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Boundedness of Rough Operators on Grand Variable Herz Spaces

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Abstract

In this paper, we prove the boundedness of Calderón-Zygmund singular integral operators T_Ω on grand Herz spaces with variable exponent under some conditions.

Keywords

Calderón-Zygmund Singular Integral Operator, Grand Herz Spaces, Variable Exponent

1. Introduction

Several years ago, the theory of function spaces with variable exponent has been extensively studied by some experts. initial work [1] by Kovóčik and Rákosník appearing in 1991. The Lebesgue spaces and some other function spaces have been studied in the variable exponent setting. Let $\Omega \in L^s\left(S^{n-1}\right)$ for $s>1$ be a homogeneous function of degree zero and satisfies

$$\int_{S^{n-1}} \Omega\left(x'\right) \mathrm{d}\sigma\left(x'\right) = 0, \tag{1.1}$$

where $x' = \dfrac{x}{|x|}$ for any $x \neq 0$. The Calderón-Zygmund singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) \mathrm{d}y. \tag{1.2}$$

This operator was firstly introduced by Calderón and Zygmund (see ([2] [3]) in which they proved that these operators are bounded on L^p, where $0<p<1$. They have proved the boundedness of Lebesgue spaces $L^p\left(\mathbb{R}^n\right)$ for all

$1 < p < \infty$. The boundedness is extended to the case on Herz spaces by Lu and Yang [4]. In [5], Lu, Ding and Yan proved that T_Ω and the commutator $[b, T_\Omega]$ are bounded on weighted $\left(L^p\left(\mathbb{R}^n\right), L^q\left(\mathbb{R}^n\right)\right)$. Recently, Humberto Rafeiro introduced Grand Lebesgue sequence spaces in [6], where various operators of harmonic analysis were studied in these spaces. In [7], Tan and Liu discussed some boundedness of homogeneous fractional integrals on variable exponent spaces. In [8], Humberto Rafeiro and Muhammad Asad Zaighum proposed grand variable Herz spaces $\dot{K}_q^{\alpha,p),\theta}\left(\mathbb{R}^n\right)$ and obtain the boundedness of sublinear operators on $\dot{K}_q^{\alpha,p),\theta}\left(\mathbb{R}^n\right)$.

Motivated by [8] our main purpose of this paper is to prove the boundedness of the Calderón-Zygmund singular integral operator T_Ω on grand Herz spaces with variable exponent. In Section 2, we first briefly recall some standard notations and lemmas in variable function spaces. Then will define the homogeneous and non-homogeneous Herz spaces with variable exponent and define the grand variable Herz space. In Section 3, the main result, we will prove the boundedness of Calderón-Zygmund singular integral operators on grand Herz spaces with variable exponent.

2. Preliminaries and Lemmas

Suppose $\Omega \subset \mathbb{R}^n$ and measurable function $p(\cdot): \Omega \to [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x < \infty. \tag{2.1}$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}. \tag{2.2}$$

These spaces are referred to as variable L^p spaces, since they generalize the standard L^p spaces:

If $p(x) = p$ is constant, $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

The space $L^{p(\cdot)}_{loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{loc}(\Omega) = \left\{f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega\right\}. \tag{2.3}$$

Define $\mathcal{P}^0(E)$ to be the set of $p(\cdot): E \to (0, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\left\{p(x) : x \in E\right\} > 0,\ p^+ = \operatorname{ess\,sup}\left\{p(x) : x \in E\right\} < \infty. \tag{2.4}$$

Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot): E \to [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\left\{p(x) : x \in \Omega\right\} > 1,\ p^+ = \operatorname{ess\,sup}\left\{p(x) : x \in \Omega\right\} < \infty. \tag{2.5}$$

2.1. Herz Space with Variable Exponent

In this part, we present definitions of Herz spaces with variable exponent and use a notation in order to define those spaces. The important property for Herz spaces with variable exponents is the boundedness of the Hardy-Littlewood maximal operator.

Let $\iota \in \mathbb{Z}$, $B_\iota := \{x \in \mathbb{R}^n : |x| \le 2^\iota\}$, $B_\iota := B_\iota \setminus B_{\iota-1}$, $\chi_i := \chi_{R_i}$.

$\mathbb{N}_0$ denotes the set of integers. For $m \in \mathbb{N}_0$, we denote $\tilde{\chi}_m := \chi R_m$ if $m \ge 1$ and $\tilde{\chi}_0 := \chi B_0$.

Definition 2.1.1. (cf [9]). Suppose $\alpha \in \mathbb{R}, 0 \le q \le \infty$ and $p(\cdot): \mathbb{R}^n \to (0,\infty)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

The homogeneous Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus 0) : \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\} \tag{2.6}$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} := \left\| \left\{ 2^{\alpha l} \|f \chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{l=-\infty}^{\infty} \right\|_{\ell^q(\mathbb{Z})}. \tag{2.7}$$

The non-homogeneous Herz space $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is

$$K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n) : \|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.8}$$

where

$$\|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} := \left\| \left\{ 2^{\alpha m} \|f \tilde{\chi}_m\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{m=0}^{\infty} \right\|_{\ell^q(\mathbb{N}_0)}. \tag{2.9}$$

Lemma 2.1.1. (cf [10]). If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \ |x-y| \le \frac{1}{2} \tag{2.10}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(|x|+e)}, \ |y| \ge |x| \tag{2.11}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, that is, the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2.1.2. (cf [1]). Suppose $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q'(\cdot)}(\Omega)$, then fg is integrable on Ω and

$$\int |f(x) g(x)| \mathrm{d}x \le r_q \|f\|_{L^{q(\cdot)}(R^n)} \|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}, \tag{2.12}$$

where $r_q = 1 + 1/q_- - 1/p_+$.

Lemma 2.1.3. (cf [10]). Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C. \tag{2.13}$$

Lemma 2.1.4. (cf [11]). Define another variable exponent $\tilde{q}(\cdot)$ by

$\dfrac{1}{p(x)} = \dfrac{1}{q} + \dfrac{1}{\tilde{q}(x)} \ (x \in \mathbb{R}^n)$. Then, we have

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q}(\mathbb{R}^n)}, \tag{2.14}$$

for all measurable functions f and g.

Lemma 2.1.5. (cf [12]). Suppose $\Omega \in L^s(S^{n-1}), s \in [1,\infty]$. If $a>0, d \in (0,s]$ and $-n+\frac{(n-1)d}{s} < v < \infty$

$$\left(\int_{|y|\le a|x|}|y|^v |\Omega(x-y)|^d \,\mathrm{d}y\right)^{1/d} \lesssim \|\Omega\|_{L^s(S^{n-1})}|x|^{(v+n)/d}. \tag{2.15}$$

Lemma 2.1.6. (cf [13] Corollary 4.5.9.). Suppose $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Then $\|\chi_Q\|_{p(\cdot)} \approx |Q|^{\frac{1}{pQ}}$ for any cube(or ball) $Q \subset \mathbb{R}^n$ where,

$$\|\chi_Q\|_{p(\cdot)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}} & \text{if } |Q| \le 2^n \text{ and } x \in Q \\ |Q|^{\frac{1}{p\infty}} & \text{if } |Q| \ge 1 \end{cases} \tag{2.16}$$

Lemma 2.1.7. (cf [8]). Suppose $D>1$ and $q \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then

$$\frac{1}{c_0}r^{\frac{n}{q(0)}} \le \left\|\chi_{R_{r,Dr}}\right\|_{q(\cdot)} \le c_0 r^{\frac{n}{q(0)}}, \quad \text{for } 0<r\le 1, \tag{2.17}$$

and

$$\frac{1}{c_\infty}r^{\frac{n}{q_\infty}} \le \left\|\chi_{R_{r,Dr}}\right\|_{q(\cdot)} \le c_\infty r^{\frac{n}{q_\infty}}, \quad \text{for } r\ge 1, \tag{2.18}$$

respectively, where $c_0 \ge 1$ and $c_\infty \ge 1$ depend on D, but do not depend on r.

Lemma 2.1.8. (cf [9]). Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}, \tag{2.19}$$

where δ_1, δ_2 are constants with $0<\delta_1,\delta_2<1$ and χ_S and χ_B are the characteristic functions of S and B, respectively.

2.2. Grand Space of Sequences

Definition 2.2.1. (cf [6]). Let $1 \le p < \infty$ and $\theta > 0$, the grand Lebesgue sequence space is given by the norm

$$\|\mathbf{x}\|_{\ell^{p),\theta}(\mathbb{X})} := \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k\in\mathbb{X}}|\chi_k|^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0}\varepsilon^{\frac{\theta}{p(1+\varepsilon)}}\|\mathbf{x}\|_{\ell^{p(1+\varepsilon)}}(\mathbb{X}). \tag{2.20}$$

where $\mathbf{x} = \{x_k\}_{k\in\mathbb{X}}$.

Note that the following nesting properties hold:

$$l^{p(1-\varepsilon)} \hookrightarrow l^p \hookrightarrow l^{p),\theta_1} \hookrightarrow l^{p),\theta_2} \hookrightarrow l^{p(1+\delta)}$$

for $0<\varepsilon<\frac{1}{p}, \delta>0$ and $0<\theta_1 \le \theta_2$.

2.3. Grand Variable Herz Spaces

Definition 2.3.1. (cf [6]). Suppose $\alpha \in \mathbb{R}, 1 \le p < \infty, q : \mathbb{R}^n \to [1,\infty), \theta > 0$. We define the homogeneous grand variable Herz space by

$$\dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right) = \left\{ f \in L_{loc}^{q(\cdot)}\left(\mathbb{R}^n \setminus \{0\}\right) : \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right)} < \infty \right\}, \tag{2.21}$$

where

$$\begin{aligned} \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right)} &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \|f\chi_k\|_{L^{q(\cdot)}}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &= \sup_{\varepsilon>0} \varepsilon^{\frac{1}{p(1+\varepsilon)}} \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p(1+\varepsilon)}\left(\mathbb{R}^n\right)}. \end{aligned} \tag{2.22}$$

In a similar way, non-homogeneous grand variable Herz spaces can be introduced.

3. Main Results

In the following theorem, we prove that Calderón-Zygmund singular integral operator T_Ω are bounded on grand Herz space with variable exponent.

Theorem 3.1. Let $1 < p < \infty, q(\cdot) \in \mathcal{P}_{0,\infty}\left(\mathbb{R}^n\right)$ and $\Omega \in L^s\left(S^{n-1}\right)\left(s > q'^-\right)$, $0 < v \le 1$ such that $\left[-n\delta_1 - \left(v + \frac{n}{s}\right) - \frac{n}{q(0)}\right] < \alpha < \left[-n\delta_1 - \left(v + \frac{n}{s}\right) + \frac{n}{q'(0)}\right]$ and $\left[-n/q_\infty - n\delta_1 - \left(v + \frac{n}{s}\right)\right] < \alpha < \left[n/q'_\infty - n\delta_1 - \left(v + \frac{n}{s}\right)\right]$. Let T_Ω bounded on $L^{q(\cdot)}\left(\mathbb{R}^n\right)$ satisfying the size condition (1.2). Then T_Ω is bounded on $\dot{K}_{q(\cdot)}^{\alpha,p,\theta}\left(\mathbb{R}^n\right)$.

Proof Theorem 3.1. Let $f \in \dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right)$.

Then, we obtain

$$\begin{aligned} \|T_\Omega f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right)} &= \sup_{\varepsilon>0}\left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \|\chi_k T_\Omega f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\le \sup_{\varepsilon>0}\left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{l=-\infty}^{\infty} \|\chi_k T_\Omega\left(f\chi_l\right)\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}^{p(1+\varepsilon)} \right)\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\le C\sup_{\varepsilon>0}\left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k T_\Omega\left(f\chi_l\right)\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + c\sup_{\varepsilon>0}\left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{l=k-1}^{k+1} \|\chi_k T_\Omega\left(f\chi_l\right)\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + c\sup_{\varepsilon>0}\left(\varepsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha p(1+\varepsilon)} \left(\sum_{l=k+2}^{\infty} \|\chi_k T_\Omega\left(f\chi_l\right)\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &=: M_1 + M_2 + M_3. \end{aligned} \tag{3.1}$$

For M_2 using the $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T_Ω, we get

$$\begin{aligned}
M_2 &\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{Z}}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\left\|T(f\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{Z}}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{Z}}2^{k\alpha p(1+\varepsilon)}\left\|f\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&= c\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.
\end{aligned}\tag{3.2}$$

We estimate M_1, for each $k\in\mathbb{Z}$ and $l\le k-2$ and a.e. $x\in B_k$ applying condition (1.2) and generalized Hölder's inequality, we have

$$\begin{aligned}
\left|T_\Omega(f)(x)\right| &\le c\int_{B_l}\frac{\Omega(x-y)}{|x-y|^n}|f(y)|\mathrm{d}y\\
&\le c2^{-kn}\int_{B_l}|\Omega(x-y)||f(y)|\mathrm{d}y\\
&\le c2^{-kn}\left\|\Omega(x-\cdot)\chi_l(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.3}$$

Observation that $s>q'^{-}$, $\tilde{q}'(\cdot)>1$ and $\frac{1}{q'(x)}=\frac{1}{\tilde{q}'(x)}+\frac{1}{s}$. Form lemmas 2.1.4 and 2.1.5, we get

$$\begin{aligned}
\left\|\Omega(x-\cdot)\chi_l(\cdot)\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} &\le \left\|\Omega(x-\cdot)\chi_l(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\left\|\chi_l(\cdot)\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le \left\|\Omega(x-\cdot)\chi_l(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-l\nu}\left(\int_{A_l}|\Omega(x-y)|^s|y|^{s\nu}\mathrm{d}y\right)^{\frac{1}{s}}\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-l\nu}2^{k\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.4}$$

When $|B_l|\le 2^n$ and $x_l\in B_l$. From Lemma 2.1.6, we obtain

$$\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\approx|B_l|^{\frac{1}{\tilde{q}'(x_l)}}\approx\left\|\chi_{B_l}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_l|^{-\frac{1}{s}}.\tag{3.5}$$

When $|B_l|\ge 1$, we get

$$\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\approx|B_l|^{\frac{1}{\tilde{q}'(\infty)}}\approx\left\|\chi_{B_l}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_l|^{-\frac{1}{s}}.\tag{3.6}$$

Consequently, we obtain

$$\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\approx\left\|\chi_{B_l}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}|B_l|^{-\frac{1}{s}}.\tag{3.7}$$

By Lemmas 2.1.3 and 2.1.8, we have

$$\begin{aligned}
&\left\|T_\Omega(f)\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}2^{-l\nu}2^{k\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_l}\right\|_{L^{\tilde{q}'(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\le C2^{-kn}2^{-j\nu}2^{k\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_{B_l}\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}|B_l|^{-\frac{1}{s}}\|\chi_{B_k}\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}$$
$$\le C2^{-kn+(k-l)\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_{B_l}\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_{B_k}\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)} \tag{3.8}$$
$$\le C2^{-kn+(k-l)\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_l\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_k\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}.$$

From Lemma 2.1.7, we get

$$2^{-kn}\|\chi_k\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_l\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}\le c2^{-kn}2^{\left(\frac{kn}{q(0)}\right)}2^{\left(\frac{ln}{q'(0)}\right)}\le c2^{\frac{(l-k)n}{q'(0)}}. \tag{3.9}$$

Therefore,

$$\begin{aligned}&\|T_\Omega(f)\chi_l\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\\ &\le c2^{-kn}2^{-l\nu}2^{k\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_{B_l}\|_{L^{\tilde{q}'(\cdot)}\left(\mathbb{R}^n\right)}\|\chi_{B_k}\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\\ &\le c2^{(k-l)\left(\nu+\frac{n}{s}\right)}2^{\frac{(l-k)n}{q'(0)}}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}.\end{aligned} \tag{3.10}$$

Moreover, splitting M_1 by means of Minkowskis's inequality, we have

$$\begin{aligned}M_1&\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_kT_\Omega(f\chi_l)\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\quad+c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_kT_\Omega(f\chi_l)\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &:=M_{11}+M_{12}.\end{aligned} \tag{3.11}$$

For M_{11} using (3.10) we get

$$\begin{aligned}M_{11}&\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}2^{(k-l)\left(\nu+\frac{n}{s}\right)}2^{\frac{(l-k)n}{q'(0)}}\|\Omega\|_{L^s\left(S^{n-1}\right)}\left\|f\frac{\chi_l}{\chi_l}\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\left\|\frac{\chi_k}{\chi_l}\right\|2^{(k-l)\left(\nu+\frac{n}{s}\right)}2^{\frac{(l-k)n}{q'(0)}}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\chi_l\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{\alpha l}\left(\sum_{l=-\infty}^{k-2}2^{(k-l)\left[\alpha+n\delta_1+\left(\nu+\frac{n}{s}\right)-\frac{n}{q'(0)}\right]}\|\Omega\|_{L^s\left(S^{n-1}\right)}\|f\chi_l\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\le c\|\Omega\|_{L^s\left(S^{n-1}\right)}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{\alpha l}2^{(l-k)\left[-\alpha-n\delta_1-\left(\nu+\frac{n}{s}\right)+\frac{n}{q'(0)}\right]}\|f\chi_l\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\ &\le c\|\Omega\|_{L^s\left(S^{n-1}\right)}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2}2^{\alpha lp(1+\omega)}\|f\chi_l\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}^{p(1+\varepsilon)}2^{(l-k)mp(1+\varepsilon)}\right)\right)^{\frac{1}{p(1+\varepsilon)}},\end{aligned} \tag{3.12}$$

where $m := \left[-\alpha - n\delta_1 - \left(v + \frac{n}{s}\right) + \frac{n}{q'(0)}\right] > 0$. Then we use hölder's inequality, Fubini's theorem for series and $2^{-p(1+\varepsilon)} < 2^{-p}$ to obtain

$$M_{11} \le c\|\Omega\|_{L^s(S^{n-1})} \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k-2} 2^{\alpha l p(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} 2^{mp(1+\varepsilon)(l-k)/2}\right)\right.$$
$$\left.\times\left(\sum_{l=-\infty}^{k-2} 2^{m(p(1+\varepsilon))'(l-k)/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$= c\|\Omega\|_{L^s(S^{n-1})} \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k-2} 2^{\alpha p(1+\varepsilon) l}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} 2^{mp(1+\varepsilon)(l-k)/2}\right)^{\frac{1}{p(1+\varepsilon)}} \tag{3.13}$$
$$= c\|\Omega\|_{L^s(S^{n-1})} \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha l p(1+\varepsilon)}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \sum_{k=l+2}^{-1} 2^{mp(l-k)/2}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\le c\|\Omega\|_{L^s(S^{n-1})} \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha l p(1+\varepsilon)}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\le c\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.$$

Now for M_{12} using Minkowski's inequality, we have

$$M_{12} \le c\sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_k T_\Omega(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$+ c\sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=0}^{k-2}\|\chi_k T_\Omega(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \tag{3.14}$$
$$:= B_1 + B_2.$$

The estimate for B_2 follows in similar manner to M_{11} with $q'(0)$ replaced by q'_∞ and using the fact that $\left[-\alpha - n\delta_1 - \left(v + \frac{n}{s}\right) + n/q'_\infty\right] > 0$. B_1 using Lemma 2.1.7, we have

$$2^{-kn}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le c2^{-kn}2^{\left(\frac{kn}{q_\infty}\right)}2^{\left(\frac{ln}{q'(0)}\right)} \le c2^{\left(\frac{-kn}{q'(\infty)}\right)}2^{\left(\frac{ln}{q'(0)}\right)}. \tag{3.15}$$

We get therefore,

$$\|T_\Omega(f)\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le c2^{-kn}2^{\left(\frac{kn}{q_\infty}\right)}2^{\left(\frac{ln}{q'(0)}\right)}2^{(k-l)\left(v+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$
$$\le c2^{\left(\frac{-kn}{q'(\infty)}\right)}2^{\left(\frac{ln}{q'(0)}\right)}2^{(k-l)\left(v+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{3.16}$$

Now using (3.16) and fact that $\left[\alpha + n\delta_1 + \left(v + \frac{n}{s}\right) - n/q'_\infty\right] < 0$ we have

$$B_1 \le \sup_{\varepsilon>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_k T_\Omega f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}$$

$$
\begin{aligned}
&\le c\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{\left(\frac{-kn}{q'(\infty)}\right)}2^{\left(\frac{ln}{q'(0)}\right)}2^{(k-l)\left(v+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\
&\le c\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}2^{n\delta_1(k-l)}2^{\left(\frac{-kn}{q'(\infty)}\right)}2^{\left(\frac{ln}{q'(0)}\right)}2^{(k-l)\left(v+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\
&\le \sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{\left[k\alpha-\frac{kn}{q'(\infty)}\right]p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}c\|\Omega\|_{L^s(S^{n-1})}2^{(k-l)\left[n\delta_1+v+\frac{n}{s}\right]}2^{\left(\frac{ln}{q'(0)}\right)}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)} \qquad (3.17)\\
&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{(k-l)\left[n\delta_1+v+\frac{n}{s}\right]}2^{\left(\frac{ln}{q'(0)}\right)}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\
&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{\alpha l}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-l)\left[n\delta_1+v+\frac{n}{s}+\frac{l}{k-l}\left(\frac{n}{q'(0)}-\alpha\right)\right]}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}.
\end{aligned}
$$

For applying Hölder's inequality and using the fact $\left[-\alpha-n\delta_1-\left(v+\frac{n}{s}\right)+\frac{n}{q'(0)}\right]>0$, we get

$$
\begin{aligned}
B_1 &\le c\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{l\alpha p(1+\varepsilon)}\left\|f(\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)\right.\\
&\quad\left.\times\left(\sum_{l=-\infty}^{-1}2^{(k-l)\left[n\delta_1+v+\frac{n}{s}+\frac{l}{k-l}\left(\frac{n}{q'(0)}-\alpha\right)\right](p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{1/p(1+\varepsilon)} \qquad (3.18)\\
&\le c\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\left(\sum_{l\in\mathbb{Z}}2^{l\alpha p(1+\varepsilon)}\left\|f(\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)\right)^{1/p(1+\varepsilon)}\\
&\le c\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Next, we estimate M_3. For each $k\in\mathbb{Z}$ and $l\ge k+2$ and a.e $x\in B_k$; the size condition (3.10) and Hölder's inequality imply

$$
\left\|T_\Omega(f)\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le c2^{-ln+(k-l)\left(v+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \qquad (3.19)
$$

Similar to M_1, splitting M_3 by means of Minkowski's inequality, we have

$$
\begin{aligned}
M_3 &\le c\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\left\|\chi_k T_\Omega(f\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\quad+c\sup_{\varepsilon>0}\left(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\left\|\chi_k T_\Omega(f\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \qquad (3.20)\\
&:=M_{31}+M_{32}.
\end{aligned}
$$

For M_{32} lemma 2.1.7 yields

$$2^{-ln}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_l\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le c2^{-ln}2^{\left(\frac{kn}{q_\infty}\right)}2^{\left(\frac{ln}{q'_\infty}\right)} \le c2^{\frac{(k-l)n}{q_\infty}}. \tag{3.21}$$

We get

$$\begin{aligned}\left\|T_\Omega(f)\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le c2^{\delta_1 n(k-l)}2^{\frac{(k-l)n}{q_\infty}}2^{(k-l)\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le c\|\Omega\|_{L^s(S^{n-1})}2^{(k-l)\left[\delta_1 n+\frac{n}{q_\infty}+\left(\nu+\frac{n}{s}\right)\right]}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{3.22}$$

Using (3.22) for M_{32}, we have

$$\begin{aligned}M_{32} &\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\|\Omega\|_{L^s(S^{n-1})}2^{(k-l)\left[\delta_1 n+\frac{n}{q_\infty}+\left(\nu+\frac{n}{s}\right)\right]}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{\alpha l}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}2^{h(k-l)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},\end{aligned} \tag{3.23}$$

where $h:=\left[\delta_1 n+\frac{n}{q_\infty}+\left(\nu+\frac{n}{s}\right)+\alpha\right]>0$. Then we use Hölder's inequality, Fubini's theorem for series and $2^{-p(1+\varepsilon)}<2^{-p}$ to obtain

$$\begin{aligned}&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{\alpha p(1+\varepsilon)l}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}2^{hp(1+\varepsilon)(k-l)/2}\right)\right. \\ &\quad\left.\times\left(\sum_{l=k+2}^{\infty}2^{d(p(1+\varepsilon))'(k-l)/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &= c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}\sum_{l=k+2}^{\infty}2^{\alpha p(1+\varepsilon)l}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}2^{hp(1+\varepsilon)(k-l)/2}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &= c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{l=0}^{\infty}2^{\alpha p(1+\varepsilon)l}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\sum_{k=0}^{l-2}2^{hp(1+\varepsilon)(k-l)/2}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &< c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{\alpha p(1+\varepsilon)l}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\sum_{k=-\infty}^{l-2}2^{hp(k-l)/2}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &= c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{\alpha p(1+\varepsilon)l}\left\|f\chi_l\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\le c\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.\end{aligned} \tag{3.24}$$

So for M_{31} using Minkowski's inequality, we have

$$\begin{aligned}M_{31} &\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=k+2}^{-1}\left\|\chi_k T_\Omega(f\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad+c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}\left\|\chi_k T_\Omega(f\chi_l)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \\ &:= V_1+V_2.\end{aligned} \tag{3.25}$$

The estimate for V_1 follows in a similar manner to M_{32} with q_∞ replaced by $q(0)$ and using the fact that $\left[\delta_1 n+\frac{n}{q(0)}+\left(\nu+\frac{n}{s}\right)+\alpha\right]>0$. For V_2 using Lemma 2.1.7, we obtain

$$\begin{aligned}&2^{-ln}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\\&\le c2^{-ln}2^{\left(\frac{kn}{q(0)}\right)}2^{\left(\frac{ln}{q'_\infty}\right)}\le c2^{\left(\frac{kn}{q(0)}\right)}2^{\left(\frac{-ln}{q_\infty}\right)}.\end{aligned}\tag{3.26}$$

By taking (3.26) and the fact that $\left[\delta_1 n+\frac{n}{q(0)}+\left(\nu+\frac{n}{s}\right)+\alpha\right]>0$, we get

$$\begin{aligned}V_2&\le\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}\|\chi_k T_\Omega f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\&\le c\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}2^{\delta_1 n(k-l)}2^{\frac{kn}{q(0)}}2^{\frac{-ln}{q_\infty}}2^{(k-l)\left(\nu+\frac{n}{s}\right)}\|\Omega\|_{L^s(S^{n-1})}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}2^{(k-l)\left[\delta_1 n+\left(\nu+\frac{n}{s}\right)\right]}2^{\frac{kn}{q(0)}}2^{\frac{-ln}{q_\infty}}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\left(\alpha+\frac{n}{q(0)}\right)p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}2^{(k-l)\left[\delta_1 n+\left(\nu+\frac{n}{s}\right)-\frac{ln}{q_\infty}\right]}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\left(\sum_{l=0}^{\infty}2^{(k-l)\left[\delta_1 n+\left(\nu+\frac{n}{s}-\frac{ln}{q_\infty}\right)\right]}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta 2^{\alpha l}\left(\sum_{l=0}^{\infty}2^{(k-l)\left[\delta_1 n+\left(\nu+\frac{n}{s}\right)-l\left(\alpha+\frac{n}{q_\infty}\right)\right]}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}.\end{aligned}\tag{3.27}$$

Finally by Hölder's inequality and $\left[\delta_1 n+\left(\nu+\frac{n}{s}\right)+\left(\alpha+\frac{n}{q_\infty}\right)\right]>0$, we get

$$\begin{aligned}V_2&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\left(\sum_{l=0}^{\infty}2^{l\alpha p(1+\varepsilon)}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)\right.\\&\left.\quad\times\left(\sum_{l=0}^{\infty}2^{(k-l)\left[\delta_1 n+\left(\nu+\frac{n}{s}\right)-l\left(\alpha+\frac{n}{q_\infty}(p(1+\varepsilon))'\right)\right]}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right)^{1/p(1+\varepsilon)}\\&\le c\|\Omega\|_{L^s(S^{n-1})}\sup_{\varepsilon>0}\left(\varepsilon^\theta\sum_{l\in\mathbb{Z}}2^{l\alpha p(1+\varepsilon)}\|f(\chi_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)}\right)^{1/p(1+\varepsilon)}\\&\le c\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}(\mathbb{R}^n)}.\end{aligned}\tag{3.28}$$

Combining the estimates for M_1, M_2 and M_3 yields

$$\left\|T_\Omega f\right\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right)} \le c\left\|f\right\|_{\dot{K}_{q(\cdot)}^{\alpha,p),\theta}\left(\mathbb{R}^n\right)}. \quad (3.29)$$

4. Conclusion

In this paper, we investigated the boundedness of rough operators on grand variable Herz space. We proved the boundedness of Calderón-Zygmund singular integral operators on grand Herz spaces with variable exponent under some conditions of variable exponent.

Founding

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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