interaction of anisotropic targets with light, electromagnetic wave also has been of great interest [11,12]. The research techniques of scattering wave from anisotropic medium and plasma can be divided into three, namely, analytical method, approximation method and numeration. The later two are based on the first. In some literatures, the changes of elements of dielectric tensor and permeability

tensor with coordinate systems are ignored, the orthonormalities of spherical vector wave functions based on the Helmholtz equation derived in the isotropic medium are also ignored and being directly used them to the anisotropic medium. So some errors are inevitable in those obtained results. On the other hand the expression of scattering field from a magnetized cold plasma target is not much published. In the present paper, the expression of the electric field inside & outside a magnetized cold plasma spherical target is presented in detail based on the scale transformation theory of the electromagnetic field by transforming the dielectric tensor in the right angle coordinate system into the spherical system. A formula of computing the scattering field from a general anisotropic target is then developed. Based on the formulae, the Rayleigh scattering features of a magnetized plasma sphere are researched. The effects induced on the feature by the factors of electric density, incident angle and outside magnetic field etc. are demonstrated. The time-harmonic factor  $e^{-j\omega t}$  is used in this paper.

## 2. Research of Rayleigh Scattering from a Magnetized Plasma Sphere

### 2.1. Expressions of Electric Fields inside and outside a Magnetized Cold Plasma Sphere

Assume a magnetized cold plasma sphere to have radius  $R_0$  and its centre to be located at the origin of the primary

coordinate system  $\Sigma$ . The outside magnetic field  $B_0$  is in z-axis. The dielectric constant tensor of this plasma sphere is given as [5]

$$\boldsymbol{\varepsilon} = \varepsilon_r \boldsymbol{\varepsilon}_0 = \varepsilon_0 \begin{bmatrix} \varepsilon & -j\varepsilon_p & 0 \\ j\varepsilon_p & \varepsilon & 0 \\ 0 & 0 & \varepsilon_1 \end{bmatrix} \quad (1)$$

By utilizing the relation between  $\mathbf{D}$  and  $\mathbf{E}$  and the relation between the vectors in right angle system and spherical system, we can obtain the expression of the dielectric constant tensor in spherical coordinate system as

$$\boldsymbol{\varepsilon} = \varepsilon_0 \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \quad (2)$$

where

$$\begin{aligned} \varepsilon_{11} &= \varepsilon + (\varepsilon_1 - \varepsilon) \cos^2 \theta \\ \varepsilon_{12} &= -(\varepsilon_1 - \varepsilon) \cos \theta \sin \theta \\ \varepsilon_{13} &= -j\varepsilon_p \sin \theta \\ \varepsilon_{22} &= \varepsilon_1 - (\varepsilon_1 - \varepsilon) \cos^2 \theta \\ \varepsilon_{23} &= -j\varepsilon_p \cos \theta, \varepsilon_{33} = \varepsilon \\ \varepsilon_{12} &= \varepsilon_{21}, \varepsilon_{13} = -\varepsilon_{31}, \varepsilon_{32} = -\varepsilon_{23} \end{aligned}$$

$$\varepsilon_p = \frac{\frac{ne^3 B_0}{m^2 \omega^3 \varepsilon_0}}{1 - \frac{e^2 B_0^2}{m^2 \omega^2}}, \quad \varepsilon_1 = 1 - \frac{ne^2}{m \omega^2 \varepsilon_0}, \quad \varepsilon = 1 - \frac{\frac{ne^2}{m \omega^2 \varepsilon_0}}{1 - \frac{e^2 B_0^2}{m^2 \omega^2}}$$

$n, m$  are the electron density and electron mass respectively,  $\omega$  the angle frequency of incident wave. Expression (2) indicates that the dielectric tensor is relative to the observing point. When the frequency is low, the condition  $\lambda \gg R_0$  is satisfied, it is so approximately considered that the magnetized cold plasma sphere locates in the electrostatic field [13,14]. The plasma has not electric charge in whole, according to the formulae  $\nabla \cdot \mathbf{D} = 0$ ,  $\mathbf{E} = -\nabla u$  and considering that the differential of potential  $u$  is not relative to the order for  $x$  and  $y$ , the differential equation of  $u$  is obtained as in the primary coordinate system

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^2 u}{\partial y^2} + \varepsilon_1 \frac{\partial^2 u}{\partial z^2} = 0 \quad (3)$$

Now, a scale coordinate system  $\Sigma'$  is introduced as a new coordinate system. The coordinates of this system are indicated with  $x'$ ,  $y'$  and  $z'$ . The relation of coordinates between the two systems is written as

$$x' = \frac{x}{\sqrt{\varepsilon}}, \quad y' = \frac{y}{\sqrt{\varepsilon}}, \quad z' = \frac{z}{\sqrt{\varepsilon_1}}$$

The differential equation of the potential in the scale coordinate system is derived by substituting the above expressions into Equation (3) and using the condition  $u = u'$  [14] at any spatial point, and it is expressed as

$$\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2} = 0 \quad (4)$$

The condition  $u = u'$  is understandable, for the potential is defined as the work done by the electric field to move a unit charge from one point to the reference point, namely  $W/q$ , so both the numerator and the denominator are scale invariants. Equation (4) shows that a magnetized cold plasma sphere in the primary coordinate system is transformed into an isotropic sphere in the scale coordinate system from the view point electric potential equation. This manipulation can greatly simplify the electromagnetic scattering problems. It is well known that the solution of Equation (4) can be obtained by using the method of separation of variables as follows:

$$u'(R', \theta', \phi') = \sum_{m,n} a_{m,n} R'^m P_n^m(\cos \theta') \cos m\phi' + \sum_{m,n} c_{m,n} R'^m P_n^m(\cos \theta') \sin m\phi' \quad (5)$$

Expression (5) is a general solution in the scale coordinate system. The parameters in the two coordinate systems can be found in literature [10]. The electric potential outside the sphere is expressed as

$$u_1(R, \theta, \phi) = \sum_{m,n} \left( e_{m,n} R^n + \frac{f_{m,n}}{R^{n+1}} \right) P_n^m(\cos \theta) \cos m\phi + \sum_{m,n} \left( g_{m,n} R^n + \frac{h_{m,n}}{R^{n+1}} \right) P_n^m(\cos \theta) \sin m\phi \quad (6)$$

On the surface of the sphere, the electric potential inside the sphere is equal to that outside the sphere and the electric displacement  $\mathbf{D}$  is continuous in the normal direction, namely

$$u|_{R=R_0} = u_1|_{R=R_0} \quad (7)$$

$$\varepsilon_0 \left( \varepsilon_{11} \frac{\partial u}{\partial R} + \varepsilon_{12} \frac{1}{R} \frac{\partial u}{\partial \theta} + \varepsilon_{13} \frac{1}{R \sin \theta} \frac{\partial u}{\partial \phi} \right)_{R=R_0} = \varepsilon_0 \frac{\partial u_1}{\partial R}|_{R=R_0}$$

Inserting Expressions (5), (6) and (7) into the above conditions yields the solution of electric potential inside and outside a magnetized cold plasma sphere as

$$u(R, \theta, \phi) = \frac{3A}{2+\varepsilon_1} R \cos \theta + \frac{3(2B+B\varepsilon+jC\varepsilon_p)}{\varepsilon^2+4\varepsilon-\varepsilon_p^2+4} R \cos \phi \sin \theta + \frac{3(2C+C\varepsilon-jB\varepsilon_p)}{\varepsilon^2+4\varepsilon-\varepsilon_p^2+4} R \sin \phi \sin \theta \quad (8)$$

$$\begin{aligned}
u_l(R, \theta, \phi) = & AR \cos \theta \\
& + BR \sin \theta \cos \phi \\
& + CR \sin \theta \sin \phi \\
& + \frac{AR_0^3}{R^2} \frac{1-\epsilon_l}{2+\epsilon_l} \cos \theta \\
& - \frac{R_0^3(B\epsilon^2 + B\epsilon - 2B - B\epsilon_p^2 - 3jC\epsilon_p)}{\epsilon^2 + 4\epsilon - \epsilon_p^2 + 4} \sin \theta \cos \phi \\
& - \frac{R_0^3(C\epsilon^2 + C\epsilon - 2C - C\epsilon_p^2 + 3jB\epsilon_p)}{\epsilon^2 + 4\epsilon - \epsilon_p^2 + 4} \sin \theta \sin \phi
\end{aligned} \quad (9)$$

where

$$A = -E_0 \cos \theta, B = -E_0 \sin \theta \cos \phi, C = -E_0 \sin \theta \sin \phi.$$

From Expression (8), the electric field is obtained as

$$\begin{aligned}
\mathbf{E} = & -\frac{3(2C + C\epsilon - jB\epsilon_p)}{\epsilon^2 + 4\epsilon - \epsilon_p^2 + 4} \hat{y} - \frac{3(2B + B\epsilon + jC\epsilon_p)}{\epsilon^2 + 4\epsilon - \epsilon_p^2 + 4} \hat{x} \\
& - \frac{3A}{2 + \epsilon_l} \hat{z} \\
= & E_x \hat{x} + E_y \hat{y} + E_z \hat{z}
\end{aligned} \quad (10)$$

The above result is obviously in agree with those in the reference [10] when the dielectric tensor is a uniform medium, which tests the correctness of Expression (10).

## 2.2. The Scattering Feature of a Magnetized Cold Plasma Sphere

The scattering field from an anisotropic target is derived as following by using the researching method in the literature [13]

$$E_s = f(\hat{i}, \hat{r}) \frac{e^{jkr}}{r} \quad (11)$$

where

$$f(\hat{i}, \hat{r}) = \frac{k^2}{4\pi} \int_v \left\{ -\hat{r} \times [\hat{r} \times (\mathbf{D} - \mathbf{E})] \right\} e^{-jkv \cdot \hat{r}} dv' \quad (12)$$

is the amplitude of scattering field and  $\mathbf{D}$   $\mathbf{E}$  are respectively electric field & the ‘electric displacement’ inside the plasma sphere, in which  $\mathbf{D} = \epsilon_r \cdot \mathbf{E}$ , has the same unit with  $\mathbf{E}$ . Expression of  $\mathbf{D}-\mathbf{E}$  written as

$$\mathbf{D}-\mathbf{E} = ((\epsilon-1)E_x - j\epsilon_p E_y) \hat{x} + ((\epsilon-1)E_y + j\epsilon_p E_x) \hat{y} + (\epsilon_l-1)E_z \hat{z}$$

Inserting the above into Expression (12) and considering that  $kr' = \frac{2\pi}{\lambda} r' \ll 1$ , the amplitude is derived as following

$$f(\hat{i}, \hat{r}) = \frac{k^2 V}{4\pi} \begin{bmatrix} ((\epsilon-1)E_x - j\epsilon_p E_y) \hat{x} + ((\epsilon-1)E_y + j\epsilon_p E_x) \hat{y} \\ + (\epsilon_l-1)E_z \hat{z} \\ - \hat{r} \left[ ((\epsilon-1)E_x - j\epsilon_p E_y) r_x \right. \\ \left. + ((\epsilon-1)E_y + j\epsilon_p E_x) r_y + (\epsilon_l-1)E_z r_z \right] \end{bmatrix} \quad (13)$$

The symbol  $V$  is the sphere’s volume and vectors  $\hat{r}, \hat{i}$  are respectively the vectors in scattering direction and incident direction.

$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} = r_x \hat{x} + r_y \hat{y} + r_z \hat{z}$ . In our knowledge, Expression (13) is a novel one. The projections of amplitude in the right coordinate system are

$$\begin{aligned}
f_x(\hat{i}, \hat{r}) &= \frac{k^2 V}{4\pi} \left[ ((\epsilon-1)E_x - j\epsilon_p E_y) - ((\epsilon-1)E_x - j\epsilon_p E_y) r_x^2 \right. \\
&\quad \left. - ((\epsilon-1)E_y + j\epsilon_p E_x) r_x r_y - (\epsilon_l-1)E_z r_x r_z \right] \\
f_y(\hat{i}, \hat{r}) &= \frac{k^2 V}{4\pi} \left[ ((\epsilon-1)E_y + j\epsilon_p E_x) - ((\epsilon-1)E_y + j\epsilon_p E_x) r_y^2 \right. \\
&\quad \left. - ((\epsilon-1)E_x - j\epsilon_p E_y) r_x r_y - (\epsilon_l-1)E_z r_y r_z \right] \\
f_z(\hat{i}, \hat{r}) &= \frac{k^2 V}{4\pi} \left[ (\epsilon_l-1)E_z - ((\epsilon_l-1)E_z) r_z^2 \right. \\
&\quad \left. - ((\epsilon-1)E_y + j\epsilon_p E_x) r_z r_y - ((\epsilon-1)E_x - j\epsilon_p E_y) r_x r_z \right]
\end{aligned}$$

The differential scattering cross section is presented as

$$\sigma_d = |f(\hat{i}, \hat{r})|^2 = f_x f_x^* + f_y f_y^* + f_z f_z^* \quad (14)$$

Since the inner electric field is dependent on the direction of the incident wave. So it can be seen from (13) that there are two parts in the differential scattering cross section, the first part is relative to the incident direction, the second is relative to both incident direction and the observing azimuth angle. After considering the orthonormalities of trigonometric functions, the scattering cross section is obtained

$$\sigma = \frac{6k^4 V^2}{5\pi} \left[ |(\epsilon-1)E_x - j\epsilon_p E_y|^2 + |(\epsilon-1)E_y + j\epsilon_p E_x|^2 \right. \\
\left. + |(\epsilon_l-1)E_z|^2 \right] \quad (15)$$

Expression (14) is an analytical one and a novel result in our knowledge.

## 2.3. Discussion

In order to test the rightness of expression (14), we assume that the incident electric field  $\mathbf{E}_0$  is in the x-direction and now obtain  $B = C = 0$ ,  $A = -E_0$ . If the medium is an isotropic one and now assume  $\epsilon_l = \epsilon, \epsilon_p = 0$  is reasonable.

$$f_x(\hat{i}, \hat{r}) = \frac{k^2 V}{4\pi} \frac{3E_0(\varepsilon - 1)}{2 + \varepsilon} (1 - r_x^2)$$

$$f_y(\hat{i}, \hat{r}) = \frac{k^2 V}{4\pi} \frac{3E_0(\varepsilon - 1)}{2 + \varepsilon} r_x r_y$$

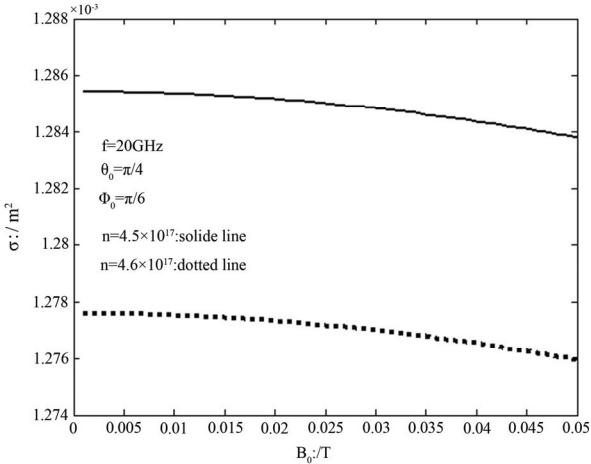
$$f_z(\hat{i}, \hat{r}) = \frac{k^2 V}{4\pi} \frac{3E_0(\varepsilon - 1)}{2 + \varepsilon} r_x r_z$$

The differential scattering cross section is presented as

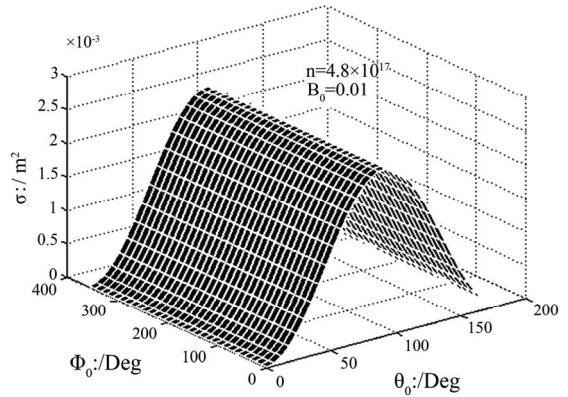
$$\sigma_d = |f(\hat{i}, \hat{r})|^2 = \frac{k^4 V^2}{(4\pi)^2} \left| \frac{3E_0(\varepsilon - 1)}{2 + \varepsilon} \right|^2 \left( 1 - (\hat{r} \cdot \hat{x})^2 \right)$$

This is in agreement with what in references [13,14]. These parameters frequency  $f = 20$  GHz,  $R_0 = 3$  mm, and so  $kR_0 \ll 1$  are used in the simulations:

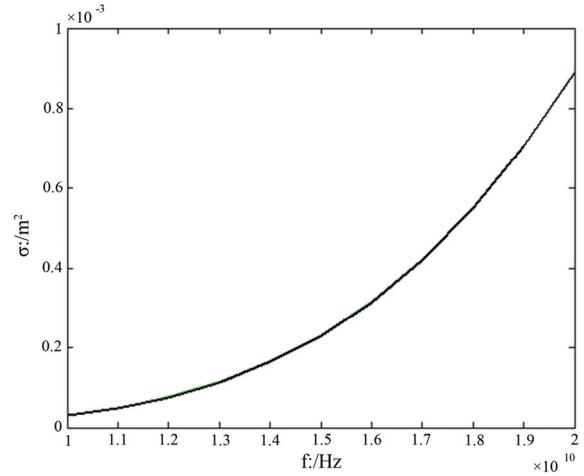
The influence of outside magnetic field on the RCS and the parameters used are all demonstrated in **Figure 1**. It indicates that the RCS decreases when the outside magnetic field increases and magnitude of electric density is given. The reason is that the anisotropy of plasma is enhanced as the outside magnetic field increase. This change in **Figure 1** is in agreement with that in the literature [15,16]. **Figure 2** shows that the angle  $\theta_0$  between electric field and  $B_0$  has an effect on RCS and  $\varphi_0$  has no impact. We know that the anisotropy has a good symmetry in the plane, x-y plane, after the isotropic plasma sphere being magnetized. Thus the plasma is isotropic medium in the plane of x-y and the RCS is not impacted by angle  $\varphi_0$ . The scattering characteristic change with frequency is presented in **Figure 3**. It is well known that the Rayleigh scattering field is radiated by the electromagnetic sources inside the plasma. Those radiating sources have the same phase and so the radiations are mainly electric dipole radiations. These kinds of radiations are proportional to  $\omega^4$ . It is seen from **Figure 4**



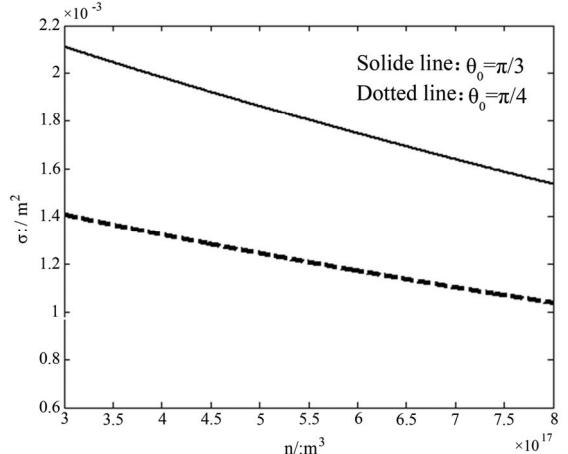
**Figure 1. Change of  $\sigma$  versus outside field  $B_0$ .**



**Figure 2. Change of  $\sigma$  versus polarization.**



**Figure 3. Change of  $\sigma$  versus frequency.**



**Figure 4. Change of  $\sigma$  versus electric density.**

that RCS will decrease as the electric density increasing, which is in agree to that in the reference [17]. This is caused by the fact that the plasma's absorbability to E.M.

wave is enhanced as the electric density being increased.

### 3. Conclusions

In this paper, the electric fields inside and outside a magnetized cold plasma sphere are investigated. We use the scale transformation theory of the electromagnetic field to reconstruct the Laplace equation and then obtain two analytical expressions of the electric potentials inside and outside the magnetized cold plasma sphere in detail. Its correctness is tested with literature. The dielectric tensor in different coordinate systems and a general formula to compute the scattering field from anisotropic target are presented. We take the magnetized cold plasma sphere as an example, its analytical RCS is obtained first in detail and simulations are presented which indicates the characteristic of electric dipole radiation. How to use the scale transformation theory to study the analytical scattering feature for a multilayer magnetized cold plasma target is our next research subject.

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