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Existence of *T*-*ν*-*p*(*x*)-Solution of a Nonhomogeneous Elliptic Problem with Right Hand Side Measure

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Abstract

Using the theory of weighted Sobolev spaces with variable exponent and the L^1-version on Minty's lemma, we investigate the existence of solutions for some nonhomogeneous Dirichlet problems generated by the Leray-Lions operator of divergence form, with right-hand side measure. Among the interest of this article is the given of a very important approach to ensure the existence of a weak solution of this type of problem and of generalization to a system with the minimum of conditions.

Keywords

Nonhomogeneous Elliptic Equations, Dirichlet Problems, Weighted Sobolev Spaces with Variable Exponent, Minty's Lemma, T-ν-$p(x)$-Solutions

1. Introduction

Consider the nonhomogeneous and nonlinear Dirichlet boundary value problem:

$$(\mathcal{P})\quad \begin{cases} -\operatorname{div}\left(a\left(x,u,\nabla u\right)\right)=\mu & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain of $I\!R^N$ ($N \geq 2$) and $Au=-\operatorname{div}\left(a\left(x,u,\nabla u\right)\right)$ is a Leray-Lions operator defined from the weighted Sobolev spaces with variable exponent $W_0^{1,p(x)}(\Omega,\nu)$ into its dual $W^{-1,p'(x)}(\Omega,\nu^*)$ with $\nu^*=\nu^{1-p'(x)}$ and $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. The datum μ is a measure that admits an L^1-dual composition.

Throughout the paper, we suppose that the exponent $p(\cdot)$ is an element of $C_+(\bar{\Omega})$ = {log-Hölder continuous function $p(\cdot):\bar{\Omega}\to I\!R$ such that $1<p_-\le p(x)\le p_+<N$ } (where for all $h\in C_+(\bar{\Omega})$, we denote h_+ and h_- by $h_+=\sup_{x\in\Omega}h(x)$ and $h_-=\inf_{x\in\Omega}h(x)$) and that ν is a weight function defined on Ω (*i.e.*, ν is a measurable function which is strictly positive a.e. in Ω) satisfying:

$$\nu\in L^1_{loc}(\Omega), \tag{1.1}$$

$$\nu^{\frac{-1}{p(x)-1}}\in L^1_{loc}(\Omega), \tag{1.2}$$

$$\nu^{-s(x)}\in L^1(\Omega) \text{ for some } s(x)\in\left(\frac{N}{p(x)},\infty\right)\cap\left(\frac{1}{p(x)-1},\infty\right). \tag{1.3}$$

The problem $(\mathcal{P})$ is studied where the following assumptions are satisfied:

(H_1) *a* is a Carathéodory function satisfying:

$$|a(x,r,\xi)|\le\beta\nu^{\frac{1}{p(x)}}\left[b(x)+|r|^{p(x)-1}+\nu^{\frac{1}{p'(x)}}(\gamma(r)|\xi|)^{p(x)-1}\right] \tag{1.4}$$

$$\left[a(x,r,\xi)-a(x,r,\eta)\right](\xi-\eta)\ge 0\quad\forall\xi,\eta\in I\!R^N \tag{1.5}$$

$$a(x,r,\xi)\xi\ge\alpha\nu|\xi|^{p(x)}, \tag{1.6}$$

where $b(\cdot)$ is a positive function in $L^{p'(x)}(\Omega)$, $\gamma(r)$ is a continuous function and α,β are strictly positive constants.

(H_2) The second member μ is supposed of the form:

$$\mu=f-\operatorname{div}F, \tag{1.7}$$

where $f\in L^1(\Omega)$ and $F\in\left(L^{p'(x)}(\Omega,\nu^*)\right)^N$.

A typical example of the problem $(\mathcal{P})$ is the following involving the so-called $p(x)$-*Laplacian* operator with weight:

$$\Delta_{\nu,p(x)}u=\operatorname{div}\left(\nu(x)|\nabla u|^{p(x)-2}\nabla u\right).$$

The operator $\Delta_{\nu,p(x)}$ becomes *p*-Laplacian when $p(x)\equiv p$ (a constant) and $\nu(x)\equiv 1$. The $p(x)$-Laplacian operator with weight possesses more complicated nonlinearities than the classical *p*-Laplacian, for example, it is inhomogeneous with some degeneracy or singularity. For the applied background of $p(x)$-Laplacian, we refer to (see [1]). The study of differential equations with variable exponents has been a very active field in recent years, we find applications in electro-rheological fluids (see [1] and [2]) and in image processing (see [3]).

Under our assumptions (in particular (1.5), the problem $(\mathcal{P})$ does not admit, in general, a weak solution since the term $a(x,u,\nabla u)$ may not belong to $\left(L^1_{loc}(\Omega)\right)^N$. To overcome this difficulty we use in this paper the framework of L^1-version of Minty's lemma (similar to the one used in [4]). And due to the assumption (1.6) it may be a degenerated or singular problem. Note also that, since the datum is a measure, then the notion of a weak solution cannot be used, hence it is replaced by another approach of solution calling T-ν-$p(x)$-solution (see definition 3.1 below).

Dirichlet problem of type $(\mathcal{P})$ was considered in ([5] [6]), where in the first work the case of $p(x) \equiv p$ (a constant) and $\nu(x) \equiv 1$ is treated, while the second work concerns the degenerated case with $p(x) \equiv p$ (a constant). Hence our present paper can be seen as a generalization of the two works ([5] [6]). We also point out that the existence of solutions for elliptic equations with variable exponents can be found in [7] [8] and [9] and.

This paper is divided into three sections, organized as follows: In Section 2, we introduce and prove some properties of the weighted Sobolev spaces with variable exponent and in Section 3, we prove the existence of T-ν-$p(x)$-solutions of our problem $(\mathcal{P})$. Among the research objectives of this article is to introduce it for applications in physics and also will be a platform for the problem systems of Dirichlet and others.

2. Weighted Sobolev Spaces with Variable Exponent

Let $p \in C_+(\bar{\Omega})$ and ν be a weighted function in Ω.

We define the weighted Lebesgue space with variable exponents $L^{p(x)}(\Omega,\nu)$ as the set of all measurable functions $u:\Omega \to I\!R$ for which the convex weight-modular

$$\rho_{\nu,p(x)}(u) = \int_\Omega \nu(x)|u|^{p(x)}\,dx$$

is finite. The expression

$$\|u\|_{p(x),\nu} = \inf\left\{\mu > 0 : \int_\Omega \nu(x)\left|\frac{u}{\mu}\right|^{p(x)} dx \le 1\right\}$$

defines a norm in $L^{p(x)}(\Omega,\nu)$, called the Luxemburg norm.

Proposition 2.1. *The space* $\left(L^{p(x)}(\Omega,\nu), \|\cdot\|_{p(x),\nu}\right)$ *is a Banach space.*

Proof. By considering the operator $M_{\nu^{\frac{1}{p(x)}}} : L^{p(x)}(\Omega,\nu) \to L^{p(x)}(\Omega)$ defined by

$$M_{\nu^{\frac{1}{p(x)}}}(f) = f\nu^{\frac{1}{p(x)}},$$

for all $f \in L^{p(x)}(\Omega,\nu)$, it's easy to show that $M_{\nu^{\frac{1}{p(x)}}}$ is an isomorphism and hence we can deduce.

Remark 2.1. *When* $\nu(x) \equiv 1$, *the weighted Lebesgue spaces with variable exponent* $L^{p(x)}(\Omega,\nu)$ *coincides with the Lebesgue space with variable exponent* $L^{p(x)}(\Omega)$.

The weight-modular $\rho_{\nu,p(x)}$ *coincides with the modular* $\rho_{p(x)}$ *defined on* $L^{p(x)}(\Omega)$ *by* $\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\,dx$ (*for more details see* [10] [11] [12] *and* [13]).

Lemma 2.1. *For all function* $u \in L^{p(x)}(\Omega,\nu)$, *the following assertions are sa-*

tisfied:

1) $\rho_{\nu,p(x)}(u)>1(=1;<1)\Leftrightarrow\|u\|_{p(x),\nu}>1(=1;<1)$, *respectively*.

2) If $\|u\|_{p(x),\nu}>1$, then $\|u\|_{p(x),\nu}^{p_-}\le\rho_{\nu,p(x)}(u)\le\|u\|_{p(x),\nu}^{p_+}$.

3) If $\|u\|_{p(x),\nu}<1$, then $\|u\|_{p(x),\nu}^{p_+}\le\rho_{\nu,p(x)}(u)\le\|u\|_{p(x),\nu}^{p_-}$.

Proof. It suffices to remark that $\rho_{\nu,p(x)}(u)=\rho_{p(x)}\left(\nu^{\frac{1}{p(x)}}u\right)$ and $\left\|\nu^{\frac{1}{p(x)}}u\right\|=\|u\|_{p(x),\nu}$, and using the analogous result in [13].

Proposition 2.2. *Let* Ω *be a bounded open domain of* $\mathbb{R}^N$ *and* ν *be a weight function on* Ω *satifying the integrability condtions* (1.1) *and* (1.2). *Then* $L^{p(x)}(\Omega,\nu)\hookrightarrow L^1_{loc}(\Omega)$.

Proof.

Let K be an included compact on Ω. By vertue of Hölder inequality we have,

$$\begin{aligned}\int_K|u|\,\mathrm{d}x&=\int_K|u|\nu^{\frac{1}{p(x)}}\nu^{\frac{-1}{p(x)}}\mathrm{d}x\\&\le 2\left\||u|\nu^{\frac{1}{p(x)}}\right\|_{L^{p(x)}(K)}\left\|\nu^{\frac{-1}{p(x)}}\right\|_{L^{p'(x)}(K)}\\&\le 2\|u\|_{p(x),\nu}\left(\int_K\nu^{\frac{-p'(x)}{p(x)}}\mathrm{d}x+1\right)^{\frac{1}{p'_-}}\\&\le 2\|u\|_{p(x),\nu}\left(\int_K\nu^{\frac{-1}{p(x)-1}}\mathrm{d}x+1\right)^{\frac{1}{p'_-}}.\end{aligned}$$

Hence, the conditions (1.1) and (1.2) allow to conclude.

We define the weighted Sobolev space with variable exponents denoted $W^{1,p(x)}(\Omega,\nu)$, by

$$W^{1,p(x)}(\Omega,\nu)=\left\{u\in L^{p(x)}(\Omega):\frac{\partial u}{\partial x_i}\in L^{p(x)}(\Omega,\nu),i=1,\cdots,N\right\},$$

equipped with the norm

$$\|u\|_{1,p(x),\nu}=\|u\|_{p(x)}+\sum_{i=1}^N\left\|\frac{\partial u}{\partial x_i}\right\|_{p(x),\nu}$$

which is equivalent to the Luxemburg norm

$$|||u|||=\inf\left\{\mu>0:\int_\Omega\left(\left|\frac{u}{\mu}\right|^{p(x)}+\nu(x)\sum_{i=1}^N\left|\frac{\frac{\partial u}{\partial x_i}}{\mu}\right|^{p(x)}\right)\mathrm{d}x\le1\right\}.$$

Proposition 2.3. *Let* ν *be a weight function on* Ω *satisfying the conditions* (1.1) *and* (1.2). *Then the space* $\left(W^{1,p(x)}(\Omega,\nu),\|\cdot\|_{1,p(x),\nu}\right)$ *is a Banach space.*

Proof. Let $(u_n)_n$ be a Cauchy sequence in $\left(W^{1,p(x)}(\Omega,\nu),\|.\|_{1,p(x),\nu}\right)$. Then $(u_n)_n$ is a Cauchy sequence in $L^{p(x)}(\Omega)$ and $\left(\frac{\partial u_n}{\partial x_i}\right)_n$ is also a Cauchy sequence in $L^{p(x)}(\Omega,\nu)$ for all $i=1,\cdots,N$. By vertue of proposition 2.1, we can deduce that there exist $u\in L^{p(x)}(\Omega)$ and $v_i\in L^{p(x)}(\Omega,\nu)$ such that:

$$u_n\to u \text{ in } L^{p(x)}(\Omega)$$

and

$$\frac{\partial u_n}{\partial x_i}\to v_i \text{ in } L^{p(x)}(\Omega,\nu) \text{ for all } i=1,\cdots,N.$$

Moreover, by using proposition 2.2, we have $L^{p(x)}(\Omega,\nu)\subset L^1_{loc}(\Omega)\subset D'(\Omega)$. Thus, for all $\varphi\in D(\Omega)$ one has,

$$\left\langle T_{v_i},\varphi\right\rangle=\lim_{n\to\infty}\left\langle T_{\frac{\partial u_n}{\partial x_i}},\varphi\right\rangle=-\lim_{n\to\infty}\left\langle T_{u_n},\frac{\partial\varphi}{\partial x_i}\right\rangle=-\left\langle T_u,\frac{\partial\varphi}{\partial x_i}\right\rangle=\left\langle T_{\frac{\partial u}{\partial x_i}},\varphi\right\rangle.$$

Hence $T_{v_i}=T_{\frac{\partial u}{\partial x_i}}$, *i.e.* $v_i=\frac{\partial u}{\partial x_i}$.

Consequently,

$$u\in W^{1,p(x)}(\Omega,\nu)$$

and

$$u_n\to u \text{ in } W^{1,p(x)}(\Omega,\nu).$$

Remark 2.2. *Since* ν *satisfies the conditions* (1.1) *and* (1.2), *it's easy to prove that* $C_0^\infty(\Omega)$ *is included in* $W^{1,p(x)}(\Omega,\nu)$; *then we can define the following space*

$$W_0^{1,p(x)}(\Omega,\nu)=\overline{C_0^\infty(\Omega)}^{\|.\|_{1,p(x),\nu}},$$

which is also a Banach space under the norm $\|.\|_{1,p(x),\nu}$.

Proposition 2.4. (*Characterization of the dual space*).

Let $p(.)\in C_+(\overline{\Omega})$ and ν be a weight function on Ω satisfying the conditions (1.1) and (1.2). Then for all $G\in\left(W_0^{1,p(x)}(\Omega,\nu)\right)^*$, there exists a unique system of functions $(g_0,g_1,\cdots,g_N)\in L^{p'(x)}(\Omega)\times\left(L^{p'(x)}\left(\Omega,\nu^{1-p'(x)}\right)\right)^N$ such that,

$$G(f)=\int_\Omega f(x)g_0(x)\mathrm{d}x+\sum_{i=1}^N\int_\Omega\frac{\partial f}{\partial x_i}g_i(x)\mathrm{d}x,\ \ \forall f\in W_0^{1,p(x)}(\Omega,\nu).$$

Proof. The proof of this proposition is similar to that used in [12] (theorem3.16).

Now, let us introduce the function p_s defined by

$$p_s(x)=\frac{p(x)s(x)}{s(x)+1}.$$

We have

$$p_s(x) < p(x) \text{ a.e. in } \Omega$$

and

$$\begin{cases} p_s^*(x) = \dfrac{Np_s(x)}{N - p_s(x)} = \dfrac{Np(x)s(x)}{N(s(x)+1) - p(x)s(x)} & \text{if } p(x)s(x) < N(s(x)+1), \\ p_s^*(x) \text{ is arbitrary}, & \text{otherwise.} \end{cases}$$

Proposition 2.5. *Let* $p, s \in C_+(\overline{\Omega})$ *and* ν *be a weight function on* Ω *which satisfies the conditions* (1.1), (1.2) *and* (1.3). *Then* $W^{1,p(x)}(\Omega,\nu) \hookrightarrow W^{1,p_s(x)}(\Omega)$.

Proof. According to the Hölder inequality and the condition (1.3), one has

$$\begin{aligned} \int_\Omega |v(x)|^{p_s(x)} \mathrm{d}x &= \int_\Omega |v(x)|^{p_s(x)} \nu^{\frac{p_s(x)}{p(x)}} \nu^{\frac{-p_s(x)}{p(x)}} \mathrm{d}x \\ &\le \left(\frac{1}{\left(\frac{p}{p_s}\right)_-} + \frac{1}{(s+1)^-} \right) \left\| |v(x)|^{p_s(x)} \nu^{\frac{p_s(x)}{p(x)}} \right\|_{\frac{p(x)}{p_s(x)}} \left\| \nu^{\frac{-p_s(x)}{p(x)}} \right\|_{s(x)+1} \\ &\le \left(\frac{1}{\left(\frac{p}{p_s}\right)_-} + \frac{1}{(s+1)^-} \right) \left(\int_\Omega |v(x)|^{p(x)} \nu(x) \mathrm{d}x \right)^{\frac{1}{\gamma_1}} \left(\int_\Omega \nu(x)^{-s(x)} \mathrm{d}x \right)^{\frac{1}{\overline{\gamma}_1}} \\ &\le C \left(\int_\Omega |v(x)|^{p(x)} \nu(x) \mathrm{d}x \right)^{\frac{1}{\gamma_1}} \left(\int_\Omega \nu(x)^{-s(x)} \mathrm{d}x \right)^{\frac{1}{\overline{\gamma}_1}} \\ &\le C \left(\int_\Omega |v(x)|^{p(x)} \nu(x) \mathrm{d}x \right)^{\frac{1}{\gamma_1}}. \end{aligned}$$

If we take $v = \dfrac{\partial u}{\partial x_i}$, we then obtain

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_s(x)} \mathrm{d}x \le C \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p(x)} \nu(x) \mathrm{d}x \right)^{\frac{1}{\gamma_1}}$$

where

$$\gamma_1 = \begin{cases} \left(\dfrac{p}{p_s}\right)_- & \text{if } \left\| \left| \dfrac{\partial u}{\partial x_i}(x) \right|^{p_s(x)} \nu^{\frac{p_s(x)}{p(x)}} \right\|_{\frac{p(x)}{p_s(x)}} \ge 1, \\ \left(\dfrac{p}{p_s}\right)^+ & \text{if } \left\| \left| \dfrac{\partial u}{\partial x_i}(x) \right|^{p_s(x)} \nu^{\frac{p_s(x)}{p(x)}} \right\|_{\frac{p(x)}{p_s(x)}} < 1. \end{cases}$$

Consequently, we can write

$$\left\| \frac{\partial u}{\partial x_i}(x) \right\|_{p_s(x)}^{\gamma_2} \le C \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p(x)} \nu(x) \mathrm{d}x \right)^{\frac{1}{\gamma_1}} \le C_0 C_1 \left\| \frac{\partial u}{\partial x_i}(x) \right\|_{p(x),\nu}^{\frac{\gamma_3}{\gamma_1}}$$

where

$$\gamma_2 = \begin{cases} (p_s)_- & \text{if } \left\| \frac{\partial u}{\partial x_i}(x) \right\|_{p_s(x)} \geq 1, \\ (p_s)^+ & \text{if } \left\| \frac{\partial u}{\partial x_i}(x) \right\|_{p_s(x)} < 1, \end{cases}$$

and

$$\gamma_3 = \begin{cases} p^+ & \text{si } \left\| \frac{\partial u}{\partial x_i}(x) \right\|_{p(x),\nu} \geq 1, \\ p_- & \text{si } \left\| \frac{\partial u}{\partial x_i}(x) \right\|_{p(x),\nu} < 1. \end{cases}$$

Thus

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{p_s(x)} \leq C \left\| \frac{\partial u}{\partial x_i} \right\|_{p(x),\nu}^{\frac{\gamma_3}{\gamma_1 \gamma_2}}, \quad i = 1, 2, \cdots, N. \tag{2.1}$$

Note that $C = c(\gamma_1, \gamma_2, \gamma_3)$ denotes some positive constant which may be changing step by step.

Since $p_s(x) < p(x)$ p.p. in Ω, then, there exists a positive constant C such that

$$\|u\|_{L^{p_s(x)}(\Omega)} \leq C \|u\|_{L^{p(x)}(\Omega)}.$$

Thus, we conclude that

$$W^{1,p(x)}(\Omega,\nu) \hookrightarrow W^{1,p_s(x)}(\Omega).$$

Corollary 2.1. *Let* $p, s \in C_+(\bar{\Omega})$ *and* ν *be a weight on* Ω *which satisfies the conditions* (1.1), (1.2) *and* (1.3). *Then* $W^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$, *for* $1 \leq r(x) < p_s^*(x)$.

Corollary 2.2. *Let* $p \in C_+(\bar{\Omega})$ *and* ν *be a weight function on* Ω *which satisfies the conditions* (1.1), (1.2) *and* (1.3). *Then*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega;\nu)}, \quad \forall u \in C_0^\infty(\Omega).$$

Proof. Let $u \in C_0^\infty(\Omega)$. Since $1 \leq p(x) < p_s^*(x)$, we deduce by vertue of the embedding $W^{1,p_s(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ that,

$$\|u\|_{L^{p(x)}(\Omega)} \leq C_1 \left(\|u\|_{L^{p_s(x)}(\Omega)} + \|\nabla u\|_{\left(L^{p_s(x)}(\Omega)\right)^N} \right).$$

Thus, in view of the proposition 2.5, we obtain

$$\|u\|_{L^{p(x)}(\Omega)} \leq C_2 \|\nabla u\|_{L^{p_s}(\Omega)} \leq C_3 \|\nabla u\|_{L^{p(x)}(\Omega;\nu)},$$

which allows to conclude that

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega;\nu)}.$$

3. Existence Result

Consider the nonhomogeneous nonlinear Dirichlet boundary problem:

$$(\mathcal{P})\begin{cases}-\operatorname{div}\left(a\left(x,u,\nabla u\right)\right)=-\operatorname{div}F & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Definition 3.1. *A function u is called a* T-ν-$p(x)$ *-solution of problem* $(\mathcal{P})$ *if.*

$$\begin{cases}u\in W_0^{1,p(x)}(\Omega,\nu),\\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\mathrm{d}x=\int_\Omega fT_k(u-\varphi)\mathrm{d}x+\int_\Omega F\nabla T_k(u-\varphi)\mathrm{d}x,\ \ \forall\varphi\in W_0^{1,p(x)}(\Omega,\nu)\cap L^\infty(\Omega).\end{cases}$$

Theorem 3.1. *Let suppose that the assumptions* (1.1)-(1.7) *are satisfied. Then the problem* $(\mathcal{P})$ *has at least one* T-ν-$p(x)$ *-solution.*

Remark 3.1. *Note that in the particular case where* $p(.)\equiv p$ *(constant),* $\gamma(r)=1$ *and* $\nu=1$*, the same result is proved in* [14] *by using the approach of pseudo-monotonicity.*

3.1. Approximate Problem

Let $(f_n)_n$ be a sequence of functions in $L^\infty(\Omega)$ which converges strongly to f in $L^1(\Omega)$ such that $\|f_n\|_{L^\infty(\Omega)}\le\|f\|_{L^\infty(\Omega)}$. For $n\ge1$, we consider the approximate problem of $(\mathcal{P})$

$$(\mathcal{P}_n)\begin{cases}u_n\in W_0^{1,p(x)}(\Omega,\nu)\\ -\operatorname{div}\left(a\left(x,T_n(u_n),\nabla u_n\right)\right)=f_n-\operatorname{div}F & \text{in } \Omega.\end{cases}$$

This section is devoted to establishing the existing solution for the approximate problem $(\mathcal{P}_n)$.

Theorem 3.2. *The operator* A_k *defined by,*

$$A_k: W_0^{1,p(x)}(\Omega,\nu)\to W^{-1,p'(x)}(\Omega,\nu^*)$$
$$u\mapsto A_k u=-\operatorname{div}\left(a\left(x,T_k(u),\nabla u\right)\right)$$

is bounded, coercive, hemicontinuous and pseudo-monotone.

Proof of Theorem 3.2

- The operator A_k is bounded. Indeed for all $u,v\in W_0^{1,p(x)}(\Omega,\nu)$, one has

$$\left|\langle A_k u,v\rangle\right|=\left|\int_\Omega a(x,T_k(u),\nabla u)\nabla v\mathrm{d}x\right|=\left|\int_\Omega a(x,T_k(u),\nabla u)\nu^{\frac{-1}{p(x)}}\nabla v\nu^{\frac{1}{p(x)}}\mathrm{d}x\right|$$

$$\le\left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\left\|a(x,T_k(u),\nabla u)\nu^{\frac{-1}{p(x)}}\right\|_{L^{p'(x)}(\Omega)}\left\|\nabla v\nu^{\frac{1}{p(x)}}\right\|_{L^{p(x)}(\Omega)}$$

$$\le2\left(\int_\Omega\left|a(x,T_k(u),\nabla u)\nu^{\frac{-1}{p(x)}}\right|^{p'(x)}\mathrm{d}x\right)^{\frac{1}{p'_-}}\|\nabla v\|_{L^{p(x)}(\Omega,\nu)}$$

$$\le2\left(\int_\Omega\left(b(x)+|T_k(u)|^{p(x)-1}+\nu^{\frac{1}{p'(x)}}\left(\gamma(T_k(u))|\nabla u|\right)^{p(x)-1}\right)^{p'(x)}\mathrm{d}x\right)^{\frac{1}{p'_-}}\|\nabla v\|_{L^{p(x)}(\Omega,\nu)}$$

$$\le C_1\left(\int_\Omega\left(b(x)^{p'(x)}+|T_k(u)|^{p(x)}+\nu(x)\left(\gamma(T_k(u))|\nabla u|\right)^{p(x)}\right)\mathrm{d}x\right)^{\frac{1}{p'_-}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}$$

$$\le \left(C_1 + C_2 + C_3 \left(\int_\Omega \left| T_k(u) \right|^{p(x)} + \nu(x) \left(\gamma\left(T_k(u)\right) |\nabla u| \right)^{p(x)} \mathrm{d}x \right)^{\frac{1}{p'_-}} \right) \|v\|_{W_0^{1,p(x)}(\Omega,\nu)}.$$

Since $\gamma(.)$ is continuous and $|T_k(u)| \le k$ a.e. in Ω, then $\gamma(T_k(u))|\nabla u|$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$; hence the operator A_k is bounded.

- The operator A_k is hemicontinuous. Indeed, let t be a reality that tends to t_0. We have

  $$a\left(x, T_k(u+tv), \nabla T_k(u+tv)\right) \to a\left(x, T_k(u+t_0 v), \nabla T_k(u+t_0 v)\right), \text{ a.e. in } \Omega.$$

Since $\left(a\left(x, T_k(u+tv), \nabla T_k(u+tv)\right)\right)_t$ is bounded in $\left(L^{p'}(\Omega)\right)^N$, we deduce that $A_k(u+tv)$ converges to $A_k(u+t_0 v)$ weakly in $W^{-1,p'(x)}\left(\Omega,\nu^*\right)$ as t tends to t_0.

- The operator A_k is coercive. Indeed, for all $u \in W_0^{1,p(x)}(\Omega,\nu)$, we have

$$\frac{\langle A_k u, u \rangle}{\|u\|_{W_0^{1,p(x)}(\Omega,\nu)}} \ge \frac{\int_\Omega \nu(x) |\nabla u|^{p(x)} \mathrm{d}x}{\|u\|_{W_0^{1,p(x)}(\Omega,\nu)}} \ge \frac{\|u\|^{\delta}_{W_0^{1,p(x)}(\Omega,\nu)}}{\|u\|_{W_0^{1,p(x)}(\Omega,\nu)}} \ge \|u\|^{\delta-1}_{W_0^{1,p(x)}(\Omega,\nu)},$$

where

$$\delta = \begin{cases} p_- & \text{if } \|u\|_{W_0^{1,p(x)}(\Omega,\nu)} \le 1, \\ p^+ & \text{if } \|u\|_{W_0^{1,p(x)}(\Omega,\nu)} > 1, \end{cases}$$

Obviously, we have $\|u\|^{\delta-1}_{W_0^{1,p(x)}(\Omega,\nu)}$ tends to infinity, when $\|u\|_{W_0^{1,p(x)}(\Omega,\nu)} \to \infty$, hence we conclude.

- It remains to show that A_k is pseudo-monotone: Let $(u_j)_j$ be a sequence in $W_0^{1,p(x)}(\Omega,\nu)$ such that

$$u_j \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega,\nu) \text{ and } \limsup_j \langle A_k u_j, u_j - u \rangle \le 0. \tag{3.1}$$

Firstly, we prove that $A_k u_j$ converges to $A_k u$ weakly in $W^{-1,p'(x)}\left(\Omega,\nu^*\right)$. Indeed, since $(u_j)_j$ is a bounded sequence in $W_0^{1,p(x)}(\Omega,\nu)$, then by the growth condition, $(A_k u_j)_j$ is bounded in $W^{-1,p'(x)}\left(\Omega,\nu^*\right)$, therefore there exists a function $h_k = (h_{ki})$ such that,

$$\begin{aligned} & A_k u_j \rightharpoonup h_k \text{ dans } W^{-1,p'(x)}\left(\Omega,\nu^*\right), \\ & a_i\left(x, T_k(u_j), \nabla u_j\right) \rightharpoonup h_{ki} \text{ in } L^{p'(x)}\left(\Omega,\nu^*\right), \text{ for } i = 1,\cdots,N. \end{aligned} \tag{3.2}$$

Hence, we can write

$$\limsup_j \langle A_k u_j, u_j \rangle \le \langle h_k, u \rangle. \tag{3.3}$$

On the one hand, by (1.5), we have

$$\sum_{i=1}^N \int_\Omega \left(a_i\left(x, T_k(u_j), \nabla v\right) - a_i\left(x, T_k(u_j), \nabla u_j\right) \right) \left(\frac{\partial v}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \right) \mathrm{d}x \ge 0,$$
$$\forall v \in W_0^{1,p(x)}(\Omega,\nu).$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla u_j\right)\frac{\partial u_j}{\partial x_i}dx \\
&\ge \sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla u_j\right)\frac{\partial v}{\partial x_i}\mathrm{d}x-\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla v\right)\frac{\partial v}{\partial x_i}\mathrm{d}x \\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla v\right)\frac{\partial u_j}{\partial x_i}\mathrm{d}x.
\end{aligned}
\tag{3.4}
$$

Since $u_j \to u$ strongly in $L^{p(x)}(\Omega)$ and a.e. in Ω, then

$$a_i\left(x,T_k\left(u_j\right),\nabla v\right)\to a_i\left(x,T_k\left(u\right),\nabla v\right) \text{ strongly in } L^{p'(x)}\left(\Omega,v^*\right) \text{ for } i=1,\cdots,N. \tag{3.5}$$

Therefore,

$$\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla v\right)\frac{\partial v}{\partial x_i}\mathrm{d}x \to \sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u\right),\nabla v\right)\frac{\partial v}{\partial x_i}\mathrm{d}x \tag{3.6}$$

and

$$\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla v\right)\frac{\partial u_j}{\partial x_i}\mathrm{d}x \to \sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u\right),\nabla v\right)\frac{\partial u}{\partial x_i}\mathrm{d}x. \tag{3.7}$$

By vertue of (3.2), we have

$$\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla u_j\right)\frac{\partial v}{\partial x_i}\mathrm{d}x \to \sum_{i=1}^{N}\int_{\Omega} h_{ki}\frac{\partial v}{\partial x_i}\mathrm{d}x. \tag{3.8}$$

Now, combining (3.4)-(3.6) and (3.7), we obtain

$$
\begin{aligned}
&\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u_j\right),\nabla u_j\right)\frac{\partial u_j}{\partial x_i}\mathrm{d}x \\
&\ge \sum_{i=1}^{N}\int_{\Omega} h_{ki}\frac{\partial v}{\partial x_i}\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u\right),\nabla v\right)\frac{\partial u}{\partial x_i}\mathrm{d}x \\
&\quad-\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u\right),\nabla v\right)\frac{\partial v}{\partial x_i}\mathrm{d}x.
\end{aligned}
$$

Due to (3.3), we deduce that

$$
\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega} h_{ki}\frac{\partial u}{\partial x_i}\mathrm{d}x &\ge \sum_{i=1}^{N}\int_{\Omega} h_{ki}\frac{\partial v}{\partial x_i}\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u\right),\nabla v\right)\frac{\partial u}{\partial x_i}\mathrm{d}x \\
&\quad-\sum_{i=1}^{N}\int_{\Omega} a_i\left(x,T_k\left(u\right),\nabla v\right)\frac{\partial v}{\partial x_i}\mathrm{d}x.
\end{aligned}
$$

This implies that,

$$\sum_{i=1}^{N}\int_{\Omega}\left(a_i\left(x,T_k\left(u_j\right),\nabla v\right)-h_{ki}\right)\left(\frac{\partial v}{\partial x_i}-\frac{\partial u_j}{\partial x_i}\right)\mathrm{d}x \ge 0, \quad \forall v\in W_0^{1,p(x)}\left(\Omega,v\right). \tag{3.9}$$

On the other hand, choose $v=u+tw$ in (3.9) (with $t\in\left]-1,1\right[$). It's easy to see that

$$\int_{\Omega}\left(a\left(x,T_k\left(u\right),\nabla\left(u+tw\right)\right)-h_k\right)\nabla w\mathrm{d}x=0, \quad \forall w\in W_0^{1,p(x)}\left(\Omega,v\right), \forall\, t\in\left]-1,1\right[.$$

Hence $A_k u = h_k \in W^{-1,p'(x)}\left(\Omega,v^*\right)$, and we deduce that $A_k u_j$ weakly converges to $A_k u$ in $W^{-1,p'(x)}\left(\Omega,v^*\right)$.

Secondly, we prove that $\left\langle A_k u_j, u_j\right\rangle \to \left\langle A_k u, u\right\rangle$. Indeed, in view of (3.2) and (3.3), we have

$$\limsup\langle A_k u_j, u_j\rangle \le \langle A_k u, u\rangle = \langle h_k, u\rangle.$$

It remains to show that,

$$\liminf\langle A_k u_j, u_j\rangle \ge \langle A_k u, u\rangle = \langle h_k, u\rangle.$$

For that, we have

$$\begin{aligned}\langle A_k u_j, u_j\rangle &= \sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u_j\right)\frac{\partial u_j}{\partial x_i}\mathrm{d}x\\ &= \sum_{i=1}^{N}\int_\Omega \left(a_i\left(x, T_k\left(u_j\right), \nabla u_j\right) - a_i\left(x, T_k\left(u_j\right), \nabla u\right)\right)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\mathrm{d}x\\ &\quad + \sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u\right)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\mathrm{d}x\\ &\quad + \sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u_j\right)\frac{\partial u}{\partial x_i}\mathrm{d}x.\end{aligned}$$

Since $\sum_{i=1}^{N}\int_\Omega \left(a_i\left(x, T_k\left(u_j\right), \nabla u_j\right) - a_i\left(x, T_k\left(u_j\right), \nabla u\right)\right)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\mathrm{d}x \ge 0$, we deduce that

$$\begin{aligned}\langle A_k u_j, u_j\rangle &\ge \sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u\right)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\mathrm{d}x\\ &\quad + \sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u_j\right)\frac{\partial u}{\partial x_i}\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf\langle A_k u_j, u_j\rangle &\ge \liminf\sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u\right)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\mathrm{d}x\\ &\quad + \liminf\sum_{i=1}^{N}\int_\Omega a_i\left(x, T_k\left(u_j\right), \nabla u_j\right)\frac{\partial u}{\partial x_i}\mathrm{d}x.\end{aligned}$$

Hence, $\liminf\langle A_k u_j, u_j\rangle \ge \sum_{i=1}^{N}\int_\Omega h_i\frac{\partial u}{\partial x_i}\mathrm{d}x \ge \langle A_k u, u\rangle$. This achieved the proof.

3.2. Proof of Theorem 3.1

The proof is divided into 4 steps.

Step 1: We will show that $(u_n)_n$ is a Cauchy sequence in measure. Using $T_k(u_n)$ as a test function in $(\mathcal{P}_n)$ leads to,

$$\int_\Omega a\left(x, T_k\left(u_n\right), \nabla u_n\right)\nabla T_k\left(u_n\right)\mathrm{d}x = \int_\Omega f_n T_k\left(u_n\right)\mathrm{d}x + \int_\Omega F\cdot\nabla T_k\left(u_n\right)\mathrm{d}x.$$

From (1.6) and (1.7), we deduce for all $k > 1$ that,

$$\begin{aligned}&\alpha\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu(x)\mathrm{d}x\\ &\le k\|f\|_{L^1} + \sum_{i=1}^{N}\int_\Omega |F_i|\nu(x)^{\frac{-1}{p(x)}}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|\nu(x)^{\frac{1}{p(x)}}\mathrm{d}x\\ &\le k\|f\|_{L^1} + \sum_{i=1}^{N}\int_\Omega |F_i|\nu(x)^{\frac{-1}{p(x)}}\left(\frac{\alpha}{2}\right)^{\frac{-1}{p(x)}}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|\nu(x)^{\frac{1}{p(x)}}\left(\frac{\alpha}{2}\right)^{\frac{1}{p(x)}}\mathrm{d}x.\end{aligned}$$

Now, by Young's inequality, we obtain

$$\begin{aligned}
&\alpha\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu\left(x\right)\mathrm{d}x\\
&\leq k\left\|f\right\|_{L^1}+\sum_{i=1}^{N}\int_{\Omega}\left|F_i\right|^{p'(x)}\nu\left(x\right)^{\frac{-p'(x)}{p(x)}}\frac{C\left(\alpha\right)}{p'\left(x\right)}\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu\left(x\right)\frac{\alpha}{2p\left(x\right)}\mathrm{d}x \qquad (3.10)\\
&\leq k\left\|f\right\|_{L^1}+\sum_{i=1}^{N}\int_{\Omega}\left|F_i\right|^{p'(x)}\nu\left(x\right)^{\frac{-p'(x)}{p(x)}}C\left(\alpha,p'^{-}\right)\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu\left(x\right)\frac{\alpha}{2p^{-}}\mathrm{d}x.
\end{aligned}$$

Then, one has

$$\begin{aligned}
&\left(1-\frac{1}{2p^{-}}\right)\alpha\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu\left(x\right)\mathrm{d}x\\
&\leq k\left\|f\right\|_{L^1}+\frac{C\left(\alpha,p'^{-}\right)}{k}+\sum_{i=1}^{N}\int_{\Omega}\left|F_i\right|^{p'(x)}\nu\left(x\right)^{\frac{-p'(x)}{p(x)}}\mathrm{d}x,
\end{aligned}$$

for $k\geq 1$, which implies that

$$\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu\left(x\right)\mathrm{d}x\leq Ck \quad \text{for all } k>1. \qquad (3.11)$$

Let $k>0$ large enough and B_R be a ball of Ω. Using (3.11) and applying Hölder's inequality and Poincaré's inequality, we obtain

$$\begin{aligned}
&k\ meas\left(\left\{\left|u_n\right|>k\right\}\cap B_R\right)\\
&=\int_{\{|u_n|>k\}\cap B_R}\left|T_k\left(u_n\right)\right|\mathrm{d}x\leq\left\|T_k\left(u_n\right)\right\|_{L^1(\Omega)}\leq C\left\|T_k\left(u_n\right)\right\|_{L^{p(x)}(\Omega)}\\
&\leq C\left\|\nabla T_k\left(u_n\right)\right\|_{p(x),\nu}\quad\text{(by vertue of Corollary 2.2)} \qquad (3.12)\\
&\leq C\left(\int_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial T_k\left(u_n\right)}{\partial x_i}\right|^{p(x)}\nu\left(x\right)\mathrm{d}x\right)^{\frac{1}{\kappa}}\quad\text{(by vertue of Lemma 2.1)}\\
&\leq Ck^{\frac{1}{\kappa}},
\end{aligned}$$

where

$$\kappa=\begin{cases}p_{-} & \text{if } \left\|\nabla T_k\left(u_n\right)\right\|_{p(x),\nu}\leq 1,\\ p_{+} & \text{if } \left\|\nabla T_k\left(u_n\right)\right\|_{p(x),\nu}>1,\end{cases}$$

which implies that,

$$meas\left(\left\{\left|u_n\right|>k\right\}\cap B_R\right)\leq\frac{C}{k^{1-\frac{1}{\kappa}}},\quad\forall k>1. \qquad (3.13)$$

So, we have, for all $\delta>0$,

$$\begin{aligned}
&meas\left(\left\{\left|u_n-u_m\right|>\delta\right\}\cap B_R\right)\\
&\leq meas\left(\left\{\left|u_n\right|>k\right\}\cap B_R\right)+meas\left(\left\{\left|u_m\right|>k\right\}\cap B_R\right) \qquad (3.14)\\
&\quad+meas\left(\left\{\left|T_k\left(u_n\right)-T_k\left(u_m\right)\right|>\delta\right\}\right).
\end{aligned}$$

Since $\left(T_k\left(u_n\right)\right)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, there exists a subsequence, still denoted by $T_k\left(u_n\right)$ and a measurable function $v_k \in W_0^{1,p(x)}(\Omega,\nu)$ such that $T_k\left(u_n\right)$ converges to v_k weakly in $W_0^{1,p(x)}(\Omega,\nu)$, strongly in $L^{p(x)}(\Omega)$ and almost everywhere in Ω. Hence $\left(T_k\left(u_n\right)\right)_n$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon>0$. Then by (3.13), there exists $k(\varepsilon)>0$ such that,

$$meas\left(\left\{\left|u_n-u_m\right|>\delta\right\}\cap B_R\right)<\varepsilon,\ \ \forall\, n, m \geq n_0\left(k(\varepsilon),\delta,R\right).$$

This proves that $\left(u_n\right)_n$ is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function *u*. Hence

$$\begin{aligned}&T_k\left(u_n\right) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p(x)}(\Omega,\nu),\\ &\text{strongly in } W^{p(x)}(\Omega),\ \text{ and a.e. in } \Omega.\end{aligned} \tag{3.15}$$

Step 2: We shall prove that

$$\begin{aligned}&\int_\Omega a\left(x,u_n,\nabla\varphi\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\\ &\leq \int_\Omega f_n T_k\left(u_n-\varphi\right)\mathrm{d}x+\int_\Omega F\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\\ &\forall\varphi\in W_0^{1,p(x)}(\Omega,\nu)\cap L^\infty(\Omega).\end{aligned} \tag{3.16}$$

Let $\varphi\in W_0^{1,p(x)}(\Omega,\nu)\cap L^\infty(\Omega)$ and let n be large enough ($n\geq k+\|\varphi\|_\infty$). Using the admissible test function $T_k\left(u_n-\varphi\right)$ in $\left(\mathcal{P}_n\right)$ leads to

$$\int_\Omega a\left(x,u_n,\nabla u_n\right)\nabla\left(T_k\left(u_n-\varphi\right)\right)\mathrm{d}x=\int_\Omega f_n T_k\left(u_n-\varphi\right)\mathrm{d}x+\int_\Omega F\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x, \tag{3.17}$$

i.e.,

$$\begin{aligned}&\int_\Omega a\left(x,u_n,\nabla u_n\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x+\int_\Omega a\left(x,u_n,\nabla\varphi\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\\ &-\int_\Omega a\left(x,u_n,\nabla\varphi\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x=\int_\Omega f_n T_k\left(u_n-\varphi\right)\mathrm{d}x+\int_\Omega F\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x,\end{aligned} \tag{3.18}$$

which implies that

$$\begin{aligned}&\int_\Omega\left(a\left(x,u_n,\nabla u_n\right)-a\left(x,u_n,\nabla\varphi\right)\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\\ &+\int_\Omega a\left(x,u_n,\nabla\varphi\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x=\int_\Omega f_n T_k\left(u_n-\varphi\right)\mathrm{d}x+\int_\Omega F\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x.\end{aligned} \tag{3.19}$$

Thanks to assumption (1.5) and the definition of truncation function, we have

$$\int_\Omega\left(a\left(x,u_n,\nabla u_n\right)-a\left(x,u_n,\nabla\varphi\right)\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\geq 0. \tag{3.20}$$

Combining (3.19) and (3.20), we obtain (3.16).

Step 3: We claim that

$$\begin{aligned}&\int_\Omega a\left(x,u,\nabla\varphi\right)\nabla T_k\left(u-\varphi\right)\mathrm{d}x\\ &\leq\int_\Omega fT_k\left(u-\varphi\right)\mathrm{d}x+\int_\Omega F\nabla T_k\left(u-\varphi\right)\mathrm{d}x\quad\forall\varphi\in W_0^{1,p(x)}(\Omega,\nu)\cap L^\infty(\Omega).\end{aligned} \tag{3.21}$$

Let $M=k+\|\varphi\|_\infty$. Since $T_M\left(u_n\right)$ converges to $T_M(u)$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$, then

$$T_k\left(u_n-\varphi\right)\rightharpoonup T_k\left(u-\varphi\right)\quad\text{weakly in } W_0^{1,p(x)}(\Omega,\nu). \tag{3.22}$$

Thanks to assumption (1.4), we have

$$\begin{aligned}
&\left|a\left(x,T_M\left(u_n\right),\nabla\varphi\right)\right|^{p'(x)}\nu^{\frac{p'(x)}{p(x)}}\\
&\le\beta\left[b(x)+\left|T_M\left(u_n\right)\right|^{p(x)-1}+\nu^{\frac{1}{p'(x)}}\left(\gamma\left(T_M\left(u_n\right)\right)\left|\nabla\varphi\right|\right)^{p(x)-1}\right]^{p'(x)}\\
&\le C\left[b(x)^{p'(x)}+\left|T_M\left(u_n\right)\right|^{p(x)}+\nu(x)\gamma_0^{p(x)}\left|\nabla\varphi\right|^{p(x)}\right],
\end{aligned}\tag{3.23}$$

where $\gamma_0=\sup\left\{\left|\gamma(s)\right|:|s|\le k+\|\varphi\|_\infty\right\}$ and C is a positive constant. Since $T_M\left(u_n\right)$ converges to $T_M(u)$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$, strongly in $L^{p(x)}(\Omega)$ and a.e. in Ω, thus

$$\left|a\left(x,T_M\left(u_n\right),\nabla\varphi\right)\right|^{p'(x)}\nu^{\frac{p'(x)}{p(x)}}\to\left|a\left(x,T_M(u),\nabla\varphi\right)\right|^{p'(x)}\nu^{\frac{p'(x)}{p(x)}}\quad\text{a.e in }\Omega$$

and

$$\begin{aligned}
&C\left[b(x)^{p'(x)}+\left|T_M\left(u_n\right)\right|^{p(x)}+\nu(x)\gamma_0^{p(x)}\left|\nabla\varphi\right|^{p(x)}\right]\\
&\to C\left[b(x)^{p'(x)}+\left|T_M(u)\right|^{p(x)}+\nu(x)\gamma_0^{p(x)}\left|\nabla\varphi\right|^{p(x)}\right].
\end{aligned}$$

Combining (3.21), (3.22) and using Vitali's theorem, we obtain

$$\int_\Omega a\left(x,u_n,\nabla\varphi\right)\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\to\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\mathrm{d}x.\tag{3.24}$$

Now, we show that

$$\int_\Omega f_nT_k\left(u_n-\varphi\right)\mathrm{d}x\to\int_\Omega fT_k(u-\varphi)\mathrm{d}x.\tag{3.25}$$

In the first time, we have $f_nT_k\left(u_n-\varphi\right)\to fT_k(u-\varphi)$ a.e in Ω, $\left|f_nT_k\left(u_n-\varphi\right)\right|\le k\left|f_n\right|$ and $k\left|f_n\right|\to k|f|$ in $L^1(\Omega)$. In the second time, by using Vitali's theorem we obtain (3.25).

Since $F\in\left(L^{p'(x)}\left(\Omega,\nu^*\right)\right)^N$, one has

$$\int_\Omega F\nabla T_k\left(u_n-\varphi\right)\mathrm{d}x\to\int_\Omega F\nabla T_k(u-\varphi)\mathrm{d}x.\tag{3.26}$$

Thanks to (3.24), (3.25) and (3.26), we obtain (3.21).

Step 4: In this step, we introduce the following generalization of Minty's lemma in weighted Sobolev space with variable exponents $W^{1,p(x)}(\Omega,\nu)$ (which is proved in [15]).

Lemma 3.1. ([15]) *Let u be a measurable function such that* $T_k(u)\in W_0^{1,p(x)}(\Omega,\nu)$ *for every* $k>0$. *Then the following statements are equivalent*:

1) $\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\mathrm{d}x\le\int_\Omega fT_k(u-\varphi)\mathrm{d}x+\int_\Omega F\nabla T_k(u-\varphi)\mathrm{d}x,$

2) $\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\mathrm{d}x=\int_\Omega fT_k(u-\varphi)\mathrm{d}x+\int_\Omega F\nabla T_k(u-\varphi)\mathrm{d}x,$

for every $\varphi\in W_0^{1,p(x)}(\Omega,\nu)\cap L^\infty(\Omega)$ *and for every* $k>0$.

Finally, the result (3.21) and the lemma 3.1 lead to the completion of the proof of theorem 3.1.

4. Conclusion

In this article, we have demonstrated the existence of a solution of a problem with a second measure member and in the space of Sobolev with variable exponent using Minty's lemma. It is a very important technique in which we use the notions of hemicontinuous and pseudo-monotonic instead of broad or strict monotony.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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