Open Journal of Statistics, 2021, 11, 506-523
https://www.scirp.org/journal/ojs
ISSN Online: 2161-7198
ISSN Print: 2161-718X

Positive Stable Frailty Approach in the Construction of Dependence Life-Tables

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How to cite this paper: Walter, O., Patrick, W., Otieno, J. and Carolyne, O. (2021) Positive Stable Frailty Approach in the Construction of Dependence Life-Tables. *Open Journal of Statistics*, **11**, 506-523.
https://doi.org/10.4236/ojs.2021.114032

Received: March 15, 2021
Accepted: August 13, 2021
Published: August 16, 2021

Abstract

Dependence may arise in insurance when the insureds are clustered into groups e.g. joint-life annuities. This dependence may be produced by sharing a common risk acting on mortality of members of the group. Various dependence models have been considered in literature; however, the focus has been on either the lower-tail dependence alone or upper-tail dependence alone. This article implements the frailty dependence approach to life insurance problems where most applications have been within medical setting. Our strategy is to use the conditional independence assumption given an observed association measure in a positive stable frailty approach to account for both lower and upper-tail dependence. The model is calibrated on the association of Kenyan insurers 2010 male and female published rates. The positive stable model is then proposed to construct dependence life-tables and generate life annuity payment streams in the competitive Kenyan market.

Keywords

Joint-Life Annuity, Life-Table Functions, Shared Frailty Model, Positive Stable Distribution, Bayesian Inference

1. Introduction and Motivation

When annuity payments are concerned, the calculation of expected present values (EPVs), needed in pricing and reserving, requires an appropriate mortality model in order to avoid biased valuations [1] [2] [3]. Frailty models are used in life insurance to represent heterogeneity in a population due to unreported risk factors [4] [5]. Heterogeneity due to reported risk factors is addressed at policy issue during the underwriting process. There has been a growing literature on dependence mortality modeling in life insurance in recent years (see, e.g., [6] [7]

[8]) where the focus has been on either the lower-tail dependence alone or upper-tail dependence alone. Frailty dependence modeling [9] [10] [11] [12] is an approach that accounts for dependence in event times of related individuals. Clustered survival times are assumed to be conditional independent with respect to the shared risk. Our strategy is to use the conditional independence assumption to account for lower and upper-tail dependence for a given association measure. Dependence frailty models have been used by several authors with applications in medical field (see e.g., [13]). This article implements the dependence frailty approach in insurance risk management setting.

Our article contributes to the existing literature in several ways. First, we apply a shared frailty approach to life insurance risk problems where most applications have been within medical setting. Second, most dependence models in literature have focused on either the lower-tail dependence alone or upper-tail dependence alone. We apply the conditional independence assumption for an observed association measure in a positive stable frailty to account for both lower and upper-tail dependence. Third, we incorporate a stochastic dependence structure via a dynamically evolving positive stable process to model time-varying shared risk.

2. Notations, Assumptions and Data

2.1. Notations

To facilitate an easier discussion of frailty dependence modeling, the following notations are used. We consider joint-life annuity contracts where x is the age of the male annuitant whose future lifetime random variable is t_{i1} and y the age of the female annuitant whose future lifetime random variable is t_{i1}. v^t is the present value factor, $s_{\overline{xy}}(t)$ is the probability of survival until the last of (x, y) dies and $a_{\overline{xy}}$ the EPV of the benefit. $D_{xy} = v^{xy} l_{xy}$ and $N_{xy} = \sum_{k=0}^{\infty} D_{xy+k}$ are commutation functions.

2.2. Assumptions

For simplicity of notation of the commutation functions, we suppose that the limiting age of our mortality table is infinity.

The force of mortality $\lambda_x(t)$ for a life aged x during time t is assumed piece-wise constant across each whole year of age $[t, t+1)$. *i.e.*
$\lambda_x(t) = \lambda_{x+z}(t); 0 < z < 1$. Similar assumptions are found in [14].

A deterministic financial structure is adopted for the present value factor v^t for illustration purposes.

Model calibration with reference to standard mortality tables [5] [15] [16] is applied due to limited joint-life mortality data-set that is available in the Kenyan annuity market.

2.3. Data

The association of Kenyan insurers (AKI) 2010 published mortality tables is based upon data collected by the association of Kenya insurers for an investigation into the mortality of assured lives in the Republic of Kenya. The AKI 2010 mortality rates are used as the baseline hazard rates in the study. Joint-life and last survivor annuitants member data for policies in-force between 2001-2013 will be used in the study to determine the average age difference for insured couples in Kenya. The annuitants data used in this article was obtained from a major Kenyan insurance company. To preserve confidentiality, we took a sub-sample of 178 joint-life policies.

3. Materials and Methods

3.1. Joint-Life Last Survivor Annuity

The proposed model can be applied to any type of joint-life annuity business. In this article, we discuss the case of joint-life last survivor annuities. This is a contract that provides level payments to two or more annuitants until the last of them dies. This can be an immediate annuity for a single lump-sum payment whose expected present value (EPV) of say amount b per annum, payable in arrears, until the last of (x, y) dies is given by:

$$ba_{\overline{xy}} = b\sum_{t=1}^{\infty} v^t s_{\overline{xy}}(t) \tag{1}$$

where the future lifetime random variable $T = Max(t_{i1}, t_{i2})$.

Under the dependence (frailty) assumption:

$$s_{\overline{xy}}(t) = s_x(t) + s_y(t) - s_{xy}(t) \tag{2}$$

Assuming independence we have:

$$s_{\overline{xy}}(t) = s_x(t) + s_y(t) - s_x(t) \cdot s_y(t) \tag{3}$$

The EPV in Equation (1) becomes:

$$b\sum_{t=1}^{\infty} v^t \left[\exp\left\{-\int_0^t \lambda_{x+s}\mathrm{d}s\right\} + \exp\left\{-\int_0^t \lambda_{y+s}\mathrm{d}s\right\} - \exp\left\{-\int_0^t \left(\lambda_{xy+s}\right)\mathrm{d}s\right\}\right] \tag{4}$$

$$b\sum_{t=1}^{\infty} v^t \left[\exp\left\{-\int_0^t \lambda_{x+s}\mathrm{d}s\right\} + \exp\left\{-\int_0^t \lambda_{y+s}\mathrm{d}s\right\} - \exp\left\{-\int_0^t \left(\lambda_{x+s} + \lambda_{y+s}\right)\mathrm{d}s\right\}\right] \tag{5}$$

respectively. Here again for simplicity of notation, we suppose that the limiting age of our mortality table is infinity. If the purchase price for the annuity is p then the future level payment stream b applying the traditional equivalence principle is:

$$b = \frac{p}{a_{\overline{xy}}} \tag{6}$$

The prevailing traditional insurance practice assumes independence in pricing joint-life annuity contracts thereby adopting the EPV shown in Equation (5) whose joint mortality is obtained by summing up the individual mortality rates. Frailty dependence models focus on modeling the EPV as shown in Equation (4) whose joint mortality accounts for heterogeneity and dependence.

3.2. Shared Frailty Model

Definition 1. *The hazard function for a shared frailty model is given by;*

$$\lambda_{ij}(t \mid \Omega_i) = \Omega_i \lambda_0(t); t > 0 \tag{7}$$

where $\lambda_{ij}(t \mid \Omega_i)$ *is the conditional hazard function for the* j^{th} *individual in the* i^{th} *group and* Ω_i *is the shared random effect associated with the* i^{th} *group.* $\lambda_0(t)$ *is the population's base force of mortality.*

Formally, the expression of the bivariate net survival function is summarized in the following proposition.

Proposition 1. *Under the assumption of independent future life-times for a given shared frailty the bivariate net survival function is;*

$$s(t_{i1}, t_{i2}) = L_{\Omega_i}\left(\Lambda_{01}(t_{i1}) + \Lambda_{02}(t_{i2})\right) \tag{8}$$

Proof. The bivariate conditional survival function for a given shared frailty Ω_i at time $t_{i1} > 0, t_{i2} > 0$ is given by; $s(t_{i1}, t_{i2} \mid \Omega_i) = s(t_{i1} \mid \Omega_i) s(t_{i2} \mid \Omega_i)$ and from $s(t \mid \Omega) = \exp\left\{-\int_0^t \lambda(u \mid \Omega) \mathrm{d}u\right\} = \exp(-\Omega \Lambda_0(t))$ we have that
$s(t_{i1}, t_{i2} \mid \Omega_i) = \exp\left\{-\Omega\left[\Lambda_{01}(t_{i1}) + \Lambda_{02}(t_{i2})\right]\right\}$ using expectation
$s(t_{i1}, t_{i2}) = E\left[\exp\left\{-\Omega\left[\Lambda_{01}(t_{i1}) + \Lambda_{02}(t_{i2})\right]\right\}\right]$ this simplifies to
$s(t_{i1}, t_{i2}) = L_{\Omega_i}\left(\Lambda_{01}(t_{i1}) + \Lambda_{02}(t_{i2})\right)$ □

3.3. Shared Frailty and Archimedean Copula Approach

Copulas have been studied in actuarial science to model joint-life survival functions [17]. Similarity between the frailty and copula dependence approach is discussed with reference to the elliptical or Archimedean dependence copula. The family of Archimedean copula [18] [19] in the bivariate case is described by reference to a generator function.

$$C_\phi(u, w) = p\{q(u) + q(w)\}$$

where the generator function $p(.)$ is any non-negative decreasing function and non-negative second derivative with $p(0) = 1$ and $q(.)$ its pseudo-inverse function. A special case showing similarity to shared frailty approach is when $p(s) = L_\Omega(s)$ where Ω is the frailty random variable and $u = s(t_{i1}), w = s(t_{i2})$ this leads to:

$$C_\phi\{s(t_{i1}), s(t_{i2})\} = L_\Omega\left[L^{-1}(s(t_{i1})) + L^{-1}(s(t_{i2}))\right]$$

Comparing this with the marginal survival from shared frailty approach *i.e.* $s(t_{i1}) = L_\Omega(\Lambda_0(t_{i1}))$ and therefore $L_\Omega^{-1}(s(t_{i1})) = \Lambda_0(t_{i1})$ shows that

$$\begin{aligned} C_\phi\{s(t_{i1}), s(t_{i2})\} &= L_\Omega\left[L^{-1}(s(t_{i1})) + L^{-1}(s(t_{i2}))\right] \\ &= L_\Omega\left(\Lambda_0(t_{i1}) + \Lambda_0(t_{i2})\right) = s(t_{i1}, t_{i2}) \end{aligned}$$

If $p(.)$ is the Laplace transform of a gamma distribution with scale parameter 1, then the Clayton copula model is obtained. Similarly, the Gumbel copula is obtained if $p(.)$ has a positive stable Laplace though the estimation strategies and association measures differ with the frailty approach. Whereas in the copula approach, the marginal survivor functions and the dependence structure have to be specified [20] [21] for the joint survivor to be constructed. In frailty models, the dependence structure is introduced indirectly.

3.4. The Positive Stable Frailty Model

The probability density function (p.d.f) for the positive stable distribution in closed-form is given by;

$$f(\Omega) = -\frac{1}{\pi\Omega}\sum_{n=1}^{\infty}\frac{\Gamma(nr+1)}{n!}\left(-\Omega^{-r}k/r\right)^{n}\sin(rn\pi); k>0, \Omega>0, 0<r\le 1 \quad (9)$$

For identifiability we assume $k = r$ (see [10] for proof), then we have the standard case with only one parameter *r*.

$$f(\Omega) = -\frac{1}{\pi\Omega}\sum_{n=1}^{\infty}\frac{\Gamma(nr+1)}{n!}\left(-\Omega^{-r}\right)^{n}\sin(rn\pi); \Omega>0, 0<r\le 1$$

Proposition 2. *The Laplace transform is a special case of the Power Variance Family* (r,k,η) *Laplace given by*:

$$L_{\Omega}(s) = \exp\left\{-\frac{k}{r}s^{r}\right\} \quad (10)$$

As indicated earlier for identifiability reasons we let $k = r$.

$$L_{\Omega}(s) = \exp\left(-s^{r}\right), 0<r\le 1 \quad (11)$$

The proposed frailty distribution has many advantages. First, it is easy to implement due to the simplified Laplace transform shown in Equation (11). Second, the positive stable has an infinite mean and variance. This allows for a much higher degree of heterogeneity to be accounted for that would not be possible by using a frailty distribution with finite variance. Third, the positive stable distribution is infinitely divisible, allowing the splitting of the shared risk into cause specific risks which may be easier to interpret. The net bivariate survival, density and hazard functions at time $t_{i1}>0, t_{i2}>0$ are:

$$s(t_{i1},t_{i2}) = \exp\left\{-\left(\Lambda_0(t_{i1})+\Lambda_0(t_{i2})\right)^{r}\right\} \quad (12)$$

$$\begin{aligned} f(t_{i1},t_{i2}) = s(t_{i1},t_{i2})\cdot\lambda_0(t_{i1})\lambda_0(t_{i2})\Big[& r^2\left(\Lambda_0(t_{i1})+\Lambda_0(t_{i2})\right)^{2r-2} \\ & -r(r-1)\left(\Lambda_0(t_{i1})+\Lambda_0(t_{i2})\right)^{r-2}\Big] \end{aligned} \quad (13)$$

$$\begin{aligned} \lambda(t_{i1},t_{i2}) = \lambda_0(t_{i1})\lambda_0(t_{i2})\Big[& r^2\left(\Lambda_0(t_{i1})+\Lambda_0(t_{i2})\right)^{2r-2} \\ & -r(r-1)\left(\Lambda_0(t_{i1})+\Lambda_0(t_{i2})\right)^{r-2}\Big] \end{aligned} \quad (14)$$

In dependence frailty models, the frailty distribution is identifiable through the [9] [22] cross-ratio function, which describes how association of the bivariate hazards evolves over time. The cross ratio measure $A(t_{i1},t_{i2})$ for the first life if the second life has experienced the event rather than being event free at a given time is given by;

$$A(t_{i1},t_{i2}) = \frac{s(t_{i1},t_{i2})\frac{\partial^2}{\partial t_{i1}\partial t_{i2}}s(t_{i1},t_{i2})}{\frac{\partial}{\partial t_{i1}}s(t_{i1},t_{i2})\frac{\partial}{\partial t_{i2}}s(t_{i1},t_{i2})} \tag{15}$$

Using the positive stable as frailty distribution, the cross ratio function from Equation (15) becomes:

$$A(t_{i1},t_{i2}) = 1-\left(1-\frac{1}{r}\right)\left(\Lambda_0(t_{i1})+\Lambda_0(t_{i2})\right)^{-r} \tag{16}$$

From Equation (16), values of r close to zero indicate high association between t_{i1} and t_{i2} because $A(t_{i1},t_{i2})$ takes values greater than 1, r close to one indicate low association between t_{i1} and t_{i2} since $A(t_{i1},t_{i2})$ takes values near 1 while $r=1$ corresponds to independence *i.e.* $A(t_{i1},t_{i2})=1$.

We present below four examples with specific baseline distributions to find the frailty dependence hazard functions with explicit expressions.

Example 1.

Let $\lambda_0(t)$ follow a Weibull(a,τ) distribution with p.d.f $f_0(t)=\tau a t^{\tau-1}\exp(-at^{\tau}); a,\tau>0; t\geq 0$ where τ is the shape parameter and a the scale parameter.

Then the survival, hazard and cumulative hazard functions are;

1) $s_0(t)=\exp(-at^{\tau}), t>0.$
2) $\lambda_0(t)=\tau a t^{\tau-1}, t>0.$
3) $\Lambda_0(t)=at^{\tau}, t>0.$

From Equation (14) the positive stable Weibull (PSW henceforth) frailty dependence hazard is described explicitly as:

$$\lambda(t_{i1},t_{i2}) = a_1\tau_1 t_{i1}^{\tau_1-1}\cdot a_2\tau_2 t_{i2}^{\tau_2-1}\cdot\left[r^2\left(a_1t_{i1}^{\tau_1}+a_2t_{i2}^{\tau_2}\right)^{2r-2}-r(r-1)\left(a_1t_{i1}^{\tau_1}+a_2t_{i2}^{\tau_2}\right)^{r-2}\right] \tag{17}$$

The Weibull distribution is widely used in the analysis of lifetime data because it is flexible enough to account for an increasing ($\tau>1$), decreasing ($\tau<1$) or constant ($\tau=1$) hazard rate. Further, the law of Weibull is useful in mortality models for annuitants see e.g. [16].

Example 2.

Let $\lambda_0(t)$ follow a Lognormal(μ,σ^2) distribution with parameters μ,σ the p.d.f is given by;

$$f_0(t)=\frac{1}{\sigma t\sqrt{2\pi}}\mathrm{e}^{\frac{-(\ln t-\mu)^2}{2\sigma^2}}; t,\sigma>0, -\infty<\mu<\infty$$

Then the survival, hazard and cumulative hazard functions are:

1) $s_0(t)=1-\Phi\left(\frac{\ln t-\mu}{\sigma}\right), t>0.$
2) $\lambda_0(t)=\frac{f_0(t)}{1-\Phi\left(\frac{\ln t-\mu}{\sigma}\right)}, t>0.$
3) $\Lambda_0(t)=-\ln\left(1-\Phi\left(\frac{\ln t-\mu}{\sigma}\right)\right), t>0.$

From Equation (14) the positive stable Lognormal frailty dependence hazard is:

$$\lambda(t_{i1},t_{i2})=\frac{f_0(t_{i1})}{1-\Phi\left(\frac{\ln t_{i1}-\mu_1}{\sigma_1}\right)}\cdot\frac{f_0(t_{i2})}{1-\Phi\left(\frac{\ln t_{i2}-\mu_2}{\sigma_2}\right)}$$
$$\cdot\left[r^2\left(-\ln\left(1-\Phi\left(\frac{\ln t_{i1}-\mu_1}{\sigma_1}\right)\right)-\ln\left(1-\Phi\left(\frac{\ln t_{i2}-\mu_2}{\sigma_2}\right)\right)\right)^{2r-2}\right.$$
$$\left.-r(r-1)\left(-\ln\left(1-\Phi\left(\frac{\ln t_{i1}-\mu_1}{\sigma_1}\right)\right)-\ln\left(1-\Phi\left(\frac{\ln t_{i2}-\mu_2}{\sigma_2}\right)\right)\right)^{r-2}\right] \tag{18}$$

The Lognormal is also used in modeling failure time data because it can take various unimodal shapes *i.e.* bathtub-shaped or hump-shaped.

Example 3.

Let $\lambda_0(t)$ follow a Gamma(p,φ) with p.d.f

$$f_0(t)=\frac{\varphi^p t^{p-1}\exp(-\varphi t)}{\Gamma(p)};t>0,\varphi>0,p>0$$

Then the survival, hazard and cumulative hazard functions are:

1) $s_0(t)=\frac{\gamma(p,\varphi t)}{\Gamma(p)},t>0.$

2) $\lambda_0(t)=\frac{\varphi^p t^{p-1}\exp(-\varphi t)}{\gamma(p,\varphi t)},t>0.$

3) $\Lambda_0(t)=-\ln\left(\frac{\gamma(p,\varphi t)}{\Gamma(p)}\right),t>0.$

The Gamma is widely used in survival analysis to generate mixtures in exponential and Poisson models. It has positive support and is also a good choice for the baseline hazard.

From Equation (14) the positive stable Gamma frailty dependence hazard is described explicitly as:

$$\lambda(t_{i1},t_{i2})=\frac{\varphi_1^{p_1}t_{i1}^{p_1-1}\exp(-\varphi_1 t_{i1})}{\gamma(p_1,\varphi_1 t_{i1})}\cdot\frac{\varphi_2^{p_2}t_{i2}^{p_2-1}\exp(-\varphi_2 t_{i2})}{\gamma(p_2,\varphi_2 t_{i2})}$$
$$\cdot\left[r^2\left(-\ln\left(\frac{\gamma(p_1,\varphi_1 t_{i1})}{\Gamma(p_1)}\right)-\ln\left(\frac{\gamma(p_2,\varphi_2 t)}{\Gamma(p_2)}\right)\right)^{2r-2}\right.$$
$$\left.-r(r-1)\left(-\ln\left(\frac{\gamma(p_1,\varphi_1 t_{i1})}{\Gamma(p_1)}\right)-\ln\left(\frac{\gamma(p_2,\varphi_2 t)}{\Gamma(p_2)}\right)\right)^{r-2}\right] \tag{19}$$

3.5. Assessment of Model Selection Criteria

The performance of the model selection criteria for the Bayesian estimation technique is validated in a comparative study with the traditional MLE method. In the Bayesian method the deviance information criteria (DIC), Bayesian Information Criterion (BIC) and Akaike Information Criterion (AIC) is applied whereas in the MLE method the Standard Error information is used.

1) $\text{DIC} = \bar{D} + pD$ where: $\bar{D}$ is the posterior mean of $-2\log L$ measuring the quality of the goodness-of-fit of the considered model to the data.

$\hat{D} = -2\log L$ is the posterior mean of stochastic nodes and $pD = \bar{D} - \hat{D}$ is the effective number of parameter. Smaller values of DIC indicate better models and could give negative values.

2) $\text{AIC} = \hat{D} + 2p$ where: p = number of parameters of the model.

3) $\text{BIC} = \hat{D} + p \times \log(n)$ where: p = number of parameters of the model and n = sample size. The advantage of the BIC is that it includes the BIC penalty for the number of parameters being estimated.

Bayesian Analysis

The Bayesian method treats all unknown parameters as random variables in a statistical model and derives their distribution conditional upon known information. This method has been applied in actuarial modeling e.g. by [23] in analysis of simultaneous equations for insurance rate making and by [24] to analyze time-varying dependent data with possible variance shifts. The Bayesian parameter estimation strategy is implemented in the following algorithm run in OpenBUGS: First, the proposal distributions for the likelihood are specified as Weibull(τ, a), Lognormal(μ, σ^2) and Gamma(p, φ) respectively. Since we do not have prior information about baseline parameters non-informative prior distribution is picked and assumed to be flat. *i.e.* gamma distributed random variables with mean 1 and variance 10,000 for positive parameters and normal distribution of mean 1 and variance 10,000 for parameter that can take on positive or negative values. Similar approach is found in [13].

The hyperparameters of initial values are chosen to be MLE estimates determined outside of OpenBUGS using standard techniques e.g. for the Weibull $\tau_1 = 0.7, a_1 = 7$. The actual data to be estimated by the model is specified to be the males and females densities obtained from the AKI 2010 mortality data through standard numerical approximations. Parameters are estimated considering only the range of ages [55, 109]. Burn in period is set at 2000 as per the BGR plot to ensure sequence of draws from the posterior distribution have minimal autocorrelation and can be found by taking values from a single run of the Markov chain. This diminishes the effect of the starting distribution. We run 3 chains in parallel and after 10,000 iterations convergence will be monitored and if stationarity has been achieved (implying estimates are not dependent on the prior distributions) the mean posterior distribution will be picked as a point estimate. Models with smaller values of the DIC, BIC or AIC are preferred.

The WinBUGS codes used to analyze the dataset using Weibull are available upon request.

Brooks-Gelman-Rubin Diagnostic and Trace Plots

The BGR convergence diagnostic plots for the monitored nodes are presented in **Figure 1**. As the MCMC simulation progresses, the values of the total-sequence (green curve) and mean within-sequence interval width (blue curve) estimates are monitored. Their ratio (red curve) is seen to converge to one beyond 2000 iterations hence a probable choice for the burn-in period. The dynamic trace

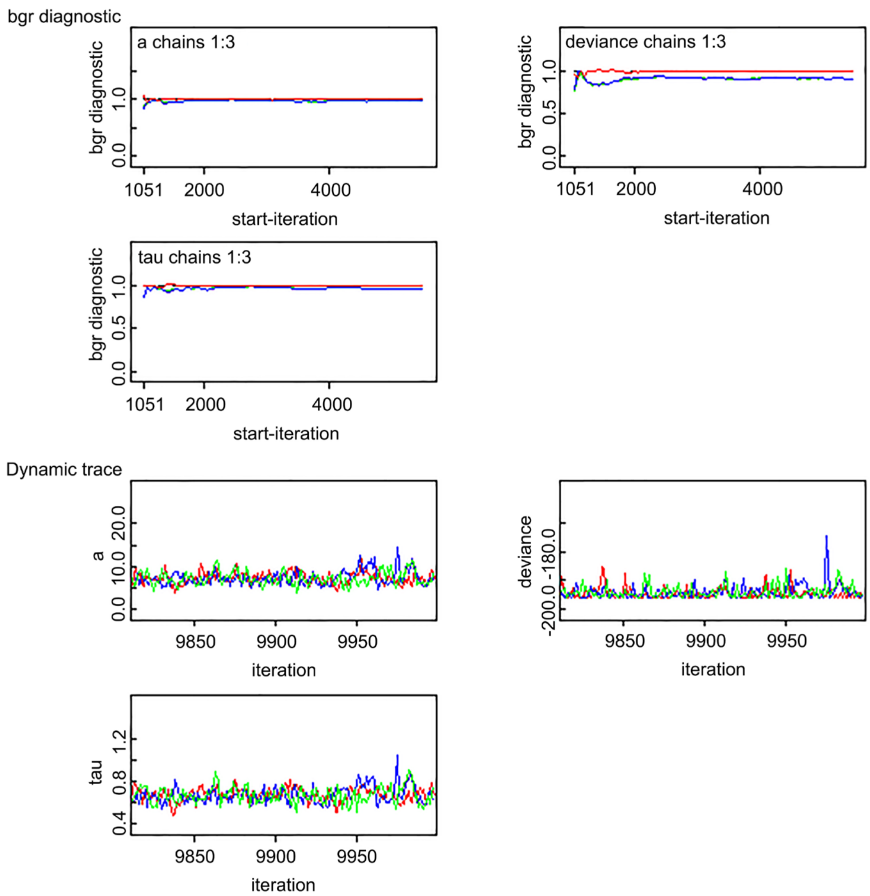

Figure 1. BGR diagnostic plot consistent with convergence and dynamic trace plots.

plots also monitored in **Figure 1** is shown to be mean-reverting and the chains appear to mix freely implying stationarity has been achieved.

Comparison with MLE

The MLE approach is concerned with obtaining parameter values say, (τ, a) that maximizes the probability of observing the data D given those parameters, $p(D \mid \tau, a)$. The likelihood function gives the probability of the observed sample generated by the model. Generally, maximization of the likelihood function to find the ML estimates is done algebraically, but can be computational intensive. In this article, the MLE algorithm is implemented using *MASS* package run in *R*. The output is given in **Table 2** using the same distribution parametrization as in the Bayesian approach **Table 1**.

Discussion

On the basis of Bayesian the Weibull distribution is chosen since the DIC, BIC and AIC are smallest compared to the other distributions. Also using the MLE approach the Weibull is also chosen because the Standard Errors is smallest compared to the other distributions. It is interesting to note from **Table 1** and **Table 2** that the parameter estimates produced by the two competing approaches agree with minimal deviations. So it may be claimed that Bayesian methods are the potential candidates to be used in any inferential procedures allowing for

Table 1. Bayesian parameter estimates, DIC, BIC and AIC values.

Baseline Model	Parameter Estimates	DIC	BIC	AIC
Weibull	$\tau_1 = 0.67$, $a_1 = 7.15$ (male)	−192.2	−192.82	−192.2
	$\tau_2 = 0.7555$, $a_2 = 10.17$ (female)	−208.9	−209.52	−209.0
Lognormal	$\mu_1 = -3.711$, $\sigma_1^2 = 2.3568$ (male)	−97.22	−95.75	−95.24
	$\mu_2 = -3.727$, $\sigma_2^2 = 1.6697$ (female)	−101.1	−99.52	−99.0
Gamma	$p_1 = 7.746$, $\varphi_1 = 0.5645$ (male)	−188.6	−189.22	−188.7
	$p_2 = 11.89$, $\varphi_2 = 0.6853$ (female)	−205.5	−206.02	−205.5

Table 2. MLE parameter estimates and standard error values.

Baseline Model	Parameter Estimates	Standard errors
Weibull	$\tau_1 = 0.6720$, $a_1 = 7.1346$ (male)	(0.0687, 0.0114)
	$\tau_2 = 0.7558$, $a_2 = 10.0388$ (female)	(0.0753, 0.0090)
Lognormal	$\mu_1 = -3.7097$, $\sigma_1^2 = 2.31248$ (male)	(0.2050, 0.1449)
	$\mu_2 = -3.726$, $\sigma_2^2 = 1.63809$ (female)	(0.1726, 0.1220)
Gamma	$p_1 = 7.830$, $\varphi_1 = 0.571$ (male)	(0.0909, 1.8737)
	$p_2 = 12.0341$, $\varphi_2 = 0.6935$ (female)	(0.1126, 2.7592)

informative priors shared by field experts. The parameter estimates used in the study are as shown in Table 1, following the implementation of Bayesian described.

Goodness of Fit Test

A chi-square goodness-of-fit test of the data for Weibull baseline distribution is as shown in Table 3. As observed in Table 3, the chi-squared p-value is greater than 5%. Thus, we fail to reject the null hypothesis that the AKI data follow a Weibull distribution at 5% level of significance. Similarly, the Kolmogorov-Smirnov (KS) hypothesis test in Table 4 between the empirical distribution function and the fitted distribution function shows a p-value greater than 0.05. We therefore fail to reject the null hypothesis that the AKI data follow a Weibull distribution at 5% level of significance. The Weibull Q-Q Plot in Figure 2 further shows a straight line through a majority of the quantiles this also shows that the Weibull provides a good fit. We can therefore conclude that the proposed distribution is a good fit for the data.

4. Model Calibration to the AKI 2010 Male and Female Published Rates

The PSW dependence model given in Equation (17) is shown in Figure 3 where $\tau_1 = 0.67, a_1 = 7.15$; $\tau_2 = 0.7555, a_2 = 10.17$ obtained from the Bayesian estimation procedure.

Inspired by [3] and for illustration purpose, we discuss the empirical results for the patterns of Equation (17) with different degrees of dependence *r*. We

Table 3. Goodness of fit of Weibull to the AKI data.

Name	Value
Chi-squared statistic	2970
Degree of freedom	2916
Chi-squared p-value	0.2384

Table 4. Goodness of fit using K-S test.

Name	p-value	Test Statistic
Kolmogorov-Smirnov test	0.1463	0.21818

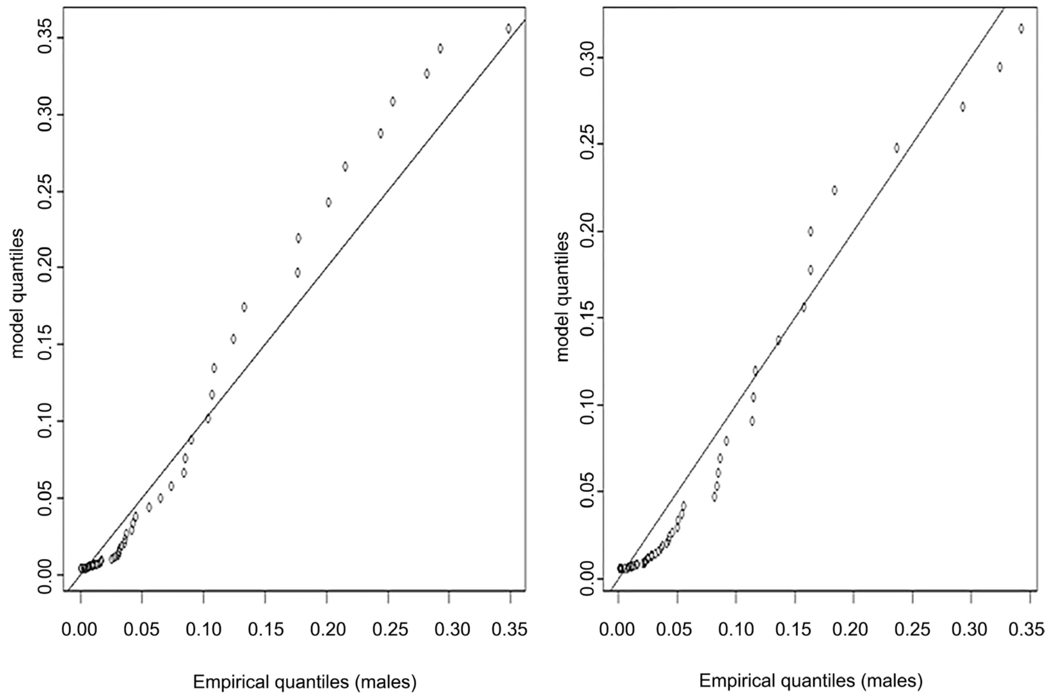

Figure 2. Empirical quantile plot of Weibull to AKI data.

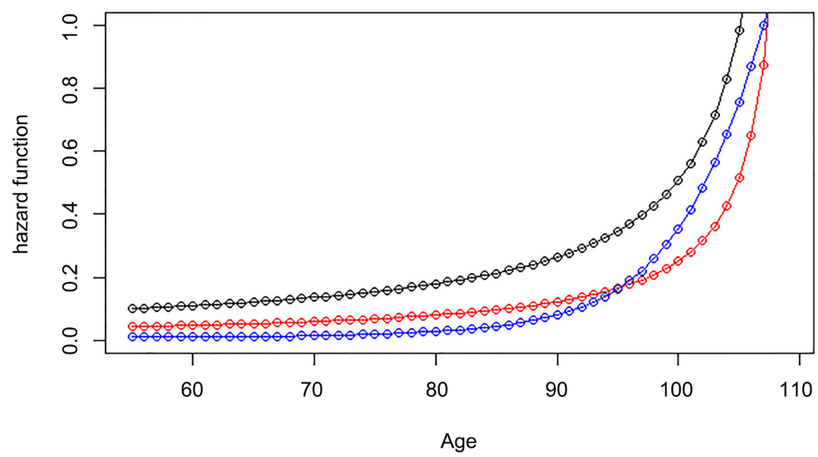

Figure 3. Dependence hazards at various empirical dependence measures.

consider two dependence measures *i.e.* Spearman's correlation $\rho = 0.74$ (black curve) and Pearson correlation $r = 0.68$ (red curve) when the age difference of insured couples is greater than four. As shown in **Figure 3**, using the above dependence measure shows a significant impact on the hazard compared with the independence (blue curve) assumption. Indeed, one cannot neglect the dependence, even moderate.

Equation (17) is then applied to generate the dependence λ_{xy} *frailty* hazard rates using Pearson correlation $r = 0.68$ (to be consistent with the parametric model chosen) shown in **Table 5**. Real-life data from a Kenyan insurer shows an average age difference of the insured couples to be 6 years. The joint survival probability is $s_{xy} = \exp\{-\lambda_{xy}\ frailty\}$. The EPV for joint-life annuity $a_{xy}\ frailty = \frac{N_{xy}}{D_{xy}}$ where $D_{xy} = v^{xy} l_{xy}$ and $N_{xy} = \sum_{k=0}^{45} D_{xy+k}$ are commutation functions.

The independence $\lambda_{xy}\ ind$ hazard rates in **Table 5** are obtained by summing the AKI 2010 male and female mortality rates. The survival probability $s_{xy} = s_x \cdot s_y = \exp\{-\lambda_{xy}\ ind\}$ and EPVs are computed as above. The density function is obtained through standard numerical approximations. Assuming a purchase price of 1000, the level annuity payments are obtained as indicated.

Table 5. Independence and dependence life-table construction.

INDEPENDENCE LIFETABLE CONSTRUCTION							PURCHASE PRICE: 1000	
AGE(y)	l_{xy}	S_{xy}	λ_{xy} ind	D_{xy}	N_{xy}	a_{xy}_ind	PAYMENT STREAM	AGE(x)
55	100,000	0.987322	0.01276	33,650.42	748,844.1	22.25363	44.9364899	55
56	98,732.15	0.989001	0.01106	32,572.34	715,193.7	21.95708	45.54338661	56
57	97,646.23	0.989673	0.010381	31,582.44	682,621.3	21.61395	46.26641343	57
58	96,637.84	0.989943	0.010108	30,643.42	651,038.9	21.24563	47.06849595	58
59	95,665.97	0.989986	0.010064	29,740.44	620,395.5	20.86033	47.93787231	59
60	94,708.01	0.989892	0.010159	28,865.32	590,655	20.46244	48.87001675	60
61	93,750.7	0.989705	0.010348	28,013.28	561,789.7	20.0544	49.86435956	61
62	92,785.53	0.989454	0.010602	27,181.26	533,776.4	19.63766	50.92256147	62
63	91,807.06	0.989153	0.010906	26,367.27	506,595.2	19.21303	52.04801895	63
64	90,811.23	0.988808	0.011255	25,569.87	480,227.9	18.781	53.24528938	64
65	89,794.87	0.988414	0.011653	24,787.94	454,658	18.34191	54.51995834	65
66	88,754.53	0.987967	0.012106	24,020.34	429,870.1	17.89608	55.87814592	66
67	87,686.54	0.98746	0.012619	23,265.99	405,849.7	17.44391	57.32659994	67
68	86,586.97	0.986891	0.013196	22,523.76	382,583.7	16.98579	58.87274996	68
69	85,451.88	0.986254	0.013841	21,792.64	360,060	16.52209	60.52501545	69
70	84,277.27	0.985547	0.014558	21,071.65	338,267.4	16.0532	62.29287286	70
71	83,059.23	0.984762	0.015355	20,359.9	317,195.7	15.57943	64.18719582	71
72	81,793.58	0.983891	0.01624	19,656.53	296,835.8	15.10113	66.22021465	72

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73	80,475.98	0.982889	0.01726	18,960.67	277,179.3	14.61864	68.40580518	73
74	79,098.91	0.981855	0.018312	18,270.81	258,218.6	14.13285	70.75714002	74
75	77,663.65	0.980663	0.019526	17,587.53	239,947.8	13.64306	73.29732669	75
76	76,161.88	0.979333	0.020884	16,909.26	222,360.3	13.15021	76.0444313	76
77	74,587.82	0.977847	0.022402	16,235.09	205,451	12.65475	79.02171141	77
78	72,935.47	0.976181	0.024108	15,564.15	189,215.9	12.15716	82.25603251	78
79	71,198.19	0.974307	0.026029	14,895.51	173,651.8	11.65799	85.77805174	79
80	69,368.91	0.97219	0.028204	14,228.24	158,756.3	11.15783	89.6231621	80
81	67,439.76	0.969793	0.030673	13,561.32	144,528	10.65737	93.8318065	81
82	65,402.58	0.967065	0.03349	12,893.8	130,966.7	10.15734	98.4509591	82
83	63,248.52	0.963949	0.036716	12,224.64	118,072.9	9.658598	103.5346902	83
84	60,968.37	0.960375	0.040431	11,552.88	105,848.3	9.162069	109.145658	84
85	58,552.52	0.95626	0.044726	10,877.55	94,295.38	8.668808	115.3561146	85
86	55,991.43	0.951503	0.049713	10,197.81	83,417.83	8.179977	122.2497347	86
87	53,276.01	0.945982	0.055531	9512.983	73,220.02	7.696852	129.9232519	87
88	50,398.16	0.939553	0.062351	8822.66	63,707.04	7.220842	138.4879971	88
89	47,351.75	0.932044	0.070376	8126.821	54,884.38	6.753487	148.0716653	89
90	44,133.89	0.923251	0.079854	7426.03	46,757.56	6.29644	158.8198986	90
91	40,746.67	0.912941	0.091084	6721.659	39,331.53	5.851461	170.8975	91
92	37,199.3	0.900837	0.104431	6016.155	32,609.87	5.420384	184.4888	92
93	33,510.49	0.886635	0.120322	5313.306	26,593.71	5.005116	199.7956	93
94	29,711.59	0.869999	0.139263	4618.593	21,280.41	4.607552	217.035	94
95	25,849.06	0.85058	0.161837	3939.385	16,661.81	4.229547	236.4319	95
96	21,986.68	0.827954	0.188798	3285.059	12,722.43	3.872816	258.2101	96
97	18,203.96	0.802014	0.22063	2666.546	9437.37	3.539173	282.5519	97
98	14,599.82	0.772303	0.258379	2096.674	6770.823	3.229317	309.663	98
99	11,275.48	0.738749	0.302797	1587.516	4674.15	2.944316	339.6374	99
100	8329.751	0.701364	0.354728	1149.78	3086.634	2.684542	372.503	100
101	5842.187	0.660341	0.414999	790.6024	1936.854	2.449845	408.189	101
102	3857.834	0.616061	0.48441	511.8304	1146.251	2.239514	446.5255	102
103	2376.66	0.569077	0.56374	309.1358	634.4207	2.05224	487.2725	103
104	1352.502	0.519526	0.654838	172.4726	325.2849	1.88601	530.2199	104
105	702.6603	0.469801	0.755445	87.84709	152.8124	1.739527	574.869	105
106	330.1108	0.419039	0.869791	40.46146	64.96529	1.605609	622.8165	106
107	138.3293	0.368476	0.99838	16.62247	24.50383	1.474139	678.3623	107
108	50.97098	0.318742	1.143375	6.004881	7.881357	1.312492	761.9096	108
109	16.24657	0.27038	1.307927	1.876476	1.876476	1	1000	109

Continued

INDEPENDENCE LIFETABLE CONSTRUCTION							PURCHASE PRICE: 1000	
AGE(y)	l_{xy}	S_{xy}	μ_{xy} *frailty*	D_{xy}	N_{xy}	a_{xy}_*frailty*	PAYMENT STREAM	AGE(x)
55	100,000	0.956844	0.044115	33,650.42	447,990.3	13.31306	75.11419	55
56	95,684.42	0.956038	0.044957	31,566.88	414,339.8	13.12578	76.18596	56
57	91,477.96	0.955202	0.045832	29,587.39	382,773	12.93703	77.2975	57
58	87,379.94	0.954334	0.046742	27,707.78	353,185.6	12.7468	78.45107	58
59	83,389.65	0.953432	0.047687	25,924	325,477.8	12.55507	79.64907	59
60	79,506.39	0.952495	0.048671	24,232.14	299,553.8	12.36184	80.89411	60
61	75,729.42	0.951519	0.049695	22,628.42	275,321.6	12.16708	82.18902	61
62	72,058.01	0.950504	0.050763	21,109.19	252,693.2	11.97077	83.53683	62
63	68,491.4	0.949445	0.051877	19,670.95	231,584	11.7729	84.94086	63
64	65,028.85	0.948341	0.053041	18,310.28	211,913.1	11.57345	86.40468	64
65	61,669.55	0.947189	0.054256	17,023.92	193,602.8	11.3724	87.93221	65
66	58,412.73	0.945985	0.055528	15,808.7	176,578.9	11.16973	89.5277	66
67	55,257.57	0.944726	0.056861	14,661.56	160,770.2	10.96542	91.1958	67
68	52,203.26	0.943407	0.058257	13,579.57	146,108.6	10.75945	92.94159	68
69	49,248.94	0.942026	0.059723	12,559.87	132,529	10.55179	94.77068	69
70	46,393.77	0.940576	0.061263	11,599.72	119,969.2	10.34242	96.6892	70
71	43,636.86	0.939053	0.062883	10,696.49	108,369.5	10.13131	98.70393	71
72	40,977.33	0.937451	0.06459	9847.62	97,672.96	9.918434	100.8224	72
73	38,414.25	0.935765	0.066391	9050.652	87,825.34	9.703759	103.0528	73
74	35,946.71	0.933986	0.068294	8303.217	78,774.69	9.48725	105.4046	74
75	33,573.73	0.932108	0.070306	7603.03	70,471.47	9.268867	107.8881	75
76	31,294.35	0.930122	0.07244	6947.889	62,868.44	9.048568	110.5147	76
77	29,107.56	0.928018	0.074704	6335.671	55,920.56	8.826304	113.2977	77
78	27,012.35	0.925786	0.077112	5764.331	49,584.89	8.602019	116.2518	78
79	25,007.65	0.923413	0.079678	5231.898	43,820.55	8.375651	119.3937	79
80	23,092.4	0.920887	0.082418	4736.475	38,588.66	8.147125	122.7427	30
81	21,265.48	0.918191	0.08535	4276.232	33,852.18	7.916357	126.3207	81
82	19,525.77	0.915308	0.088495	3849.409	29,575.95	7.683245	130.1533	82
83	17,872.09	0.912218	0.091876	3454.308	25,726.54	7.447669	134.2702	83
84	16,303.24	0.908899	0.095522	3089.296	22,272.23	7.209484	138.7062	84
85	14,817.99	0.905323	0.099464	2752.801	19,182.94	6.968515	143.5026	85
86	13,415.07	0.90146	0.103739	2443.308	16,430.14	6.724546	148.7089	36
87	12,093.15	0.897275	0.108393	2159.358	13,986.83	6.477309	154.3851	87
88	10,850.89	0.892726	0.113476	1899.547	11,827.47	6.226468	160.6047	88
89	9686.864	0.887762	0.119051	1662.524	9927.922	5.971597	167.4594	89
90	8599.633	0.882327	0.125193	1446.986	8265.398	5.712147	175.0655	90

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91	7587.685	0.876348	0.131992	1251.681	6818.412	5.447404	183.5737	91
92	6649.455	0.869743	0.139558	1075.401	5566.731	5.176426	193.1835	92
93	5783.314	0.862406	0.148029	916.982	4491.33	4.897948	204.1671	93
94	4987.565	0.845001	0.157576	775.3048	3574.348	4.61025	216.908	94
95	4214.5	0.834576	0.168417	642.2878	2799.044	4.357927	229.4669	95
96	3517.318	0.82268	0.180832	525.5272	2156.756	4.103985	243.6656	96
97	2893.626	0.808983	0.195189	423.8632	1631.229	3.848479	259.8429	97
98	2340.894	0.793049	0.211977	336.1746	1207.366	3.591484	278.4365	98
99	1856.444	0.774291	0.23187	261.3755	871.1909	3.3331	300.021	99
100	1437.429	0.751898	0.255807	198.4125	609.8154	3.073472	325.3649	100
101	1080.8	0.724725	0.285154	146.2608	411.4029	2.812803	355.5173	101
102	783.2826	0.6911	0.321963	103.9204	265.142	2.551394	391.9425	102
103	541.3263	0.648499	0.369471	70.41114	161.2216	2.289717	436.7352	103
104	351.0496	0.592961	0.433095	44.76624	90.81045	2.028548	492.9635	104
105	208.1588	0.518012	0.522626	26.02416	46.04421	1.769287	565.1995	105
106	107.8287	0.41282	0.657758	13.21649	20.02004	1.514778	660.1627	106
107	44.5138	0.260909	0.884744	5.349046	6.803557	1.27192	786.2131	107
108	11.61405	0.064307	1.343584	1.368249	1.454511	1.063046	940.6933	108
109	0.746861	1	2.744092	0.086262	0.086262	1	1000	109

Impact on Mortality Rates

As shown in Table 5 λ_{xy} *ind* is lower than λ_{xy} *frailty* during the early annuitants ages. This can be attributed to negative effects of dependence accounted for in the frailty model, thereby accounting for lower-tail dependence that is present. Thereafter, there is an overestimation of mortality risk in the independence model compared to the dependence model due to longevity risk (positive effects of dependence). Here, the upper-tail dependence has been accounted for. Thus the independence assumption underestimates deceleration in the mortality increase at very old ages.

Impact on Annuity EPVs

Consequently, comparing the annuity EPVs and payment streams shows that the independence assumptions lead to an overestimation of the insurer's liability at the initial stages of the contract thereafter there is an underestimation of liability due to deceleration in the mortality increase at very old ages (longevity risk).

5. Concluding Remarks

Although there is rich literature in frailty dependence modeling, most applications have been in medical field. Various dependence models have been considered in actuarial literature; however, the focus has been on either the lower-tail dependence or upper-tail dependence.

This article presents the frailty dependence approach calibrated on the AKI 2010 male and female published mortality rates due to limited joint-life mortality data-set available in the Kenyan market. This methodology offers greater flexibility than the lower-tail or upper-tail dependence models while preserving closed-form expressions for the net survival functions. Our strategy is to apply the conditional independence assumption in a positive stable frailty approach to account for lower and upper-tail dependence. A positive stable frailty approach is then applied to construct dependence life-tables. The frailty joint-life mortality rates are proposed to generate life annuity payment streams in the competitive Kenyan market.

The conclusion reached is that comparing the independence mortality assumption with the dependence frailty model shows a decrease in the insurer's expected liability at the early annuitant's ages followed by an increase at later ages when dependence is accounted for. This can be explained by the fact that the frail couples shall have died during the early stages of the contract, thereafter deceleration in the mortality increase at very old ages (longevity risk), underscoring the importance of dependence modeling in pricing insurance products. Thus, assuming the joint lives to be independent could lead to biased annuity valuations.

Acknowledgements

We would like to acknowledge the Association of Kenya Insurer for the study conducted on assured lives during 2010 whose mortality rates are currently being used by insurance companies in Kenya and have been used in the study.

Authors' Contribution

All authors have contributed equally in the development of this article.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Olivieri, A. and Pitacco, E. (1999) Funding Sickness Benefits for the Elderly. *Proceedings of the* 30*th ASTIN Colloquium*, Tokyo, 22-25 August 1999, 135-155.

[2] Coppola, M., Di Lorenzo, E. and Sibillo, M. (2000) Risk Sources in a Life Annuity Portfolio: Decomposition and Measurement Tools. *Journal of Actuarial Practice*, **8**, 43-61.

[3] Gildas, R., Francois, D., Enkelejd, H. and Youssouf, T. (2018) On Age Difference in Joint Lifetime Modeling with Life Assurance Annuity Applications. *Annals of Actuarial Science*, **12**, 350-371. https://doi.org/10.1017/S1748499518000076

[4] Meyricke, R. and Sherris, M. (2013) The Determinants of Mortality Heterogeneity and Implications for Pricing Annuities. *Insurance*: *Mathematics and Economics*, **53**,

379-387. https://doi.org/10.1016/j.insmatheco.2013.06.002

[5] Olivieri, A. and Pitacco, E. (2016) Frailty and Risk Classification for Life Annuity Portfolios. *Risks*, **4**, 39. https://doi.org/10.3390/risks4040039

[6] Luciano, E., Vigna, E. and Spreeuw, J.S. (2016) Dependence across Generations and Pricing Impact on Reversionary Annuities. *Risks*, **4**, 16. https://doi.org/10.3390/risks4020016

[7] Yang, L. (2017) Broken-Heart, Common Life, Heterogeneity: Analyzing the Spousal Mortality Dependence. *ASTIN Bulletin*: *The Journal of the IAA*, **47**, 837-874. https://doi.org/10.1017/asb.2017.8

[8] D'Amato, V., Haberman, S. and Piscopo, G. (2017) The Dependency Premium Based on a Multifactor Model for Dependent Mortality Data. *Communications in Statistics—Theory and Methods*, **48**, 50-61. https://doi.org/10.1080/03610926.2017.1366523

[9] Clayton, D.G. (1978) A Model for Association in Bivariate Lifetables and Its Application in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence. *Biometrika*, **65**, 141-151. https://doi.org/10.1093/biomet/65.1.141

[10] Hougaard, P. (2000) Analysis of Multivariate Survival Data. Springer, New York. https://doi.org/10.1007/978-1-4612-1304-8

[11] Fulla, S. and Laurent, P. (2008) Mortality Fluctuations Modelling with a Shared Frailty Approach.

[12] Wienke, A. (2010) Frailty Models in Survival Analysis. CRC Press, New York. https://doi.org/10.1201/9781420073911

[13] Hanagal, D. (2020) Correlated Positive Stable Frailty Models. *Communications in Statistics—Theory and Methods*, **47**, 1-17. https://doi.org/10.1080/03610926.2020.1736305

[14] Brouhns, N., Denuit, M. and Vermunt, K.J. (2002) A Poisson Log-Bilinear Regression Approach to the Construction of Projected Lifetables. *Insurance*: *Mathematics and Economics*, **31**, 373-393. https://doi.org/10.1016/S0167-6687(02)00185-3

[15] Wang, S.S. and Brown, R.L. (1998) A Frailty Model for Projection of Human Mortality Improvements. *Journal of Actuarial Practice*, **6**, 221-241. http://digitalcommons.unl.edu/joap/95

[16] Olivieri, A. (2001) Uncertainty in Mortality Projections: An Actuarial Perspective. *Insurance*: *Mathematics and Economics*, **29**, 231-245. https://doi.org/10.1016/S0167-6687(01)00084-1

[17] Carriere, J.F. (2000) Bivariate Survival Models for Coupled Lives. *Scandinavian Actuarial Journal*, **1**, 17-31. https://doi.org/10.1080/034612300750066700

[18] Cossette, H., Marceau, E., Mtalai, I. and Veilleux, D. (2017) Dependent Risk Models with Archimedean Copulas: A Computational Strategy Based on Common Mixtures and Applications. *Insurance*: *Mathematics and Economics*, **78**, 53-71. https://doi.org/10.1016/j.insmatheco.2017.11.002

[19] Hong, L. and Yang, L. (2018) Modeling Cause-of-Death Mortality Using Hierarchical Archimedean Copula. *Scandinavian Actuarial Journal*, No. 3, 247-272. https://doi.org/10.1080/03461238.2018.1546224

[20] Nelsen, R.B. (2007) An Introduction to Copulas. Springer Science and Business Media, Berlin.

[21] Czado, C., Kastenmeier, R., Brechmann, E.C. and Min, A. (2012) A Mixed Copula Model for Insurance Claims and Claim Sizes. *Scandinavian Actuarial Journal*, **4**, 278-305. https://doi.org/10.1080/03461238.2010.546147

[22] Oakes, D. (1989) Bivariate Survival Models Induced by Frailties. *Journal of the American Statistical Association*, **84**, 487-493. https://doi.org/10.1080/01621459.1989.10478795

[23] Scollnik, D.P.M. (1993) A Bayesian Analysis of a Simultaneous Equations Model for Insurance Rate-Making. *Insurance*: *Mathematics and Economics*, **12**, 265-286. https://doi.org/10.1016/0167-6687(93)90238-K

[24] Rosenberg, M. and Young, V.R. (1999) A Bayesian Approach to Understanding Time Series Data. *North American Actuarial Journal*, **3**, 130-143. https://doi.org/10.1080/10920277.1999.10595808