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Certain *m*-Convexity Inequalities Related to Fractional Integrals with Exponential Kernels

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Abstract

In this paper, we establish some mid-point type and trapezoid type inequalities via a new class of fractional integral operators which is introduced by Ahmad *et al.* We derive a new fractional-type integral identity to obtain Dragomir-Agarwal inequality for *m*-convex mappings. Moreover, some inequalities of Hermite-Hadamard type for *m*-convex mappings are given related to fractional integrals with exponential kernels. The results presented provide extensions of those given in earlier works.

Subject Areas

Mathematical Analysis, Numerical Mathematics

Keywords

Hermite-Hadamard Type Inequality, Fractional Integrals, *m*-Convex Mappings

1. Introduction

Recently, Ahmad *et al.* [1] presented a new fractional integral operator, which is named fractional integrals with exponential kernels, as follows.

Definition 1.1. *Let* $\varphi \in L^1([a,b])$. *The fractional integrals* $I_{\gamma_1^+}^{\mu}\varphi$ *and* $I_{\gamma_2^-}^{\mu}\varphi$ *of order* $\mu \in (0,1)$ *are defined respectively by*

$$I_{\gamma_1^+}^{\mu}\varphi(x) = \frac{1}{\mu}\int_{\gamma_1}^{x} \mathrm{e}^{-\frac{1-\mu}{\mu}(x-\tau)}\varphi(\tau)\,\mathrm{d}\tau, \quad x \ge \gamma_1,$$

and

$$I_{\gamma_2^-}^{\mu}\varphi(x) = \frac{1}{\mu}\int_{x}^{\gamma_2} \mathrm{e}^{-\frac{1-\mu}{\mu}(\tau-x)}\varphi(\tau)\,\mathrm{d}\tau, \quad x \le \gamma_2.$$

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Note that

$$\lim_{\mu\to 1} I^{\mu}_{\gamma_1^+}\varphi(x)=\int_{\gamma_1}^{x}\varphi(\tau)\mathrm{d}\tau \text{ and } \lim_{\mu\to 1} I^{\mu}_{\gamma_2^-}\varphi(x)=\int_{x}^{\gamma_2}\varphi(\tau)\mathrm{d}\tau.$$

In [1], the authors obtained new versions of Hermite-Hadanard inequality based on this fractional integral operators as follows.

Theorem 1.1. *Let* $\varphi:[\gamma_1,\gamma_2]\to\mathbb{R}$ *be a non-negative convex mapping and* $0\le\gamma_1<\gamma_2<\infty$. *If* $\varphi\in L^1([\gamma_1,\gamma_2])$, *then the following double inequalities hold*:

$$\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\le\frac{1-\mu}{2\left(1-\mathrm{e}^{-\rho}\right)}\left[I^{\mu}_{\gamma_1^+}\varphi(\gamma_2)+I^{\mu}_{\gamma_2^-}\varphi(\gamma_1)\right]\le\frac{\varphi(\gamma_1)+\varphi(\gamma_2)}{2}, \tag{1.1}$$

where $\rho=\frac{1-\mu}{\mu}(\gamma_2-\gamma_1)$.

Taking $\mu\to 1$ *i.e.* $\rho=\frac{1-\mu}{\mu}(\gamma_2-\gamma_1)\to 0$ in Theorem 1.1, we can recapture classical Hermite-Hadamard inequality for a convex function φ on $[\gamma_1,\gamma_2]$:

$$\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\le\frac{1}{\gamma_2-\gamma_1}\int_0^1\varphi(\tau)\mathrm{d}\tau\le\frac{\varphi(\gamma_1)+\varphi(\gamma_2)}{2}. \tag{1.2}$$

This generalized fractional integral operators had attracted the attention of many scholars. For example, Wu *et al.* [2] gave three fundamental integral identities via fractional integrals with exponential kernels to establish several Hermite-Hadamard-type inequalities. Zhou *et al.* [3] derived some parameterized fractional integrals with exponential kernels inequalities for convex mappings. For more information associated with fractional integrals with exponential kernels see reference in [4] [5].

The concept of *m*-convex mappings was introduced by Toader in [6]. It is defined as follows.

Definition 1.2. *The mapping* $\varphi:[0,\gamma_2]\to\mathbb{R}$, $\gamma_2>0$ *is named m-convex mapping, where* $m\in(0,1]$, *if for all* $k_1,k_2\in[0,\gamma_2]$ *and* $\tau\in[0,1]$, *we have*

$$\varphi(\tau k_1+m(1-\tau)k_2)\le\tau\varphi(k_1)+m(1-\tau)\varphi(k_2). \tag{1.3}$$

Due to the wide applications of *m*-convex mapping, many authors have established various integral inequalities related to *m*-convex mappings. In [7], Dragomir presented some properties and inequalities for *m*-convex mappings. In [8], Jleli *et al.* extended partial results presented in [7] via generalized fractional integrals. In [9], Farid and Abbas gave some general fractional integral inequalities for *m*-convex mappings associated with generalized Mittag-Leffer mapping. For other works involving *m*-convex mappings, we refer an interseted reader to [10] [11] [12].

These studies motivated us to establish some fractional integrals with exponential kernels inequalities for *m*-convex mappings. We considered two forms of *m*-convex combination to get certain midpoint type and trapezoid type inequalities. We gave new bounds for these inequalities and laid a foundation for their application in numerical integration. Some results of this article would provide generalizations of those given in earlier works.

2. Main Results

In this part, we mainly establish some fractional integral inequalities based on the properties of *m*-convex functions.

Theorem 2.1. *Let* $\varphi:[0,\infty)\to\mathbb{R}$ *be a m-convex function with* $m\in(0,1]$ *and* $0\le\gamma_1<m\gamma_2$. *If* $\varphi\in L^1[\gamma_1,m\gamma_2]$, *then the following inequality exists:*

$$\frac{\mu}{m\gamma_2-\gamma_1}\left[I^{\mu}_{\gamma_1^+}(m\gamma_2)+I^{\mu}_{m\gamma_2^-}(\gamma_1)\right]\le\frac{1-\mathrm{e}^{-\delta}}{\delta}\left[\varphi(\gamma_1)+m\varphi(\gamma_2)\right],\tag{2.1}$$

where $\delta=\frac{1-\mu}{\mu}(m\gamma_2-\gamma_1)$.

Proof. By means of *m*-convexity of φ, one has

$$\varphi(\tau\gamma_1+m(1-\tau)\gamma_2)\le\tau\varphi(\gamma_1)+m(1-\tau)\varphi(\gamma_2)$$

and

$$\varphi((1-\tau)\gamma_1+m\tau\gamma_2)\le(1-\tau)\varphi(\gamma_1)+m\tau\varphi(\gamma_2).$$

Adding the above inequalities, we deduce

$$\varphi(\tau\gamma_1+m(1-\tau)\gamma_2)+\varphi((1-\tau)\gamma_1+m\tau\gamma_2)\le\varphi(\gamma_1)+m\varphi(\gamma_2).\tag{2.2}$$

We can obtain the desired inequality by multiplying (2.2) with $\mathrm{e}^{-\theta\tau}$ and then integrating over $[0,1]$ with respect to $\mathrm{d}\tau$. Since

$$\begin{aligned}
&\int_0^1\mathrm{e}^{-\delta\tau}\left[\varphi(\tau\gamma_1+m(1-\tau)\gamma_2)+\varphi((1-\tau)\gamma_1+m\tau\gamma_2)\right]\mathrm{d}\tau\\
&=\int_0^1\mathrm{e}^{-\delta\tau}\varphi(\tau\gamma_1+m(1-\tau)\gamma_2)\mathrm{d}\tau+\int_0^1\varphi((1-\tau)\gamma_1+m\tau\gamma_2)\mathrm{d}\tau\\
&=\frac{1}{m\gamma_2-\gamma_1}\int_{\gamma_1}^{m\gamma_2}\mathrm{e}^{-\frac{1-\mu}{\mu}(m\gamma_2-\gamma_1)\frac{m\gamma_2-x}{m\gamma_2-\gamma_1}}\varphi(x)\mathrm{d}x\\
&\quad+\frac{1}{m\gamma_2-\gamma_1}\int_{\gamma_1}^{m\gamma_2}\mathrm{e}^{-\frac{1-\mu}{\mu}(m\gamma_2-\gamma_1)\frac{x-\gamma_1}{m\gamma_2-\gamma_1}}\varphi(x)\mathrm{d}x\\
&=\frac{\mu}{m\gamma_2-\gamma_1}\left[I^{\mu}_{\gamma_1^+}(m\gamma_2)+I^{\mu}_{m\gamma_2^-}(\gamma_1)\right]
\end{aligned}$$

and

$$\int_0^1\mathrm{e}^{-\delta}\left[\varphi(\gamma_1)+m\varphi(\gamma_2)\right]\mathrm{d}\tau=\frac{1-\mathrm{e}^{-\delta}}{\delta}\left[\varphi(\gamma_1)+m\varphi(\gamma_2)\right].$$

This ends the proof.

Corollary 2.1. *If we consider* $m=1$ *in Theorem* 2.1, *then we have right part of inequality* (8) *in* [1].

To obtain trapezoid type inequality related to fractional integrals with exponential kernels, we need the following lemma.

Lemma 2.1. *Assuming* $\varphi:[\gamma_1,m\gamma_2]\to\mathbb{R}$ *is a differentiable mapping with* $0\le\gamma_1<m\gamma_2<\infty$ *and* $0<m\le1$. *If* $\varphi'\in L^1([\gamma_1,m\gamma_2])$, *then the following identity holds:*

$$\Omega(\gamma_1,\gamma_2,\delta,\tau)=\frac{m\gamma_2-\gamma_1}{2\left(1-\mathrm{e}^{-\delta}\right)}\left[\int_0^1 \mathrm{e}^{-\delta\tau}\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\tau\right.$$
$$\left.-\int_0^1 \mathrm{e}^{-\delta(1-\tau)}\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\tau\right], \tag{2.3}$$

where $\delta=\frac{1-\mu}{\mu}(m\gamma_2-\gamma_1)$ and

$$\Omega(\gamma_1,\gamma_2,\delta,\tau):=\frac{\varphi(\gamma_1)+\varphi(m\gamma_2)}{2}-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\delta}\right)}\left[I_{\gamma_1^+}^{\mu}\varphi(m\gamma_2)+I_{m\gamma_2^-}^{\mu}\varphi(\gamma_1)\right]. \tag{2.4}$$

Proof. Integrating the following formula by parts, we have

$$\int_0^1 \mathrm{e}^{-\delta\tau}\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\tau-\int_0^1 \mathrm{e}^{-\delta(1-\tau)}\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\tau$$
$$=\frac{1}{\gamma_1-m\gamma_2}\int_0^1 \mathrm{e}^{-\delta\tau}\mathrm{d}\left(\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\right)$$
$$-\frac{1}{\gamma_1-m\gamma_2}\int_0^1 \mathrm{e}^{-\delta(1-\tau)}\mathrm{d}\left(\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\right)$$
$$=\frac{1}{\gamma_1-m\gamma_2}\left[\mathrm{e}^{-\delta\tau}\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\Big|_0^1-\int_0^1\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\left(\mathrm{e}^{-\delta\tau}\right)\right]$$
$$-\frac{1}{\gamma_1-m\gamma_2}\left[\mathrm{e}^{-\delta(1-\tau)}\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\Big|_0^1\right.$$
$$\left.-\int_0^1\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\left(\mathrm{e}^{-\delta(1-\tau)}\right)\right]$$
$$=\frac{1}{m\gamma_2-\gamma_1}\left[\left(1-\mathrm{e}^{-\delta}\right)\left(\varphi(\gamma_1)+\varphi(m\gamma_2)\right)-\delta\int_0^1\mathrm{e}^{-\delta\tau}\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\tau\right.$$
$$\left.-\delta\int_0^1\mathrm{e}^{-\delta(1-\tau)}\varphi\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\mathrm{d}\tau\right]$$
$$=\frac{1}{m\gamma_2-\gamma_1}\left(\left(1-\mathrm{e}^{-\delta}\right)\left(\varphi(\gamma_1)+\varphi(m\gamma_2)\right)-(1-\mu)\left[I_{\gamma_1^+}^{\mu}\varphi(m\gamma_2)+I_{m\gamma_2^-}^{\mu}\varphi(\gamma_1)\right]\right). \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{m\gamma_2-\gamma_1}{2\left(1-\mathrm{e}^{-\delta}\right)}$, we have the conclusion (2.3). The proof is completed.

Theorem 2.2. *Let φ be defined as in Lemma* 2.1. *If φ' is m-convex on $[\gamma_1,m\gamma_2]$ for some fixed $m\in[0,1]$, then the following inequality for fractional integrals with exponential kernels holds*:

$$\left|\Omega(\gamma_1,\gamma_2,\delta,\tau)\right|\le\frac{m\gamma_2-\gamma_1}{2\delta}\tanh\left(\frac{\delta}{4}\right)\left[\left|\varphi'(\gamma_1)\right|+m\left|\varphi'(\gamma_2)\right|\right]. \tag{2.6}$$

Proof. Applying Lemma 2.1 and convexity of $|\varphi'|$, we obtain

$$\left|\Omega(\gamma_1,\gamma_2,\delta,\tau)\right|$$
$$\le\frac{m\gamma_2-\gamma_1}{2\left(1-\mathrm{e}^{-\delta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\delta\tau}\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)-\mathrm{e}^{-\delta(1-\tau)}\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\right|\mathrm{d}\tau\right]$$
$$\le\frac{m\gamma_2-\gamma_1}{2\left(1-\mathrm{e}^{-\delta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\delta\tau}-\mathrm{e}^{-\delta(1-\tau)}\right|\left|\varphi'\left(\tau\gamma_1+m(1-\tau)\gamma_2\right)\right|\mathrm{d}\tau\right] \tag{2.7}$$
$$\le\frac{m\gamma_2-\gamma_1}{2\left(1-\mathrm{e}^{-\delta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\delta\tau}-\mathrm{e}^{-\delta(1-\tau)}\right|\left(\tau\varphi'(\gamma_1)+m(1-\tau)\varphi'(\gamma_2)\right)\mathrm{d}\tau\right].$$

By calculation, we have

$$\int_0^1 t\left|e^{-\delta\tau}-e^{-\delta(1-\tau)}\right|\varphi'(\gamma_1)\,d\tau=\frac{1-2e^{-\frac{\delta}{2}}+e^{-\delta}}{\delta}\varphi'(\gamma_1) \tag{2.8}$$

and

$$\int_0^1 m(1-t)\left|e^{-\delta\tau}-e^{-\delta(1-\tau)}\right|\varphi'(\gamma_2)\,d\tau=m\frac{1-2e^{-\frac{\delta}{2}}+e^{-\delta}}{\delta}\varphi'(\gamma_2). \tag{2.9}$$

Utilizing inequality (2.8) and inequality (2.9) in inequality (2.7), we have

$$\left|\Omega(\gamma_1,\gamma_2,\delta,\tau)\right|\le\frac{m\gamma_2-\gamma_1}{2\delta}\tanh\left(\frac{\delta}{4}\right)\left[\left|\varphi'(\gamma_1)\right|+m\left|\varphi'(\gamma_2)\right|\right].$$

The proof is completed.

Corollary 2.2. *If we consider* $m=1$ *in Theorem* 2.2, *then we can deduce Theorem* 3 *in* [1].

Theorem 2.3. *Let* $\varphi:[0,\infty)\to\mathbb{R}$ *be a m-convex function with* $m\in(0,1]$ *and* $0\le\gamma_1<\gamma_2$. *If* $\varphi\in L^1[\gamma_1,\gamma_2]$, *then the following inequality exists:*

$$\begin{aligned}&\frac{\mu}{\gamma_2-\gamma_1}\left[I^{\mu}_{\gamma_2^-}\varphi(\gamma_1)+I^{\mu}_{\gamma_1^+}\varphi(\gamma_+)\right]\\&\le\frac{-\rho e^{-\rho}-e^{\rho}+1}{\rho^2}\left[\varphi(\gamma_1)+\varphi(\gamma_2)\right]+m\frac{e^{-\rho}+\rho-1}{\rho^2}\left[\varphi\left(\frac{\gamma_1}{m}\right)+\varphi\left(\frac{\gamma_2}{m}\right)\right],\end{aligned} \tag{2.10}$$

where $\rho=\frac{1-\mu}{\mu}(\gamma_2-\gamma_1)$.

Proof. By means of *m*-convexity of φ, we deduce

$$\varphi(\tau\gamma_1+(1-\tau)\gamma_2)\le\tau\varphi(\gamma_1)+m(1-\tau)\varphi\left(\frac{\gamma_2}{m}\right)$$

and

$$\varphi(\tau\gamma_2+(1-\tau)\gamma_1)\le\tau\varphi(\gamma_2)+m(1-\tau)\varphi\left(\frac{\gamma_1}{m}\right).$$

Multiplying above-mentioned inequalities with $e^{-\rho\tau}$ and then integrating over $[0,1]$ with respect to $d\tau$, we get

$$\begin{aligned}&\int_0^1 e^{-\rho\tau}\varphi(\tau\gamma_1+(1-\tau)\gamma_2)\,d\tau\\&=\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}e^{-\frac{1-\mu}{\mu}(\gamma_2-x)}\varphi(x)\,dx=\frac{\mu}{\gamma_2-\gamma_1}I^{\mu}_{\gamma_2^-}\varphi(\gamma_1)\\&\le\int_0^1 e^{-\rho\tau}\left[\tau\varphi(\gamma_1)+m(1-\tau)\varphi\left(\frac{\gamma_2}{m}\right)\right]d\tau\\&=\frac{-\rho e^{-\rho}-e^{\rho}+1}{\rho^2}\varphi(\gamma_1)+m\frac{e^{-\rho}+\rho-1}{\rho^2}\varphi\left(\frac{\gamma_2}{m}\right)\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\int_0^1 \mathrm{e}^{-\rho\tau}\varphi\left(\tau\gamma_2+(1-\tau)\gamma_1\right)\mathrm{d}\tau \\
&=\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathrm{e}^{-\frac{1-\mu}{\mu}(x-\gamma_1)}\varphi(x)\,\mathrm{d}x=\frac{\mu}{\gamma_2-\gamma_1}I_{\gamma_1^+}^{\mu}\varphi(\gamma_2) \\
&\le\int_0^1 \mathrm{e}^{-\rho\tau}\left[\tau\varphi(\gamma_2)+m(1-\tau)\varphi\left(\frac{\gamma_1}{m}\right)\right]\mathrm{d}\tau \\
&=\frac{-\rho\mathrm{e}^{-\rho}-\mathrm{e}^{\rho}+1}{\rho^2}\varphi(\gamma_2)+m\frac{\mathrm{e}^{-\rho}+\rho-1}{\rho^2}\varphi\left(\frac{\gamma_1}{m}\right).
\end{aligned}\tag{2.12}$$

By adding (2.11) and (2.12) together, we have completed the proof.

Corollary 2.3. *If we consider* $\mu\to1$ *i.e.* $\rho=\frac{1-\mu}{\mu}(\gamma_2-\gamma_1)\to0$ *in Theorem* 2.3, *observe that*

$$\lim_{\rho\to0}\frac{-\rho\mathrm{e}^{-\rho}-\mathrm{e}^{\rho}+1}{\rho^2}=1$$

and

$$\lim_{\rho\to0}\frac{\mathrm{e}^{-\rho}+\rho-1}{\rho^2}=\frac{1}{2},$$

then we have

$$\frac{2}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\varphi(\tau)\,\mathrm{d}\tau\le\left[\varphi(\gamma_1)+\varphi(\gamma_2)\right]+\frac{m}{2}\left[\varphi\left(\frac{\gamma_1}{m}\right)+\varphi\left(\frac{\gamma_2}{m}\right)\right].\tag{2.13}$$

Theorem 2.4. *Under the assumptions of Theorem* 2.3, *if we take* $\theta=\frac{\mu}{1-\mu}\frac{\gamma_2-\gamma_1}{2}$, *then the resulting expression holds*:

$$\begin{aligned}
&\frac{\mu}{\gamma_2-\gamma_1}\left[I_{\gamma_1^+}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)+I_{\gamma_2^-}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right] \\
&\le\frac{-2\theta\mathrm{e}^{-\theta}-\mathrm{e}^{-\theta}+\theta+1}{4\theta^2}\left[\varphi(\gamma_1)+\varphi(\gamma_2)\right]+m\frac{\mathrm{e}^{-\theta}+\theta-1}{4\theta^2}\left[\varphi\left(\frac{\gamma_1}{m}\right)+\varphi\left(\frac{\gamma_2}{m}\right)\right].
\end{aligned}\tag{2.14}$$

Proof. Since φ is m-convex, we have

$$\varphi\left(\frac{1+\tau}{2}\gamma_1+\frac{1-\tau}{2}\gamma_2\right)\le\frac{1+\tau}{2}\varphi(\gamma_1)+m\frac{1-\tau}{2}\varphi\left(\frac{\gamma_2}{m}\right)\tag{2.15}$$

and

$$\varphi\left(\frac{1+\tau}{2}\gamma_2+\frac{1-\tau}{2}\gamma_1\right)\le\frac{1+\tau}{2}\varphi(\gamma_2)+m\frac{1-\tau}{2}\varphi\left(\frac{\gamma_1}{m}\right).\tag{2.16}$$

Adding inequality (2.15) and inequality (2.16) together and then multiplying by $\mathrm{e}^{-\theta\tau}$, we get

$$\begin{aligned}
&\mathrm{e}^{-\theta\tau}\left[\varphi\left(\frac{1+\tau}{2}\gamma_2+\frac{1-\tau}{2}\gamma_1\right)+\varphi\left(\frac{1+\tau}{2}\gamma_1+\frac{1-\tau}{2}\gamma_2\right)\right] \\
&\le\mathrm{e}^{-\theta\tau}\left(\frac{1+\tau}{2}\left[\varphi(\gamma_1)+\varphi(\gamma_2)\right]+m\frac{1-\tau}{2}\left[\varphi\left(\frac{\gamma_1}{m}\right)+\varphi\left(\frac{\gamma_2}{m}\right)\right]\right).
\end{aligned}\tag{2.17}$$

Integrating on both sides of inequality (2.17) respect to τ over $[0,1]$, we have completed the proof. Since

$$
\begin{aligned}
&\int_0^1 \mathrm{e}^{-\theta\tau}\left[\varphi\left(\frac{1+\tau}{2}\gamma_2+\frac{1-\tau}{2}\gamma_1\right)+\varphi\left(\frac{1+\tau}{2}\gamma_1+\frac{1-\tau}{2}\gamma_2\right)\right]\mathrm{d}\tau \\
&=\int_0^1 \mathrm{e}^{-\theta\tau}\varphi\left(\frac{1+\tau}{2}\gamma_2+\frac{1-\tau}{2}\gamma_1\right)\mathrm{d}\tau+\int_0^1 \mathrm{e}^{-\theta\tau}\varphi\left(\frac{1+\tau}{2}\gamma_1+\frac{1-\tau}{2}\gamma_2\right)\mathrm{d}\tau \\
&=\frac{2}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\frac{\gamma_1+\gamma_2}{2}} \mathrm{e}^{-\frac{1-\mu}{\mu}\frac{\gamma_2-\gamma_1}{2}\frac{(\gamma_1+\gamma_1)-2x}{\gamma_2-\gamma_1}}\varphi(x)\mathrm{d}x \\
&\quad+\frac{2}{\gamma_2-\gamma_1}\int_{\frac{\gamma_2+\gamma_1}{2}}^{\gamma_2} \mathrm{e}^{-\frac{1-\mu}{\mu}\frac{\gamma_2-\gamma_1}{2}\frac{2x-(\gamma_1+\gamma_1)}{\gamma_2-\gamma_1}}\varphi(x)\mathrm{d}x \\
&=\frac{2\mu}{\gamma_2-\gamma_1}\left[I_{\gamma_1^+}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)+I_{\gamma_2^-}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \mathrm{e}^{-\theta\tau}\left(\frac{1+\tau}{2}\left[\varphi(\gamma_1)+\varphi(\gamma_2)\right]+m\frac{1-\tau}{2}\left[\varphi\left(\frac{\gamma_1}{m}\right)+\varphi\left(\frac{\gamma_2}{m}\right)\right]\right)\mathrm{d}\tau \\
&=\frac{-2\theta\mathrm{e}^{-\theta}-\mathrm{e}^{-\theta}+\theta+1}{2\theta^2}\left(\varphi(\gamma_1)+\varphi(\gamma_2)\right)+m\frac{\mathrm{e}^{-\theta}+\theta-1}{2\theta^2}\left(\varphi\left(\frac{\gamma_1}{m}\right)+\varphi\left(\frac{\gamma_2}{m}\right)\right).
\end{aligned}
$$

We now use the following two lemmas, which are presented in [13], to obtain some mid-point type and trapezoid type inequalities.

Lemma 2.2. *Assuming* $\varphi:[\gamma_1,\gamma_2]\to\mathbb{R}$ *is a differentiable mapping, such that* $\varphi\in L^1([\gamma_1,\gamma_2])$ *with* $0\le\gamma_1<\gamma_2<\infty$, *then the following identity holds.*

$$
\begin{aligned}
&-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^-}\varphi(\gamma_1)+I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^+}\varphi(\gamma_2)\right]+\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right) \\
&=\frac{\gamma_2-\gamma_1}{4\left(1-\mathrm{e}^{-\theta}\right)}\left\{\int_0^1\left[\mathrm{e}^{-\theta\tau}-1\right]\varphi'\left(\frac{\tau}{2}\gamma_1+\frac{2-\tau}{2}\gamma_2\right)\mathrm{d}\tau\right. \\
&\quad\left.-\int_0^1\left[\mathrm{e}^{-\theta\tau}-1\right]\varphi'\left(\frac{2-\tau}{2}\gamma_1+\frac{\tau}{2}\gamma_2\right)\mathrm{d}\tau\right\}.
\end{aligned}
\tag{2.18}
$$

Lemma 2.3. *Assuming* $\varphi:[\gamma_1,\gamma_2]\to\mathbb{R}$ *is a positive convex mapping, such that* $\varphi\in L^1([\gamma_1,\gamma_2])$ *with* $0\le\gamma_1<\gamma_2<\infty$, *then the following identity exists.*

$$
\begin{aligned}
&\frac{\varphi(\gamma_1+\gamma_2)}{2}-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I^{\mu}_{\gamma_2^-}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)+I^{\mu}_{\gamma_1^+}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right] \\
&=\frac{\gamma_2-\gamma_1}{4\left(\mathrm{e}^{-\theta}-1\right)}\left\{\int_0^1\left[\mathrm{e}^{-\theta\tau}-1\right]\varphi'\left(\frac{1-\tau}{2}\gamma_1+\frac{1+\tau}{2}\gamma_2\right)\mathrm{d}\tau\right. \\
&\quad\left.-\int_0^1\left[\mathrm{e}^{-\theta\tau}-1\right]\varphi'\left(\frac{1+\tau}{2}\gamma_1+\frac{1-\tau}{2}\gamma_2\right)\mathrm{d}\tau\right\}.
\end{aligned}
\tag{2.19}
$$

Theorem 2.5. *Under the assumptions of lemma* 2.2, *if* $|\varphi'|$ *is m-convex on* $[\gamma_1,\gamma_2]$, *then the resulting expression holds.*

$$\begin{aligned}&\left|-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^-}\varphi(\gamma_1)+I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^+}\varphi(\gamma_2)\right]+\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right|\\&\le\frac{\gamma_2-\gamma_1}{4\theta\left(1-\mathrm{e}^{-\theta}\right)}\left(\mathrm{e}^{-\theta}-1+\theta\right)\left[\left|\varphi'(\gamma_1)\right|+m\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right].\end{aligned}\tag{2.20}$$

Proof. Applying Lemma 2.2 and the convexity of $|\varphi'|$, one has

$$\begin{aligned}&\left|-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^-}\varphi(\gamma_1)+I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^+}\varphi(\gamma_2)\right]+\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right|\\&\le\frac{\gamma_2-\gamma_1}{4\left(1-\mathrm{e}^{-\theta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left|\varphi'\left(\frac{\tau}{2}\gamma_1+m\frac{2-\tau}{2}\frac{\gamma_2}{m}\right)\right|\mathrm{d}\tau\right.\\&\quad\left.+\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left|\varphi'\left(\frac{2-\tau}{2}\gamma_1+m\frac{\tau}{2}\frac{\gamma_2}{m}\right)\right|\mathrm{d}\tau\right]\\&\le\frac{\gamma_2-\gamma_1}{4\left(1-\mathrm{e}^{-\theta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left(\frac{\tau}{2}\left|\varphi'(\gamma_1)\right|+m\frac{2-\tau}{2}\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right)\mathrm{d}\tau\right.\\&\quad\left.+\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left(\frac{2-\tau}{2}\left|\varphi'(\gamma_1)\right|+m\frac{\tau}{2}\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right)\mathrm{d}\tau\right],\end{aligned}$$

where we use the fact that

$$\int_0^1\frac{\tau}{2}\left|\mathrm{e}^{-\theta\tau}-1\right|\mathrm{d}\tau=\frac{1}{4}-\frac{-\theta\mathrm{e}^{-\theta}-\mathrm{e}^{-\theta}+1}{2\theta^2}$$

and

$$\int_0^1\frac{2-\tau}{2}\left|\mathrm{e}^{-\theta\tau}-1\right|\mathrm{d}\tau=\frac{-\theta\mathrm{e}^{-\theta}-\mathrm{e}^{-\theta}+1}{2\theta^2}+\frac{\mathrm{e}^{-\theta}-1}{\theta}+\frac{3}{4}.$$

After suitable arrangements, we obtain

$$\begin{aligned}&\left|-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^-}\varphi(\gamma_1)+I^{\mu}_{\frac{\gamma_2+\gamma_1}{2}^+}\varphi(\gamma_2)\right]+\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right|\\&\le\frac{\gamma_2-\gamma_1}{4\theta\left(1-\mathrm{e}^{-\theta}\right)}\left(\mathrm{e}^{-\theta}-1+\theta\right)\left[\left|\varphi'(\gamma_1)\right|+m\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right].\end{aligned}$$

This ends the proof.

Theorem 2.6. *Under the assumptions of Lemma* 2.3, *if* $|\varphi'|$ *is m-convex on* $[\gamma_1,\gamma_2]$, *then the resulting expression holds.*

$$\begin{aligned}&\left|\frac{\varphi(\gamma_1+\gamma_2)}{2}-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I^{\mu}_{\gamma_2^-}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)+I^{\mu}_{\gamma_1^+}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right]\right|\\&\le\frac{\gamma_2-\gamma_1}{4\theta\left(1-\mathrm{e}^{-\theta}\right)}\left(\mathrm{e}^{-\theta}-1+\theta\right)\left[\left|\varphi'(\gamma_1)\right|+m\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right].\end{aligned}\tag{2.21}$$

Proof. Applying Lemma 2.3 and the convexity of $|\varphi'|$, one has

$$\begin{aligned}
&\left|\frac{\varphi(\gamma_1+\gamma_2)}{2}-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I_{\gamma_2^-}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)+I_{\gamma_1^+}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right]\right| \\
&\le \frac{\gamma_2-\gamma_1}{4\left(1-\mathrm{e}^{-\theta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left|\varphi'\left(\frac{1-\tau}{2}\gamma_1+m\frac{1+\tau}{2}\frac{\gamma_2}{m}\right)\right|\mathrm{d}\tau\right. \\
&\quad\left.+\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left|\varphi'\left(\frac{1+\tau}{2}\gamma_1+m\frac{1-\tau}{2}\frac{\gamma_2}{m}\right)\right|\mathrm{d}\tau\right] \\
&\le \frac{\gamma_2-\gamma_1}{4\left(1-\mathrm{e}^{-\theta}\right)}\left[\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left(\frac{1-\tau}{2}\left|\varphi'(\gamma_1)\right|+m\frac{1+\tau}{2}\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right)\mathrm{d}\tau\right. \\
&\quad\left.+\int_0^1\left|\mathrm{e}^{-\theta\tau}-1\right|\left(\frac{1+\tau}{2}\left|\varphi'(\gamma_1)\right|+m\frac{1-\tau}{2}\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right)\mathrm{d}\tau\right],
\end{aligned}$$

where we use the fact that

$$\int_0^1\frac{1-\tau}{2}\left|\mathrm{e}^{-\theta\tau}-1\right|\mathrm{d}\tau=\frac{1}{4}+\frac{\mathrm{e}^{-\theta}-1}{\theta}+\frac{-\theta\mathrm{e}^{-\theta}-\mathrm{e}^{-\theta}+1}{\theta^2}$$

and

$$\int_0^1\frac{1+\tau}{2}\left|\mathrm{e}^{-\theta\tau}-1\right|\mathrm{d}\tau=\frac{3}{4}+\frac{\mathrm{e}^{-\theta}-1}{\theta}-\frac{-\theta\mathrm{e}^{-\theta}-\mathrm{e}^{-\theta}+1}{2\theta^2}.$$

After suitable arrangements, we obtain

$$\begin{aligned}
&\left|\frac{\varphi(\gamma_1+\gamma_2)}{2}-\frac{1-\mu}{2\left(1-\mathrm{e}^{-\theta}\right)}\left[I_{\gamma_2^-}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)+I_{\gamma_1^+}^{\mu}\varphi\left(\frac{\gamma_1+\gamma_2}{2}\right)\right]\right| \\
&\le \frac{\gamma_2-\gamma_1}{4\theta\left(1-\mathrm{e}^{-\theta}\right)}\left(\mathrm{e}^{-\theta}-1+\theta\right)\left[\left|\varphi'(\gamma_1)\right|+m\left|\varphi'\left(\frac{\gamma_2}{m}\right)\right|\right].
\end{aligned}$$

This ends the proof.

3. Conclusion

In this article, taking different exponential kernels parameters, we established three fractional integrals inequalities for m-convex mappings. Furthermore, we constructed a new lemma to obtain Dragomir-Agarwal inequality for m-convex mappings. We emphasized that certain results proved in this article generalize and extend parts of the results provided by Ahmad *et al.* in [1]. Finally, we gave mid-point type and trapezoid type inequalities for m-convex mappings.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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