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Geometric Proof of Riemann Conjecture (Continued)

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Abstract

This paper will prove Riemann conjecture(RC): All zeros of $\xi(\tau)$ lie on critical line. Denote $\tau=\beta+it$, $\beta=\sigma-1/2$, and $\xi(0+it)=u(t)$ on critical line. We have found two mysteries in Riemann's paper. *The first mystery* is the equivalence: $\xi(\beta+it)\equiv u(t-i\beta)$ is uniquely determined by its initial value $u(t)$. *The second mystery* is Riemamm conjecture 2 (RC2): Using all zeros t_j of $u(t)$ can uniquely express $u(t)=u(0)\prod_{j=1}^{\infty}\left(1-t^2/t_j^2\right)$. We find that the proof of RC is hidden in it. Our basic idea as follows. Consider functional equation $\xi(s)=G(s)\zeta(s)$, $G(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)$. It is known that on critical line $|G(t)|=Ct^{7/4}e^{-t\pi/4}\left(1+O\left(t^{-1}\right)\right)$ and $|\zeta(t)|\le Ct^{1/6}$, then we have the upper bound of growth

$$\ln\left(|u(t)|/|G(t)|\right)\le\frac{1}{6}\ln t+O(1),\quad t\ge T\gg 0.$$

To prove RC2 (or RC), by contradiction. If $\xi(\tau)$ has conjugate complex roots $t'\pm i\beta'$, $\beta'>0$, $R^2=t'^2+\beta'^2$, by symmetry $\xi(\tau)=\xi(-\tau)$, then $-(t'\pm i\beta')$ do yet. So ξ must contain four factors. Then $u(t)$ contains a real factor $p(t)=\left(1-t^2/R^2\right)^2+4t^2\beta'^2/R^4>0$ and $\ln|u(t)|$ contains a term (the lower bound)

$$\ln p(t)\ge 4\ln t+O(1),\quad t\ge T\gg 0,$$

which contradicts to the growth above. So ξ can not have the complex roots and $u(t)$ does not have the factor $p(t)$. Therefore both RC2 and RC are proved. We have seen that the two-dimensional problem is reduced to one-dimension and the one-dimensional $u(t)$ is reduced to its product expression. Perhaps this is close to the original idea of Riemann. Other results are also discussed by geometric analysis in the last section.

Keywords

RC, Equivalence, RC2, Product Expression, Single Peak, Multiple Zeros

1. Introduction

Riemann hypothesis (RH) is one of the most difficult problems in mathematics, which is reviewed in [1] [2] [3] [4] [5]. We shall consider ξ. Although $\xi = u + iv$ and $\zeta = U + iV$ have the same zeros, but their properties are quite different. ξ has the symmetry, *i.e.* $v \equiv 0$ on the critical line, and $\{u, v\}$ are alternative oscillation with single peak outside critical line, which geometrically implies RC true. Whereas the property of ζ is bad, even if on critical line $\{U, V\}$ are not alternative oscillation, sometimes almost tangent and multiple peak. Studying ζ is very hard.

To study Riemann conjecture (RC), we have proposed a framework of geometric analysis for ξ in previous papers. If three theorems are proved, then RC holds, also see section 3. Firstly we have proved theorems 1 and 2 by the symmetry of ξ. But to prove theorem 3: "on critical line $u(t)$ is single peak", we have met essential difficulty. The symmetry is not enough and the stronger tool is needed. Thus we have to focus our attention on $u(t)$ and re-investigate Riemann's thought. We have found two mysteries in it finally proved RC by method of analysis.

Denote $\beta = \sigma - 1/2, \tau = \beta + it$. Riemann ξ-function has an integral expression [4], p. 17,

$$\begin{cases} \xi(\tau) = \int_1^{\infty} \left(x^{\tau/2} + x^{-\tau/2}\right) x^{-3/4} f(x) \mathrm{d}x = u(t,\beta) + iv(t,\beta), \\ f(x) = \sum_{n=1}^{\infty} \left(3\left(n^2 \pi x\right)^2 - 2n^2 \pi x\right) \mathrm{e}^{-n^2 \pi x}. \end{cases} \tag{1.1}$$

But Riemann did not use (1.1), he had directly taken $\tau = it$ to get the real function [4], p. 301,

$$\xi(it) = 2\int_1^{\infty} \cos\left(\frac{t}{2} \ln x\right) x^{-3/4} f(x) \mathrm{d}x = u(t,0) = u(t), \quad v(t,0) = 0, \tag{1.2}$$

Why Riemann preferred (1.2) rather than (1.1)? This is the first mystery.

Riemann also regarded t as a complex variable (very important!). Taking $\tau = \beta + it = i(t - i\beta)$ and using the uniqueness of analytic function, we get

$$\xi(\tau) \equiv u(z), \quad \tau = iz, \quad z = t - i\beta, \quad \xi(\tau) = \xi(-\tau). \tag{1.3}$$

Thus the first mystery is formulated as:

Equivalence. $\xi(\tau) \equiv u(t - i\beta)$ is uniquely determined by its initial value

$u(t)$, *i.e.*, two dimensional problem is reduced to one dimension.

We also consider the initial value problem of Cauchy-Riemann system

$$\begin{cases} u_\beta = v_t, & v_\beta = -u_t, \quad \Omega = \{\beta \in [0,1/2], t \in [0,\infty)\}, \\ u(t,0) = g(t), & v(t,0) = 0, \quad t \in [0,\infty). \end{cases} \tag{1.4}$$

As $g(t)$ is analytic, Cauchy-Kovalevshkaya theorem confirms that it has a unique analytic solution. This solution just is $\xi(\tau) = u + iv = g(t - i\beta)$. Actually, by direct verification,

$$g_t - ig_\beta = u_t + iv_t - i(u_\beta + iv_\beta) = u_t + v_\beta + i(v_t - u_\beta) = 0,$$

then (1.3) and the equivalence hold yet. Here $\xi(\tau) = u(t - i\beta)$ resembles a traveling-wave solution of the wave equation, where $u(t)$ as an initial value. We had used it in previous papers.

We see that Riemann had studied ξ rather than ζ, his thought can be formulated as:

Riemann conjecture(RC). All zeros of $\xi(\tau)$ lie on critical line $\beta = 0$.

To study $u(t)$, we find the second mystery in Riemann's paper [4] (pp. 301-302).

Riemann conjecture 2 (RC2). Using all zeros $\{t_j\}$ of $u(t)$ can uniquely determine

$$u(t) = u(0)\prod_j \left(1 - \frac{t^2}{t_j^2}\right), \quad 0 \le t < \infty. \tag{1.5}$$

For *k*-ple zeros, should take *k*-ple products.

The greatest mystery is that RC2 implies RC. Actually, if $\beta \ne 0$, replacing *t* by $z = t - i\beta$ in (1.5), each factor $1 - z^2/t_j^2 \ne 0$, then $\xi(\tau) \equiv u(z) \ne 0$ and RC holds.

We rigorously have proved RC2 by the method of analysis in section 2. Here we briefly formulate our basic idea as follows. Consider functional equation

$$\xi(s) = G(s)\zeta(s), \quad G(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2). \tag{1.6}$$

It is known that on critical line

$$|G(t)| = Ct^{7/4}e^{-t\pi/4}\left(1 + O(t^{-1})\right) \text{ and } |\zeta(t)| \le Ct^{1/6},$$

we get the upper bound of growth

$$\ln\left(|u(t)|/|G(t)|\right) \le \frac{1}{6}\ln t + O(1), \quad t \ge T \gg 0. \tag{1.7}$$

To prove RC2, by contradiction. If $\xi(\tau)$ has conjugate complex roots $t' \pm i\beta'$, $\beta' > 0$, by symmetry $\xi(\tau) = \xi(-\tau)$, then $-(t' \pm i\beta')$ do yet. Thus by equivalence $\xi(\tau)$ must contain four factors

$$p(z) = \left(1 - \frac{z}{t' + i\beta'}\right)\left(1 - \frac{z}{t' - i\beta'}\right)\left(1 + \frac{z}{t' + i\beta'}\right)\left(1 + \frac{z}{t' - i\beta'}\right), \quad z = t - i\beta.$$

Letting $\beta = 0$, $u(t)$ contains a real factor

$$p(t)=\left(1-\frac{t^2}{R^2}\right)^2+4\frac{t^2\beta'^2}{R^4}>0,\ R^2=t'^2+\beta'^2, \tag{1.8}$$

and then $\ln|u(t)|$ contains a term (the lower bound)

$$\ln p(t)\ge 4\ln t+O(1),\ t\ge T\gg 0, \tag{1.9}$$

which contradicts (1.7). So ξ can not have complex roots and $u(t)$ does not have the factor $p(t)$. Therefore both RC2 and RC are proved.

Our main contribution is that we for the first time have regarded RC as an initial value problem, found these mysteries and proposed the newest method to prove Riemann conjecture.

Similar work has not been found in other papers and books.

We shall continue to complete the geometric analysis of ξ in section 3.

2. Analytical Proof of RC and RC2

2.1. Two Holes in Riemann's Analysis

Riemann denoted $u(t)$ by $\xi(t)$ in his paper. He pointed out, see [4] (pp. 301-302).

"If one denotes by α the roots of the equation $\xi(\alpha)=0$, then one can express $\log\xi(t)$ as

$$\sum\log\left(1-\frac{t^2}{\alpha^2}\right)+\log\xi(0), \tag{R}$$

because, since the density of roots of size t grows only like $\log(t/2\pi)$ as t grows, this expression converges and for infinite t is only infinite like $t\log t$; Thus it differs from $\log\xi(t)$ by a function of t^2 which is continuous and finite for finite t and which, when divided by t^2, is infinitely small for infinite t. This difference is therefore a constant, the value of which can be determined by setting $t=0$."

Because Riemann did not use $\bar{\alpha}$, we must regard α to be real roots, this is extremely important! But then it is misunderstood.

We now explain his analysis. On critical line $u(t)$ is an even entire function and has infinite number of zeros $\pm t_j$. We consider formally the remainder of $\ln u(t)$,

$$\sum_{j=n+1}^{\infty}\ln\left(1-t^2/t_j^2\right)\approx-\sum_{j=n+1}^{\infty}\frac{t^2}{t_j^2}=-t^2\gamma_n,\quad \gamma_n=\sum_{j=n+1}^{\infty}\frac{1}{t_j^2}$$

One knows $t_n=O(n/\ln n)$ (We have better $t_n=B(n)2\pi n/\ln n$, $B(n)=1.46\downarrow 1,\ n\ge 10^3$). Thus

$$\gamma_n=C\sum_{j=n+1}^{\infty}\frac{\ln^2 n}{\left(B(n)n\right)^2}\approx C\int_n^{\infty}\frac{\ln^2 x}{\left(B(x)x\right)^2}\mathrm{d}x\approx C\frac{\ln^2 n}{B(n)n},\ C=(2\pi)^{-2}. \tag{2.1}$$

This series (R) converges for finite t. **(Riemann said)** $u(t)$ **has the growth** $\ln u(t)=O(t\ln t)$ **(This expression is not suitable. We shall use** $\ln|u(t)|$, **which admits** $u(t)$ **= 0, see (2.6))**. The series (R) differs from $\ln u(t)$ by a function of t^2, which, when divided by t^2, is infinitely small for infinite t. (Riemann said) "This difference is therefore a constant." This is not rigorous, in general, it should be $O(\ln t)$ rather than a constant. There is a hole of the uniqueness.

Hadamard in 1893 proved product formula for general entire function, see [4] (p. 20), [5] (p. 16). Denoting $\tau = \beta + it = iz$, $z = t - i\beta$, the zeros of ξ are conjugate, one gets

$$\xi(iz) = \mathrm{e}^{A+Bz}\prod_j \left(1 - \frac{z^2}{z_j^2}\right),\ z = t - i\beta,\ z_j = t_j - i\beta_j.$$

which is even if $\beta = 0$, then $B = 0$. Taking $z = 0$, then $\mathrm{e}^A = \xi(0)$.

But this Hadamard's way from $\beta \neq 0$ to $\beta = 0$ implies a serious contradiction. If all zeros are real, then which itself assumes RC. If conjugate complex zeros are admitted, which just denies RC. Hadamard's theorem was referred by Von Mangoldt "The first real progression in the field in 34 years", see [4], p. 39. We think this is a misunderstanding. Actually, Hadamard's way is independent of proving RC and far from Riemann's thought. Our idea is to consider $u(t)$ at $\beta = 0$ as an initial value. We shall directly prove RC2 by method of analysis, so this contradiction is cast off.

2.2. Analytical Proof of RC and RC2

We consider the functional equation

$$\xi(s) = G(s)\zeta(s),\ G(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right),\ \xi(s) = \xi(1-s). \tag{2.2}$$

Firstly, $\Gamma(s/2)$ has asymptotic expansion

$$\Gamma(s/2) = \sqrt{2\pi}(s/2)^{s/2-1/2}\,\mathrm{e}^{-s/2}(1 + O(1/s)).$$

Take logarithm and decompose the real part, $c = \ln(\sqrt{2\pi})$,

$$\ln\Gamma(s/2) = c + (s/2 - 1/2)\ln(s/2) - s/2 + O(1/s),\quad s = \sigma + it,$$

$$= c + ((\sigma-1)/2 + it/2)\left\{\ln(t/2) + i\left(\pi/2 - \arctan\frac{\sigma}{t}\right)\right\} - (\sigma + it)/2 + O(1/s),$$

$$Re\ln\Gamma = c + ((\sigma-1)/2)\ln(t/2) - t\pi/4 + (t/2)(\sigma/t) - \sigma/2 + O(1/t),$$

Thus for $\sigma = 1/2$

$$|\Gamma(s/2)| = \sqrt{2\pi}\left(\frac{t}{2}\right)^{-1/4}\mathrm{e}^{-t\pi/4}(1 + O(1/t)). \tag{2.3}$$

Besides, $s(1-s)/2 = (t^2 + 1/4)/2$ and $|\pi^{-s/2}| = \pi^{-1/4}$, we get

$$|G(t)| = 2^{3/2}\pi^{1/2-1/4}\left(\frac{t}{2}\right)^{7/4}\mathrm{e}^{-t\pi/4}(1 + O(1/t)). \tag{2.4}$$

Secondly, there are growths of $\zeta(s)$ for large t, [4], p. 185, p. 200,

$$\begin{cases} |\zeta(\sigma)| \le Ct^{(1-\sigma)/2}\ln t, & 0 \le \sigma \le 1, \\ |\zeta(1/2 + it)| \le Ct^{\lambda}, & \lambda = 1/5 \text{ or } \lambda = 19/116. \end{cases} \tag{2.5}$$

We only need the estimate of $\sigma = 1/2$, also see Remark 1.

Thus, we get an upper bound of growth (note: not $\ln|u| = O(t\ln t)$)

$$\ln\left(|u(t)|/|G(t)|\right) \le \frac{1}{6}\ln t + C,\ \text{ if } t \ge T \gg 0. \tag{2.6}$$

So $u(t)=0$ is admitted.

Finally, to prove RC2 (or RC), by contradiction. If $\xi(\tau)$ has conjugate complex roots $t' \pm i\beta'$, $\beta' > 0$, $R^2 = \beta'^2 + t'^2$, by symmetry $\xi(\tau) = \xi(-\tau)$, then $-(t' \pm i\beta')$ do yet. Denoting $z = t - i\beta$ and using the equivalence, $\xi(\tau) \equiv u(z)$ must contain four factors

$$p(z) = \left(1 - \frac{z}{t' + i\beta'}\right)\left(1 - \frac{z}{t' - i\beta'}\right)\left(1 - \frac{z}{-t' + i\beta'}\right)\left(1 - \frac{z}{-t' - i\beta'}\right).$$

Letting $\beta = 0$, then $u(t)$ must contain a real polynomial of fourth degree

$$\begin{aligned} p(t) &= \left(1 - \frac{t}{t' + i\beta'}\right)\left(1 - \frac{t}{t' - i\beta'}\right)\left(1 + \frac{t}{t' - i\beta'}\right)\left(1 + \frac{t}{t' + i\beta'}\right) \\ &= \left(1 + \frac{t^2 - 2tt'}{R^2}\right)\left(1 + \frac{t^2 + 2tt'}{R^2}\right) = \left(1 - \frac{t^2}{R^2}\right)^2 + \frac{4t^2\beta'^2}{R^4} > 0, \end{aligned} \tag{2.7}$$

and $\ln|u(t)|$ contains a term (as a lower bound)

$$\ln p(t) \ge 4\ln t + O(1),\ \ t \ge T \gg 0. \tag{2.8}$$

Its growth contradicts (2.6). Thus $\xi(\tau)$ can not have complex roots and $u(t)$ does not have the factor $p(t)$. Therefore both RC and RC2 are rigorously proved. □

Originally we want to prove only RC2, fortunately, both RC and RC2 are proved.

Remark 1. In period of Riemann, no estimates (2.5), but it is possible to prove RC. As

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \int_0^1 \left(n^{-s} - (n+x)^{-s}\right)\mathrm{d}x + \int_1^{\infty} y^{-s}\mathrm{d}y \\ &= \frac{1}{s-1} + \sum_{n=1}^{\infty} s\int_0^1 x^{-s-1}(1-x)\mathrm{d}x,\ \text{ by integration by part} \end{aligned}$$

is already continued analytically to $Re(s) > 0$ (actually this is Euler's method). Thus

$$|\zeta(s)| \le \left|\frac{1}{s-1}\right| + |s|\int_0^{\infty} x^{-\sigma-1}\mathrm{d}x \le Ct,$$

and gets an coarse estimate (e.g. $|u(t)| \le Ct^3 \mathrm{e}^{-t\pi/4}$ in [5], p.27)

$$\ln\left(|u(t)|/|G(t)|\right) \le \ln t + O(1). \tag{2.9}$$

By (2.8) one can still prove that $\xi(\tau)$ does not have complex roots. But nobody noted it. Therefore I feel, Riemann had already approached to prove RC. Our proof is completed to follow Riemann's thought.

Numerical experiments 1. Using the data of the first 10^5 zeros in Odlyzko [6], we have computed $w_n(t) = u(0)\prod_{j=1}^{n}\left(1 - t^2/t_j^2\right)$ and $u(t)$ in (1.2) for $t \in [0, 50]$. We draw these curves by variable scale u/M, here $M(t) = (t/2+1)^{23/12}\mathrm{e}^{-t\pi/4}$, $M(0) = 1$. We see in **Figure 1** that $w_n(t)$ approximates $u(t)$ very well, of course, larger is t, then larger is its deviation.

We have for the first time computed w_n and Figure 1, which make us believe the correctness of RC2, and then consider its analytic proof as before.

3. Continuation of Geometric Analysis

In previous papers [7] [8] [9], we have proposed geometric analysis and proved three results:

Theorem 1. If $u(t)$ is single peak and single zero, then the peak-valley structure for $\beta > 0$ and RC hold.

Theorem 2 (old). If two roots of $u(t)$ are very close to each other (including double zeros), then the peak-valley structure for $\beta > 0$ and RC still hold.

Theorem 3. $u(t)$ is single peak.

RC can be derived by these three theorems. We at present re-examine these theorems. Theorem 1 is correct, see section 3.1. Theorem 2 is also correct but uncomplete, which is generalized in section 3.4. The original proof of theorem 3 [9] has defect (see section 3.2), which is derived by RC2 in section 3.3.

3.1. A Concise Proof of Theorem 1

Consider a root-interval $I_j = \left[t_j, t_{j+1}\right]$ of $u(t,\beta)$, obviously t_j and t_{j+1} depend on β. Assume $u(t,\beta) > 0$ inside I_j for $\beta \in (0,1/2]$. At the left end t_j, $u(t_j,\beta) = 0$ and $u_t(t_j,\beta) > 0$, then the slop $u_t(t_j,r) > 0$, $r \in (0,\beta]$ (which was proved in [9]) we have

$$v(t_j,\beta) = v(t_j,0) + \int_0^\beta u_\beta(t_j,r)\mathrm{d}r = -\int_0^\beta u_t(t_j,r)\mathrm{d}r < 0.$$

At the right end t_{j+1}, $u(t_{j+1},\beta) = 0$ and $u_t(t_{j+1},\beta) < 0$, similarly,

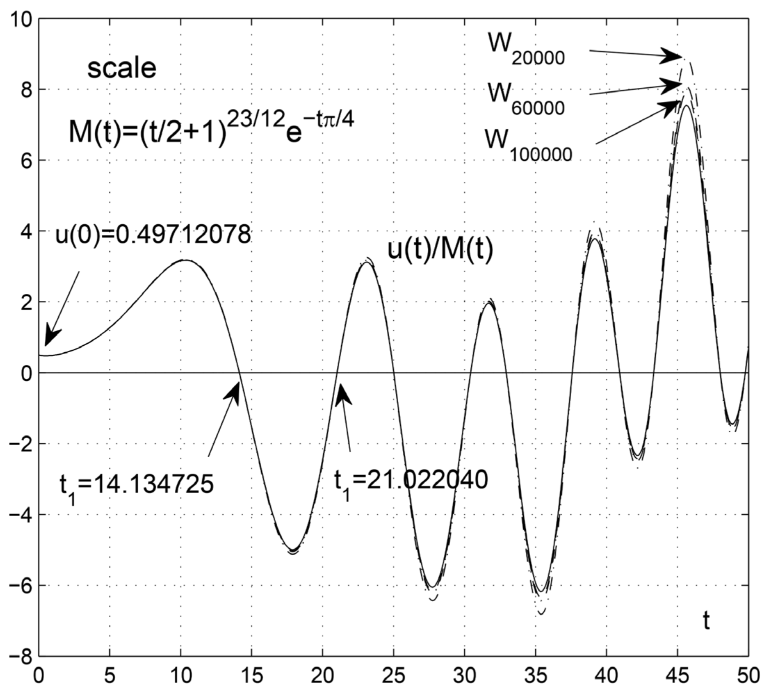

Figure 1. $w_n(t)$ approximates $u(t)$ by $n = 10^5$ zeros t_j.

$$v\left(t_{j+1},\beta\right)=v\left(t_{j+1},0\right)+\int_0^{\beta}u_{\beta}\left(t_{j+1},r\right)\mathrm{d}r=-\int_0^{\beta}u_t\left(t_{j+1},r\right)\mathrm{d}r>0.$$

Because $v(t,\beta)$ has opposite signs at two ends of I_j, there surely exists some inner point $t'=t'(\beta)$ such that $v(t',\beta)=0$. Then $|v(t,\beta)|$ is valley and $\{|u|,|v|\}$ form a peak-valley structure. There is a positive lower bound independent of $t\in I_j$.

$$\min_{t\in I_j}\left(|u(t,\beta)|+|v(t,\beta)|\right)=\mu_j(\beta)>0,\ \beta\in(0,1/2].$$

So RC holds in I_j. Because the zeros $\{t_j\}$ of analytical function $u(t,\beta)$ do not have finite condensation point, then any finite t surely falls in some I_j. RC holds for any t. □

3.2. Defect in Original Proof of Theorem 3

To prove $u(t)$ to be single peak, we discussed monotone growth of the argument $\phi+\psi$ of ξ in [9] and used Riemann's estimate $\psi(t)=\pi S(t)=O(\ln t)$. As $\phi(t)=\frac{t}{2}\left(\ln t-\ln(2e\pi)\right)+\frac{7}{8}\pi+O\left(t^{-1}\right)$ is of super-linear growth, when t increases to $t+1$, the increment $\delta\phi=\frac{1}{2}\left(\ln t-\ln(2\pi)\right)+O\left(t^{-1}\right)$ slowly increases, whereas $\delta\psi=O(1)$, then claimed that $\phi+\psi$ monotone increases. But here $\delta\psi=O(1)$ is not correct, which may be $O(\ln t)$. We consider Riemann's symmetrization

$$Z(t)=e^{i\phi(t)}\zeta(1/2+it),\ \zeta(1/2+it)=U(t)+iV(t),$$

(here the decay factor $|G(s)|\neq 0$ is reduced) and have

$$Z(t)=U\cos\phi-V\sin\phi,\ g(t)=U\sin\phi+V\cos\phi\equiv 0,\ \frac{V}{U}=-\frac{\sin\phi}{\cos\phi}.$$

Thus the local argument $\pi S(t)=\arctan(V/U)=-\phi$ in root-interval $I_j=\left[t_j,t_{j+1}\right]$.

We see in **Figure 2** (also see [6] [10]) that $S(t)$ jumps by 1 at zero t_j' of $\cos\phi$, then linearly decreases in $\left(t_j',t_{j+1}'\right)$ with the slope $\psi'\approx-\frac{1}{2}\ln t$. Whereas ϕ has slope $\phi'(t)\approx\frac{1}{2}\ln t$. It seems $\phi'+\psi'\approx 0$, and difficult to prove $\phi'+\psi'>0$. Besides, as $g(t)=0$, this research is not suitable.

This defect makes us turn to $u(t)$ on critical line and re-investigate RC2.

3.3. Revision Proof of Theorem 3

By RC2, take logarithm $\ln u(t)$ and derivation to get

$$\ln u(t)=\ln u(0)+\sum_{j=1}^{\infty}\ln\left(1-\frac{t^2}{t_j^2}\right),\ \frac{u'(t)}{u(t)}=\sum_{j=1}^{\infty}\frac{2t}{t^2-t_j^2}. \tag{3.1}$$

Consider root-interval $I_n=\left[t_n,t_{n+1}\right]$ and decompose the summation into three parts.

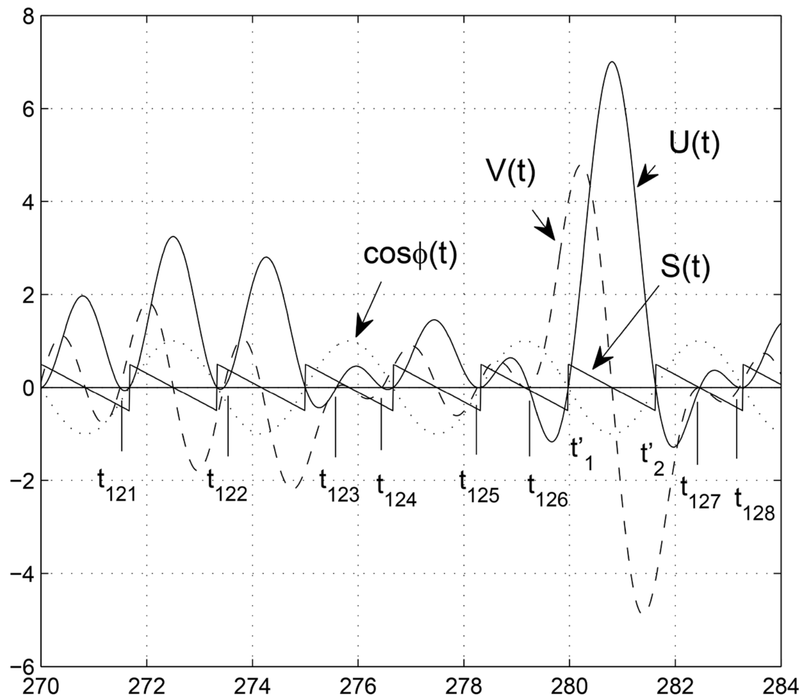

Figure 2. Curves $U, V, S, \cos\phi$. S jumps by 1 twice in $I_{126} = [t_{126}, t_{127}]$.

$$\begin{aligned} u_t(t) &= 2tu(t)\left\{\sum_{j\le n-1}\frac{1}{t^2-t_j^2}+\left(\frac{1}{t^2-t_n^2}-\frac{1}{t_{n+1}^2-t^2}\right)-\sum_{j\ge n+2}\frac{1}{t_j^2-t^2}\right\}, \\ &= 2tu(t)g(t), \quad g(t)=g_1(t)+g_2(t)+g_3(t), \ t_n<t<t_{n+1}. \end{aligned} \tag{3.2}$$

Note that $u(t)$ inside I_n has same sign. We investigate each sum

$$\begin{cases} g_1(t)=\sum\limits_{j\le n-1}\dfrac{1}{t^2-t_j^2}>0, & t>t_n>t_j, \\ g_2(t)=-\sum\limits_{j\ge n+2}\dfrac{1}{t_j^2-t^2}<0, & t<t_{n+1}<t_j. \\ g_3(t)=\dfrac{1}{t^2-t_n^2}-\dfrac{1}{t_{n+1}^2-t^2}>0, & t_n<t<t_{n+1}. \end{cases} \tag{3.3}$$

For $t\in I_n$, $g_1(t)>0, g_2(t)<0$ are finite. If t is close to t_n+0, then $g_3(t)$ tends to $+\infty$. If t is close to $t_{n+1}-0$, then $g_3(t)$ tends to $-\infty$. There surely exists some inner point $t^*\in I_n$ such that $g(t^*)=0$. We show this point t^* is unique. For this we consider their derivatives

$$\begin{cases} g_1'(t)=-\sum\limits_{j\le n-1}\dfrac{2t}{\left(t^2-t_j^2\right)^2}<0, & t>t_n>t_j, \\ g_2'(t)=-\sum\limits_{j\ge n+2}^{\infty}\dfrac{2t}{\left(t_j^2-t^2\right)^2}<0, & t<t_{n+1}<t_j. \\ g_3'(t)=-\left\{\dfrac{2t}{\left(t^2-t_n^2\right)^2}+\dfrac{2t}{\left(t_{n+1}^2-t^2\right)^2}\right\}<0, & t_n<t<t_{n+1}. \end{cases} \tag{3.4}$$

Thus all g_1, g_2, g_3 are monotone decreasing for $t \in I_n$, their sum $g(t)$ does yet. Therefore this zero t^* of $g(t)$ is unique and then $u(t)$ is single peak. This proves theorem 3. □

Using Lagarias's positivity [11] (RC is assumed), we get monotone growth [8], p. 344,

$$\left|\xi(\beta+it)\right| > \left|\xi(\beta_0+it)\right| \ge 0, \text{ for } \beta > \beta_0.$$

This is a clear description for RC. Therefore Riemann ξ-function has mathematical beauty: The symmetry, single peak and monotone growth (*i.e.* the ordering).

3.4. Theorem 2 Holds for *m*-ple Zeros

In large scale computations [6] [12] [13] [14] all zeros of $u(t)$ are single, no multiple zeros are found. People believe there are only single zeros, but so far do not prove. We have to skirt round the difficulty to prove theorem 2. We shall extend theorem 2 (old) as:

Theorem 2. If $u(t)$ has m-ple zeros on critical line, then $\{u, v\}$ for small $\beta > 0$ will bifurcate into m alternative oscillations with single peak, and RC still holds.

Proof. Assume that there are three consecutive zeros $\{t_{j-1}, t_j, t_{j+1}\}$ of $u(t)$ on critical line $\{\beta = 0, 0 < t < \infty\}$, which form two root-intervals, and $t_j = t^*$ is a *m*-ple zero, $m \ge 1$. Denote $H = \min(t_{j+1} - t_j, t_j - t_{j-1})$ and $h = H/2$. Denoting $y = t - t^*$ and the origin $O(y = 0, \beta = 0)$, and fixing a small $\beta > 0$, we discuss a circle $K_\delta(t^*)$ with $r = \sqrt{y^2 + \beta^2} \le \delta \le h$ (δ to be defined). Assume that the real function $g(y) = u(t)$ has *m*-ple zero at O

$$g(y) = a(t^*)y^m + b(t^*)y^{m+1} + \cdots, \ a(t^*) \ne 0, \tag{3.5}$$

These coefficients $a(t^*)$ and $b(t^*)$ are of same order. Thus

$$g(y - i\beta) = a(t^*)\left\{(y - i\beta)^m + K(t^*)(y - i\beta)^{m+1}\right\}, \ K \approx b(t^*)/a(t^*) \ne 0, \tag{3.6}$$

Below we discuss $P(y - i\beta) = (y - i\beta)^m$, and temporarily omit high-order remainder.

Using $y - i\beta = r(y/r - i\beta/r) = r(\cos\phi - i\sin\phi)$ and De Moivre formula we have

$$(y - i\beta)^m = r^m(\cos\phi - i\sin\phi)^m = r^m(\cos(m\phi) - i\sin(m\phi)) = 0,$$

and discuss $r^m\cos(m\phi) = 0$ and $-r^m\sin(m\phi) = 0$ respectively.

1) The zeros of $\cos(m\phi)$ satisfy $m\phi_j = (2j-1)\pi/2$, *i.e.*

$$\phi_j = \frac{2j-1}{2m}\pi, \quad 0 < \phi_j < \pi, \quad j = 1, 2, \cdots, m,$$

which are symmetric with respect to $\pi/2$. If $m = 2n$ even, then all $\phi_j \ne \pi/2$. If $m = 2n-1$ odd, the middle argument $\phi_n = \pi/2$ satisfies $\cos(\phi_n) = 0$, *i.e.* original point $y = 0$.

Taking the roots $y/r = \cos\phi_j$, *i.e.* $y^2 = (y^2+\beta^2)\cos^2\phi_j$, we have $y/\beta = \pm\cos\phi_j/\sin\phi_j = \pm\cot\phi_j$ and re-arrange the ordering of these m zeros as

$$\begin{aligned} &y/\beta = \{-\cot(\phi_1),\cdots,-\cot(\phi_{n-1}),0,\cot(\phi_{n-1}),\cdots,\cot(\phi_1)\}, && \text{if } m=2n-1. \\ &y/\beta = \{-\cot(\phi_1),\cdots,-\cot(\phi_n),\cot(\phi_n),\cdots,\cot(\phi_1)\}, && \text{if } m=2n, \end{aligned} \tag{3.7}$$

2) The zeros of $\sin(m\psi)$ satisfy $m\psi_k = k\pi$, *i.e.*,

$$\psi_k = \frac{k}{m}\pi, \quad k = 1,2,\cdots,m-1,$$

here no $k=m$, as $\psi_m = \pi$ corresponds a trivial zero $\beta/r = \sin(\pi) = 0$ (*i.e.* $\beta = 0$). Thus there are only $m-1$ nontrivial zeros, whose arguments are symmetric with respect to $\pi/2$. When $m=2n$ even, $\psi_n = \pi/2$ for $k=n$, *i.e.* $\sin(\psi_n) = 1$, which corresponds the original $y=0$.

Taking the roots $\beta/r = \sin\psi_k$, *i.e.* $\beta^2 = (y^2+\beta^2)\sin^2\psi_k$, we have $y/\beta = \pm\cos(\psi_k)/\sin(\psi_k) = \pm\cot(\psi_k)$ and re-arrange the ordering of these $m-1$ roots as

$$\begin{aligned} &y/\beta = \{-\cot(\psi_1),\cdots,-\cot(\psi_{n-1}),\cot(\psi_{n-1}),\cdots,\cot(\psi_1)\}, && \text{if } m=2n-1, \\ &y/\beta = \{-\cot(\psi_1),\cdots,-\cot(\psi_{n-1}),0,\cot(\psi_{n-1}),\cdots,\cot(\psi_1)\}, && \text{if } m=2n. \end{aligned} \tag{3.8}$$

Comparing (3.7) and (3,8) we see that for fixing $\beta > 0$, the maximum of these roots is $|y| = \beta\cot(\phi_1) = \beta\cot(\pi/2m)$, thus the radius of circle $K_\delta(t^*)$ satisfies $r = (y^2+\beta^2)^{1/2} = \beta/\sin(\pi/2m) \le h$, *i.e.* we should confine $\beta \le h\sin(\pi/2m) < h\pi/2m = Hh\pi/4m = d$.

For $g(y) = a(t^*)(y-i\beta)^m = u_1 + iv_1$, we have the following conclusions:

1) The real part $u_1(y,\beta)$ has m zeros $y/\beta = \cot\phi_j$, and the imaginary part $v_1(y,\beta)$ has $m-1$ zeros $y/\beta = \cot\psi_k$. Due to

$$\phi_j = \frac{2j-1}{2m}\pi < \psi_j = \frac{j}{m}\pi,\ j = 1,2,\cdots,n-1,$$

obviously $\cot\phi_j > \cot\psi_j$. Hence all zeros of (3.7) and (3.8) are alternatively arranged.

2) At zeros $\psi_j = j\pi/m$ of v_1, the peak values $u_1 = a(t^*)r^m\cos(j\pi) = a(t^*)r^m(-1)^j$ alternatively change their signs. At zeros $\phi_j = (j-1/2)\pi/m$ of u_1, the peak values $v_1 = a(t^*)r^m\sin((j-1/2)\pi) = a(t^*)r^m(-1)^{j-1}$ also alternatively change their signs.

3) Because $v_1 \ne 0$ at zero of u_1 and $u_1 \ne 0$ at zero of v_1, they all are single peak. Thus in the m root-intervals of u_1, all $\{|u_1|,|v_1|\}$ form local peak-valley structures, and RH locally holds.

Finally, in $K_\delta(t^*)$ with $\beta \in (0,\delta]$ suitably small we discuss a general case

$$g(y-i\beta) = a(t^*)\{(y-i\beta)^m + R_m\},\ R_m = O(r^{m+1}),\ r \le \delta, \tag{3.9}$$

The actual zeros of $\xi(\tau)$ are small perturbations of these zeros mentioned above, which do not change these m peak-valley structures in $K_\delta(t^*)$. When β increases, they will continue to develop toward locally convex direction by

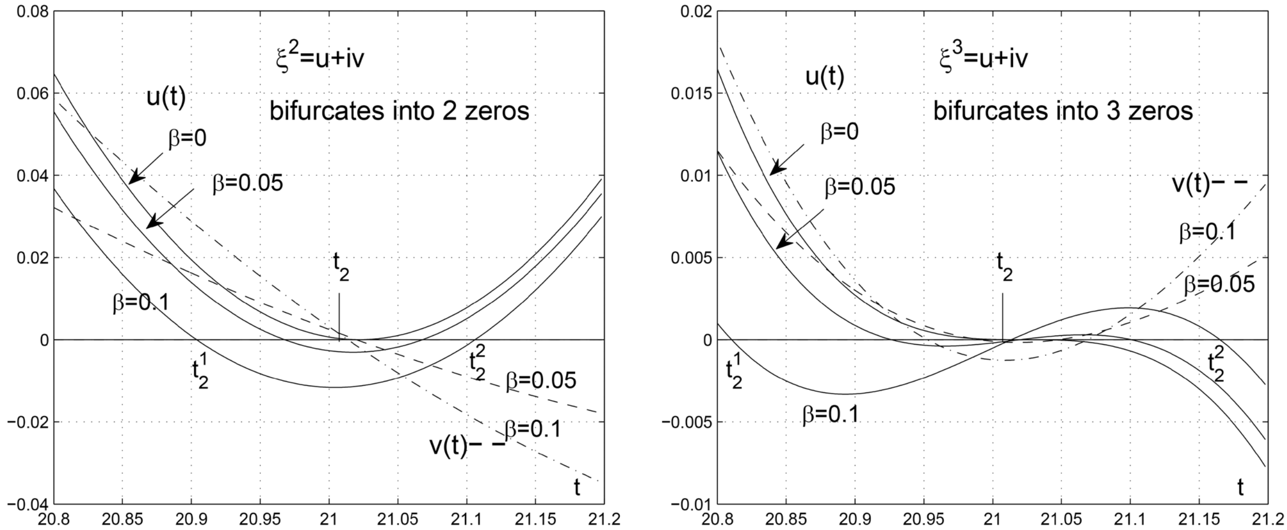

Figure 3. $Re(\xi^2), Re(\xi^3)$ at zero point t_2 bifurcate into 2 or 3 zeros.

theorem 1, so RC holds. Hence theorem 2 is proved. □

Numerical experiments 2. With the suggestion of Dr.XM Jiao, we have computed $(\xi)^m = u + iv, m = 2,3$ at the second zero $t_2 = 21.0220$ of $\xi(s)$. We see in Figure 3 for $\beta = 0.05, 0.1$, $u(t,\beta)$ indeed bifurcate into m curves of single peak, and $\{u,v\}$ are alternative oscillation. The peak of $u(t,\beta)$ develops toward its convex direction.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Bombieri, E. (2000) Problems of the Millennium: The Riemann Hypothesis. *AMS*, 107-124.
https://www.claymath.org/sites/default/files/official_problem_description.pdf

[2] Conrey, J. (2003) The Riemann Hypothesis. *Notices of the AMS*, **50**, 341-353.

[3] Sarnak, P. (2004) Problems of the Millennium: The Riemann Hypothesis. http://xcivzuj.claymath.org/sites/default/files/sarnak_rh_0.pdf

[4] Edwards, H.M. (2001) Riemann's Zeta Function. Dover Publication Inc., Mineola.

[5] Borwein, P., Choi, S., Rooney, B. and Weirathmuller, A. (2008) The Riemann Hypothesis. Springer, New York.

[6] Odlyzko, A. (n.d.) Tables of Zeros of the Riemann Zeta Function. URL. http://www.dtc.umn.edu/~odlyzko/

[7] Chen, C.M. (2020) The Symmetry of Riemann ξ-Function. *Advances in Pure Mathematics*, **10**, 464-470. https://doi.org/10.4236/apm.2020.108028

[8] Chen, C.M. (2020) Local Geometric Proof of Riemann Conjecture. *Advances in Pure Mathematics*, **10**, 589-610. https://doi.org/10.4236/apm.2020.1010036

[9] Chen, C.M. (2021) Geometric Proof of Riemann Conjecture. *Advances in Pure Mathematics*, **11**, 334-345. https://doi.org/10.4236/apm.2021.114021

[10] Riemann Hypothesis. Wikipedia. https://en.wikipedia.org/w/index.php?title=Riemann_hypothesis&oldid=1025138468

[11] Lagarias, J. (1999) On a Positivity Property of the Riemann ξ-Function. *Acta Arithmetica*, **89**, 213-234. https://doi.org/10.4064/aa-89-3-217-234

[12] Lune, J. and Riele, H. (1983) On the Zeros of the Riemann Zeta Function in the Critical Strip. *Mathematics of Computation*, **41**, 759-767. https://doi.org/10.1090/S0025-5718-1983-0717719-3

[13] Lune, J., Riele, H. and Winter, D. (1986) On the Zeros of the Riemann Zeta Function in the Critical Strip. *Mathematics of Computation*, **46**, 667-681. https://doi.org/10.1090/S0025-5718-1986-0829637-3

[14] Platt, D. and Trudgian, J. (2021) The Riemann Hypothesis is True up to $3\cdot 10^{12}$. *Bulletin of the London Mathematical Society*, **53**, 792-797. https://doi.org/10.1112/blms.12460

[15] Chen, C.M. and Hu, S.F. (2013) The Highest Order Superconvergence for bi-k Degree Rectangular Elements at Nodes: A Proof of 2k-Conjecture. *Mathematics of Computation*. **82**, 1337-1355. https://doi.org/10.1090/S0025-5718-2012-02653-6

[16] Schleich, W.P., Bezdekova, I., Kim, M.B., Abbott, P.C., Maler, H., Montgomery, H.L. and Neuberger, J.W. (2018) Equivalent Formations of the Riemann Hypothesis Based on Lines of Constant Phase. *Physica Scripta*, **93**, Article No. 065201. https://doi.org/10.1088/1402-4896/aabca9