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Supereulerian Digraph Strong Products

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Abstract

A vertex cycle cover of a digraph H is a collection $\mathcal{C}=\{C_1,C_2,\cdots,C_k\}$ of directed cycles in H such that these directed cycles together cover all vertices in H and such that the arc sets of these directed cycles induce a connected subdigraph of H. A subdigraph F of a digraph D is a circulation if for every vertex in F, the indegree of v equals its out degree, and a spanning circulation if F is a cycle factor. Define $f(D)$ to be the smallest cardinality of a vertex cycle cover of the digraph obtained from D by contracting all arcs in F, among all circulations F of D. Adigraph D is supereulerian if D has a spanning connected circulation. In [International Journal of Engineering Science Invention, 8 (2019) 12-19], it is proved that if D_1 and D_2 are nontrivial strong digraphs such that D_1 is supereulerian and D_2 has a cycle vertex cover $\mathcal{C}'$ with $|\mathcal{C}'|\le|V(D_1)|$, then the Cartesian product D_1 and D_2 is also supereulerian. In this paper, we prove that for strong digraphs D_1 and D_2, if for some cycle factor F_1 of D_1, the digraph formed from D_1 by contracting arcs in F_1 is hamiltonian with $f(D_2)$ not bigger than $|V(D_1)|$, then the strong product D_1 and D_2 is supereulerian.

Keywords

Supereulerian Digraph, Direct Product, Strong Product, Cycle Factors, Eulerian Digraph

1. Introduction

We consider finite graphs and digraphs. Undefined terms and notation will follow [1] for graphs and [2] for digraphs. We will often write $D=(V(D),A(D))$ with $V(D)$ and $A(D)$ denoting the vertex set and arc set of D, respectively. As we are to discuss products, for digraphs D_1 and D_2 with $u\in V(D_1)$ and

$v \in V(D_2)$, we save the notation (u,v) for a vertex in the product of D_1 and D_2. Thus, throughout this article, for vertices $u,v \in V(D)$ of a digraph D, we use the notation uv to denote the arc oriented from u to v in D, where u is the **tail** and v is a **head** of the arc, and use $[u,v]$ to denote either u,v or (v,u). When $[u,v] \in A(D)$, we say that u and v are adjacent. Using the terminology in [2], digraphs do not have parallel arcs (arcs with the same tail and the same head) or loops (arcs with same tail and head). If D is a digraph, we often use $G(D)$ to denote the underlying undirected graph of D, obtained from D by erasing all orientation on the arcs of D.

For a positive integer n, we define $[n] = \{1,2,\cdots,n\}$. Throughout this paper, we use paths, cycles and trails as defined in [1] when the discussion is on an undirected graph G, and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph D. A **walk** in D is an alternating sequence $W = x_1, a_1, x_2, \cdots, x_{k-1}, a_{k-1}, x_k$ of vertices x_i and arcs a_j from D such that $a_j = x_j x_{j+1}$ for every $i \in [k]$ and $j \in [k-1]$. A walk W is **closed** if $x_1 = x_k$, and is open otherwise. We use $V(W) = \{x_i : i \in [k]\}$ and $A(W) = \{a_j : j \in [k-1]\}$. We say that W is a walk from x_1 to x_k or an (x_1, x_k)-walk. If $x_1 = x_k$, then we say that the vertex x_1 is the **initial vertex** of W, the vertex x_k is the **terminal vertex** of W, and x_1 and x_k are end-vertices of W. The length of a walk is the number of its arcs. When the arcs of W are understood from the context, we will denote W by $x_1 x_2 \cdots x_k$. A **trail** in D is a walk in which all arcs are distinct. Always we use a trail to denote an open trail. If the vertices of W are distinct, then W is a path. If the vertices $x_1 x_2 \cdots x_{k-1}$ of the path W are distinct satisfying $k \geq 3$ and $x_1 = x_k$, then W is a **cycle**.

A digraph D is **strong** if, for every pair x, y of distinct vertices in D, there exists an (x,y)-walk and a (y,x)-walk; and is **connected** if $G(D)$ is connected. For the digraphs H and D, by $H \subset D$ we mean that H is a subdigraph of D. Following [3], for a digraph D with $X, Y \subset V(D)$, define

$$(X,Y)_D = \{xy \in A(D) : x \in X, y \in Y\}.$$

when $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X,Y)_D \text{ and } \partial_D^-(X) = (Y,X)_D$$

For a vertex v in D, $d_D^+(v) = |\partial_D^+\{v\}|$ and $d_D^-(v) = |\partial_D^-\{v\}|$ are the **out-degree** and the **in-degree** of v in D, respectively. We use the following notation:

$$N_D^+(v) = \{u \in V(D) - v : vu \in A(D)\} \text{ and } N_D^-(v) = \{w \in V(D) - v : wv \in A(D)\}$$

The sets $N_D^+(v), N_D^-(v)$ and $N_D(v) = N_D^+(v) + N_D^-(v)$ are called the **out-neighbourhood**, **in-neighbourhood** and **neighbourhood** of v. We called the vertices in $N_D^+(v)$, $N_D^-(v)$ and $N_D(v)$ the **out-neighbours**, **in-neighbours** and **neighbours** of v.

Let D be a digraph. We define D to be a **circulation** if for any $v \in V(D)$ we have $d_D^+(v) = d_D^-(v)$; and a strong digraph D is eulerian if for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. D is eulerian if D is a connected circulation. Thus, by definition, an eulerian digraph is also a strong digraph. It is known [3] that a digraph D is a circulation if and only if D is an arc-disjoint union of cycles. A subdigraph F of D is a **cycle factor** of D if F is spanning circulation of D. Define $f(D)=\min\{k:D$ has a cycle factor with k components$\}$. The following is well-known or immediately from the definition.

Theorem 1.1. (Euler, see Theorem 1.7.2 of [2] and Veblen [3]) Let D be a digraph. The following are equivalent.

(i) D is eulerian.

(ii) D is a spanning closed trail.

(iii) D is a disjoint union of cycles and D is connected.

The supereulerian problem was introduced by Boesch, Suffel, and Tindell in [4], seeking to characterize graphs that have spanning Eulerian subgraphs. Pulleyblank in [5] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. There have been lots of research on this topic. For more literature on supereulerian graphs, see Catlin's informative survey [6], as well as the later updates in [7] and [8]. The supereulerian problem in digraphs is considered by Gutin [9] [10]. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph, or equivalently, a connected cycle factor. Thus, supereulerian digraphs must be strong, and every hamiltonian digraph is also a supereulerian digraph.

The supereulerian digraph problem is to characterize the strong digraphs that contain a spanning closed trail.

Other than the researches on hamiltonian digraphs, a number of studies on supereulerian di-graphs have been conducted recently. In particular, Hong et al in [11] [12] and Bang-Jensen and Maddaloni [13] presented some best possible sufficient degree conditions for supereulerian digraphs. Several researches on various conditions of supereulerian digraphs can be found in [13]-[23], among others.

Following [24], some digraph products are defined as follows.

Definition 1.2. Let $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ be two digraphs,

$$V_1=\{u_1,u_2,\cdots,u_{n_1}\},\ V_2=\{v_1,v,\cdots,v_{n_2}\} \tag{1}$$

Then the Cartesian product, the Direct product and the Strong product of D_1 and D_2 are defined as following,

(i) **The Cartesian product** denoted by $D_1 \boxdot D_2$ is the digraph with vertex set $V_1\times V_2$ and

$$A(D_1 \boxdot D_2)=\{((u_i,v_j),(u_s,v_t)):u_i=u_s \text{ and } v_jv_t\in A_2$$
$$\text{or } u_iu_s\in A_1 \text{ and } v_j=v_t\}.$$

(ii) **The Direct product** denoted by $D_1\times D_2$ is the digraph with vertex set $V_1\times V_2$ and

$$A(D_1\times D_2)=\{((u_i,v_j),(u_s,v_t)):u_iu_s\in A_1 \text{ and } v_jv_t\in A_2\}.$$

(iii) **The Strong product** denoted by $D_1\boxtimes D_2$ is the digraph with vertex set

$V_1 \times V_2$ and

$$A(D_1 \boxtimes D_2) = \{((u_i, v_j),(u_s, v_t)) : u_i = u_s \text{ and } v_j v_t \in A_2 \text{ or } u_i u_s \in A_1$$
$$\text{and } v_j = v_t \text{ or both } u_i u_s \in A_1 \text{ and } v_j v_t \in A_2\}.$$

It is often of interest to investigate natural conditions on the factors of a product to assure hamiltonicity of the product, as seen in Problem 6 of [25]. Researchers have investigated conditions on factors of digraph products to warrant the product to be supereulerian. Alsatami, Liu, and Zhang in [17] introduced eulerian vertex cover of a digraph D to study the supereulerian digraph problem.

Definition 1.3. Let D be a digraph, $C_1, C_2, \cdots, C_k$ be eulerian subdigraphs of D and set $\mathcal{F} = \{C_1, C_2, \cdots, C_k\}$ where $k > 0$ is an integer.

(i) $\mathcal{F}$ is called a **cycle vertex cover** of D, if each C_i in $\mathcal{F}$ is a cycle, and both (i-1) and (i-2) hold:

(i-1) $V(D) = \bigcup_{C_i \in \mathcal{F}} V(C_i)$.

(i-2) $F = \bigcup_{C_i \in \mathcal{F}} C_i$ is weakly connected.

(ii) For any $u, v \in V(D)$, $\mathcal{F}$ is called an **eulerian chain** joining u and v, if each of the following holds.

(ii-1) $u \in V(C_1)$ and $v \in V(C_k)$.

(ii-2) $V(C_i) \cap V(C_{i+1}) \neq \emptyset$ for any i with $1 \le i \le k-1$.

A subdigraph F of a digraph D is a **circulation** if $d_F^-(v) = d_F^+(v) > 0$ holds for every $v \in V(F)$, and a spanning circulation of D is a **cycle factor** of D.

Let $e = [v_1, v_2] \in A(D)$ denote an arc of D which is either $v_1 v_2$ or $v_2 v_1$. Define D/e to be the digraph obtained from $D - e$ by identifying v_1 and v_2 into a new vertex v_e, and deleting the possible resulting loop(s). If $W \subseteq A(D)$ is a symmetric arc subset, then define the **contraction** D/W to be the digraph obtained from D by contracting each arc $e \in W$, and deleting any resulting loops. Thus even D does not have parallel arcs, a contraction D/W is loopless but may have parallel arcs, with $A(D/W) \subseteq A(D) - W$. If H is a subdigraph of D, then we often use D/H for $D/A(H)$. If L is a connected symmetric component of H and v_L is the vertex in D/H onto which L is contracted, then L is the **contraction preimage** of v_L. We adopt the convention to define $D/\emptyset = D$, and define a vertex $v \in V(D/W)$ to be a **trivial vertex** if the preimage of v is a single vertex (also denoted by v) in D. Hence, we often view trivial vertices in a contraction D/W as vertices in D.

Definition 1.4. Let F be a circulation of a digraph D and D/F denote the digraph formed from D by contracting arcs in $A(F)$. For any circulation F of D, define

(i) $f_{D(F)} = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a cycle vertex cover of } D/F\}$ and,

(ii) $f(D) = \min\{f_{D(F)} : F \text{ is a circulation of } D\}$.

By definition, if D is a circulation, then every component of D is eulerian. By Theorem 1.1, we observe the following.

$$\text{Every circulation is an arc-disjoint union of cycles.} \tag{2}$$

There have been some former results concerning the Cartesian products of digraphs to be eulerian and to be supereulerian.

Theorem 1.5. Let D_1 and D_2 be nontrivial strong digraphs.

(i) (Xu [26]) If D_1 and D_2 are eulerian digraphs. Then the Cartesian product $D_1 \Box D_2$ is eulerian.

(ii) (Alsatami, Liu, and Zhang [17]) If such that D_1 is supereulerian and D_2 has a cycle vertex cover $\mathcal{C}'$ with $|\mathcal{C}'| \le |V(D_1)|$, then the Cartesian product $D_1 \Box D_2$ is supereulerian.

The current research is motivated by Problem 6 of [25] and Theorem 1.5. We prove the following.

Theorem 1.6. Let D_1 and D_2 be strong digraphs. If $f(D_2) \le |V(D_1)|$ and if for some cycle factor F of D_1, D_1/F is hamiltonian, then the strong product $D_1 \boxtimes D_2$ is supereulerian.

In the next section, we develop some lemmas which will be used in ourarguments. The proof of the main result will be given in the last section.

2. Lemmas

Let $k \ge 0$ be an integer. We use $\mathbb{Z}_k = \{1, 2, \cdots, k\}$ to denote the cyclic group of order k and with the additive binary operation $+_k$ and with k being the additive identity in $\mathbb{Z}_k$. Let H and H' denote two digraphs. Define $H \cup H'$ to be the digraph with $V(H \cup H') = V(H) \cup V(H')$ and $A(H \cup H') = A(H) \cup A(H')$.

Let $T = v_1 v_2 \cdots v_k$ denote a trail. We use $T[v_1, v_k]$ to emphasize that T is oriented from v_1 to v_k. For any $1 \le i \le j \le k$, we use $T[v_i, v_j] = v_i v_{i+1} \cdots v_{j-1} v_j$ to denote the sub-trail of T. Likewise, if $Q = u_1 u_2 \cdots u_k u_1$ is a closed trail, then for any i, j with $1 \le i < j \le k$, $Q[u_i, u_j]$ denotes the sub-trail $u_i u_{i+1} \cdots u_{j-1} u_j$. If $T' = w_1 w_2 \cdots w_{k'}$ is a trail with $v_k = w_1$ and $V(T) \cap V(T') = \{v_k\}$, then we use TT' or $T[v_1, v_k] T'[v_k, w_{k'}]$ to denote the trail $v_1 v_2 \cdots v_k w_2 \cdots w_{k'}$. If $V(T) \cap V(T') = \varnothing$ and there is a path $z_1 z_2 \cdots z_t$ with $z_2, \cdots, z_{t-1} \notin V(T) \cup V(T')$ and with $z_1 = v_k$ and $z_t = w_1$, then we use $T z_1 \cdots z_t T'$ to denote the trail $v_1 v_2 \cdots v_k z_2 \cdots z_t w_2 \cdots w_{k'}$. In particular, if T is a (v, w)-trail of a digraph D and $uv, wz \in A(D) - A(T)$, then we use $uvTwz$ to denote the (u, z)-trail $D[A(T) \cup \{uv, wz\}]$. The subdigraphs uvT and Twz are similarly defined.

Lemma 2.1. Let $J_1, J_2, \cdots, J_k$ be vertex disjoint strong subdigraphs of a digraph D, and $J = \bigcup_{i=1}^{k} J_i$ is the disjoint union of these subdigraphs. Let $v_1, v_2, \cdots, v_k$ be vertices in $V(D/J)$ such that for each $i \in [k]$, J_i is the preimage of v_i. Suppose that $C' = v_{i_1}, v_{i_2}, \cdots, v_{i_s}$ be a cycle of D/J. Each of the following holds.

(i) D has a cycle C with $A(C') \subseteq A(C)$ such that for each $i \in [k]$, $V(C) \cap V(J_i) \ne \varnothing$. (Such a cycle C is called a **lift** of the cycle C'.)

(ii) If for each $i \in \mathbb{Z}_s$, $e_i = v_i'' v_{i+1}' \in A(C')$ is an arc in D with $v_i'' \in V(J_i)$ and $v_{i+1}' \in V(J_{i+1})$, then $C[v_i', v_i'']$ is a path in J_i.

Proof. As (i) implies (ii), it suffices to prove (i). Let $C' = v_1 v_2 \cdots v_s v_1$ be a cycle of D/J, and for each $i \in \mathbb{Z}_s$. By definition, the arc $e_i = v_i v_{i+1} \in A(C')$ is an arc in D, and so we may assume that there exist vertices $v_i', v_i'' \in V(J_i)$ such that $e_i = v_i'' v_{i+1}' \in A(D)$. If J_i is trivial, then we have $v_i' = v_i''$. Since J_i is strong, J_i contains a (v_i', v_i'')-path P_i. Thus

$$C := P_1 v_1'' v_2' P_2 v_2'' v_3' \cdots v_{i-1}'' v_i' P_i v_i'' v_{i+1}' P_{i+1} \cdots v_{s-1}'' v_s' P_s v_s'' v_1'$$

is a cycle of D with $C[v_i', v_i'']$ being a path in J_i, for each $i \in \mathbb{Z}_s$. ■

Following [2], we define a digraph to be **cyclically connected** if for every pair x, y of distinct vertices of D there is a sequence of cycles $C_1, C_2, \cdots, C_k$ such that x is in C_1, y is in C_k, and C_i and C_{i+1} have at least one common vertex for every $i \in [k-1]$. The following results are useful. Lemma 2.2 (ii) follows immediately from definition of strong digraphs.

Lemma 2.2. Let D be a digraph.

(i) (Exercise 1.17 of [2]) A digraph D is strong if and only if it is cyclically connected.

(ii) If H_1 and H_2 are strong subdigraphs of D with $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also strong.

Proposition 2.3. (Alsatami, Liu and Zhang, Proposition 2.1 of [17]) Let D be a weakly connected digraph.

Then the following are equivalent.

(i) D has a cycle vertex cover.

(ii) D is strong.

(iii) D is cyclically connected.

(iv) For any vertices $u, v \in V(D)$, there exists an eulerian chain joining u and v.

Lemma 2.4. Let D_1 and D_2 be digraphs. Each of the following holds.

(i) If D_1 and D_2 are cycles, then $D_1 \times D_2$ is a circulation.

(ii) If H_1 and H_2 are arc-disjoint subdigraphs of D_1, then $H_1 \times D_2$ and $H_2 \times D_2$ are arc-disjoint subdigraphs of $D_1 \times D_2$.

(iii) If each of D_1 and D_2 has a cycle factor, then $D_1 \times D_2$ has a cycle factor.

Proof. For (i), let V_1 and V_2 be the vertex sets of D_1 and D_2, respectively. It suffices to prove that for each $(u_i, v_j) \in V_1 \times V_2$, $d^+_{D_1 \times D_2}((u_i, v_j)) = d^-_{D_1 \times D_2}((u_i, v_j))$. Let $(u_i, v_j) \in V_1 \times V_2$. Since D_1 and D_2 are cycles, we have $|N^+_{D_1}(u_i)| = |N^-_{D_1}(u_i)|$ and $|N^+_{D_2}(v_j)| = |N^-_{D_2}(v_j)|$. By Definition 1.2, we have the following, which implies (i).

$$\begin{aligned}
d^+_{D_1 \times D_2}((u_i, v_j)) &= \left|N^+_{D_1 \times D_2}((u_i, v_j))\right| \\
&= \left|\{(u_s, v_t) \in V_1 \times V_2 : (u_i, v_j)(u_s, v_t) \in A(D_1 \times D_2)\}\right| \\
&= \left|\{(u_s, v_t) \in V_1 \times V_2 : u_i u_s \in A(D_1) \text{ and } v_j v_t \in A(D_2)\}\right| \\
&= \sum_{u_s \in N^+_{D_1}(u_i)} \sum_{v_t \in N^+_{D_2}(v_j)} \left|\{(u_s, v_t) \in V_1 \times V_2\}\right| \\
&= \left|N^+_{D_1}(u_i)\right| \cdot \left|N^+_{D_2}(v_j)\right| = \left|N^-_{D_1}(u_i)\right| \cdot \left|N^-_{D_2}(v_j)\right|
\end{aligned}$$

$$= \sum_{u_s \in N^-_{D_1}(u_i)} \sum_{v_t \in N^-_{D_2}(v_j)} \left|\left\{(u_s, v_t) \in V_1 \times V_2\right\}\right|$$

$$= \left|\left\{(u_s, v_t) \in V_1 \times V_2 : u_s u_i \in A(D_1) \text{ and } v_t v_j \in A(D_2)\right\}\right|$$

$$= \left|N^-_{D_1 \times D_2}\left((u_i, v_j)\right)\right|$$

$$= \left|\left\{(u_s, v_t) \in V_1 \times V_2 : (u_s, v_t)(u_i, v_j) \in A(D_1 \times D_2)\right\}\right|$$

$$= d^-_{D_1 \times D_2}\left((u_i, v_j)\right)$$

To prove (ii), let H_1 and H_2 be an arc-disjoint subdigraph of D_1. If there exists an arc

$$(u_i, v_j)(u_s, v_t) \in A(H_1 \times D_2) \cap A(H_2 \times D_2),$$

then by Definition 1.2, we must have $u_i u_s \in H_1 \cap H_2$. Hence if H_1 and H_2 are arc-disjoint subdigraphs of D_1, then $H_1 \times D_2$ and $H_2 \times D_2$ are arc disjoint subdigraphs of $D_1 \times D_2$.

To prove (iii), let F_1 and F_2 be the spanning circulations of D_1 and D_2, respectively. By Definition 1.2, $F_1 \times F_2$ is spanning subdigraph of $D_1 \times D_2$. By (i), $F_1 \times F_2$ is a circulation, and so $F_1 \times F_2$ is the spanning circulation of $D_1 \times D_2$. Thus $F_1 \times F_2$ is a cycle factor of $D_1 \times D_2$. ■

Lemma 2.5. Let D_1, D_2 be digraphs and F be a subdigraph of D_1. Then $A(F \square D_2) \cap A(F \times D_2) = \varnothing$.

Proof. Suppose that there exists an arc $(u_i, v_j)(u_s, v_t) \in A(F \square D_2) \cap A(F \times D_2)$. By Definition 1.2 (i), as $(u_i, v_j)(u_s, v_t) \in A(F \square D_2)$, we have either $u_i = u_s$ and $v_j v_t \in A(D_2)$ or $u_i u_s \in A(F)$ and $v_j = v_t$. By Definition 1.2 (ii), if $u_i = u_s$ or if $v_j = v_t$, then $(u_i, v_j)(u_s, v_t) \notin A(F \times D_2)$. It follows that $A(F \square D_2) \cap A(F \times D_2) = \varnothing$. ■

Theorem 2.6. (Hammack, Theorem 10.3.2 of [24]) Let m and n be integers with $m \ge n \ge 2$ and let C_m and C_n denote the cycles of order m and n, respectively. Let $gcd(m,n)$ and $lcm(m,n)$ be the greatest common divisor and the least common multiplier of m and n, respectively. Then the direct product $C_m \times C_n$ is a vertex disjoint union of $gcm(m,n)$ cycles, each of which has length $lcm(m,n)$.

We can show a bit more structural properties in the direct product revealed by Theorem 2.6, which are stated in Lemma 2.7.

Lemma 2.7. Let D_1 and D_2 be digraphs with vertex set notation in (1).

(i) Suppose that D_1 and D_2 are cycles and $v \in V(D_2)$ is an arbitrarily given vertex. Then for any cycle C in $D_1 \times D_2$, there exists a vertex $u \in V(D_1)$ such that the vertex $(u,v) \in V(C)$.

(ii) Suppose that D_1 and D_2 are circulations and $v \in V(D_2)$ is an arbitrarily given vertex. Then $D_1 \times D_2$ is also a circulation. Moreover, for any eulerian subdigraph F in $D_1 \times D_2$, there exists a vertex $u \in V(D_1)$ such that the vertex $(u,v) \in V(F)$.

Proof. Suppose $D_1 = u_1u_2\cdots u_{n_1}u_1$ and $D_2 = v_1v_2\cdots v_{n_2}v_1$ are cycles, and by symmetry, assume that $v = v_1$. Let C be a cycle in $D_1 \times D_2$. Thus C contains a vertex (u_i, v_j). It follows by Definition 1.2 that

$$C = \cdots(u_i, v_j)(u_{i+1}, v_{j+1})\cdots(u_{i+n_2-j}, v_{n_2})(u_{i+n_2-j+1}, v_1)\cdots$$

where the subscripts of vertices in D_1 are taken in $\mathbb{Z}_{n_1}$ and those of vertices in D_2 are taken in $\mathbb{Z}_{n_2}$. It follows that $u = u_{i+n_2-j+1}$. This proves (i). Suppose that D_1 and D_2 are circulations. By (2), each of D_1 and D_2 is an arc-disjoint union of cycles. By Lemma 2.4, $D_1 \times D_2$ is also a circulation. Let F be an eulerian subdigraph in $D_1 \times D_2$. By (2), F is also an arc-disjoint union of cycles $C_1, C_2, \cdots$. Applying Lemma 2.7 (i) to each cycle C_i, we conclude that (ii) holds as well. ■

3. Proofs of Theorem 1.6

Assume that D_1 and D_2 are two strong digraphs, and for some cycle factor F of D_1, D_1/F is hamiltonian with $f(D_2) \le |V(D_1)|$. We start with some notation for the copies of factors in the Cartesian product.

Definition 3.1. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two strong digraphs with $V_1 = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \cdots, v_{n_2}\}$. For $i \in \{1,2\}$, let H_i be a subdigraph of D_i.

(i) For each $u \in V_1$, let D_2^u be the subdigraph of $D_1 \boxdot D_2$ induced by $V(D_2^u) = \{(u, v_i) : 1 \le i \le n_2\}$. The subdigraph D_2^u is called the *u*-copy of D_2 in $D_1 \boxdot D_2$.

(ii) For each $v \in V_2$, let D_1^v be the subdigraph of $D_1 \boxdot D_2$ induced by $V(D_1^v) = \{(u_i, v) : 1 \le i \le n_1\}$. The subdigraph D_1^v is called the *v*-copy of D_1 in $D_1 \boxdot D_2$.

(iii) More generally, for each $u \in V_1$ (or $v \in V_2$, respectively), let H_2^u (or H_1^v, respectively) be the subdigraph of D_2^u (or D_1^v, respectively) induced by $A(H_2^u) = \{(u, v_i)(u, v_i') : v_iv_i' \in A(H_2)\}$ (or $A(H_1^v) = \{(u_i, v)(u_i', v) : u_iu_i' \in A(H_1)\}$, respectively). The subdigraph H_1^v is called the *v*-copy of H_1 in $D_1 \boxdot D_2$ and the subdigraph H_2^u is called the *u*-copy of H_2 in $D_1 \boxdot D_2$.

If two digraphs D and H are isomorphic, then we write $D \cong H$. The following is an immediate observation from Definition 3.1 for the Cartesian product $D_1 \boxdot D_2$ of two digraphs D_1 and D_2.

$$\text{For any } v \in V(D_2),\ D_1 \cong D_1^v, \text{ and for any } u \in V(D_1),\ D_2 \cong D_2^u. \tag{3}$$

Let F be a cycle factor of D_1 such that D_1/F has a Hamilton cycle. Since F is a cycle factor of D_1, each component of F is an eulerian subdigraph of D_1. Let

$$F_1, F_2, \cdots, F_k \text{ be the components of } F, \text{ and } J = D_1/F. \tag{4}$$

Then $V(J) = \{w_1, w_2, \cdots, w_k\}$, where for each $i \in [k]$, w_i is the contraction image in J of the eulerian subdigraph F_i in D_1. Since J is hamiltonian, we may by symmetry assume that $C' = w_1w_2\cdots w_kw_1$ is a hamilton cycle of J. It follows by Lemma 2.1 that

$$D_1 \text{ has a cycle } C \text{ with } A(C') \subseteq A(C). \tag{5}$$

Now we consider D_2. Let $f(D_2) = m \le |V(D_1)|$ and F' be a circulation of D_2 such that D_2/F' has a cycle vertex cover $\mathcal{C}' = \{C_1', C_2', \cdots, C_m'\}$. Let $F_1', F_2', \cdots, F_{k'}'$ be the components of F', $w_{k'+1}', \cdots, w_t'$ be the vertices in $V(D_2) - V(F')$. We define, for each i with $k'+1 \le i \le t$, F_i' to be the digraph with $V(F_i') = \{w_i'\}$ and $A(F_i') = \varnothing$. With these definitions, we have

$$V(D_2/F') = \{w_1', w_2', \cdots, w_{k'}', w_{k'+1}', \cdots, w_t'\} \tag{6}$$

By Lemma 2.1, for each $j \in [m]$, C_j' in $\mathcal{C}'$ can be lifted to a cycle C_j in D_2. To construct a spanning eulerian subdigraph of $D_1 \boxtimes D_2$, we start by justifying the following claims.

Claim 1. Each of the following holds.

(i) For any $i \in [k]$, and $j \in [t]$, $F_i \times F_j'$ is a circulation.

(ii) For any $i \in [k]$, and $j \in [t]$, $F_i \boxdot F_j'$ is an eulerian digraph.

(iii) For any $i \in [k]$, and for each $j \in [t]$, if $v \in V(F_j')$, then $F_i^v \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $F_i \boxtimes F_j'$.

Proof. For each $i \in [k]$, F_i is an eulerian subdigraph of D_1, so F_i is a disjoint union of cycles. Similarly, for each $j \in [k']$, F_j' is an eulerian sudigraph of D_2, so F_j' is a disjoint union of cycles. By Lemma 2.7, $F_i \times F_j'$ is a circulation.

By assumption, for each $i \in [k]$, F_i is an eulerian subdigraph of D_1. If $j \in [k']$, then as F_j' is an eulerian sudigraph of D_2, it follows by Theorem 1.5 (i) that $F_i \boxdot F_j'$ is an eulerian digraph.

Now assume that $k'+1 \le j \le t$. Then $V(F_j') = \{w_j'\}$, and so by (3), $F_i \boxdot F_j' = F_i^{w_j'} \cong F_i$ is eulerian. This proves (ii).

For each $i \in [k]$, each $j \in [t]$ and a fixed vertex $v \in V(F_j')$, let $J' = F_i^v \cup (F_i \times F_j')$. By (i), $F_i \times F_j'$ is a circulation. By (3), $F_i^v \cong F_i$ is an eulerian digraph. By Lemma 2.5, $A(F_i^v) \cap A(F_i \times F_j') = \varnothing$. It follows that for any vertex $z \in V(J)$,

$$d_J^+(z) = d_{F_i^v}^+(z) + d_{F_i \times F_j'}^+(z) = d_{F_i^v}^-(z) + d_{F_i \times F_j'}^-(z) = d_J^-(z)$$

and so J' is a circulation. Without loss of generality, we denote $V(F_i) = \{u_{i_1}, u_{i_2}, \cdots, u_{i_{t_i}}\}$ and $V(F_j') = \{v_{j_1}, v_{j_2}, \cdots, v_{j_{s_j}}\}$ with $v = v_{j_1}$. To prove that J' is connected, let $z_0 = (u_{i_1}, v_{j_1}) \in V(J')$ and let J_1 be the connected component of J' that contains z_0. If J' is not connected, then by symmetry, we may assume that there exists a vertex $(u_{i_2}, v_{j_2}) \in V(J') - V(J_1)$. As $F_i \times F_j'$ is a circulation, there must be an eulerian subdigraph F of $F_i \times F_j'$ with $(u_{i_2}, v_{j_2}) \in V(F)$. By Lemma 2.7 (ii), there exists a vertex $u' \in V(D_1)$ such that $(u', v_{j_1}) \in V(F)$. Thus by Definition 3.1 (ii), $V(F) \cap V(F_i^v) \ne \varnothing$. By (3) and (4), $F_i^v \cong F_i$ is connected, and so both (u_{i_1}, v_{j_1}) and (u', v_{j_1}) must be in the same component of J'. This implies that $(u', v_{j_1}) \in V(J_1)$. Since (u_{i_2}, v_{j_2}) and (u', v_{j_1}) are in the same component of J', It follows that $(u_{i_2}, v_{j_2}) \in V(J_1)$ also, contrary to the assumption that $(u_{i_2}, v_{j_2}) \in V(J') - V(J_1)$. Hence J'

must be connected, and so $F_i^v \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $F_i \boxtimes F_j'$. ■

Claim 2. Let C' be a Hamilton cycle of J and C be a lift of C' in D_1 as warranted by (5). For each $v \in V(D_2)$, let C^v denote the v-copy of C in $D_1 \boxdot D_2$. For each $j \in [t]$, if $v, v' \in V(F_j')$ are two distinct vertices, then

$$H_{v,v';j} := \bigcup_{i=1}^{k} \left(F_i^{v'} \cup (F_i \times F_j')\right) \cup C^v$$

is a spanning eulerian subdigraph $D_1 \boxtimes F_j'$.

Proof. By Lemma 2.1. for any $v \in V(D_2)$, C^v has the property that for any $i \in [k]$, $V(C^v) \cap V(F_i^v) \neq \varnothing$. By Claim 1 (iii), for any $i \in [k]$ and for any $j \in [t]$, $F_i^{v'} \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $F_i \boxtimes F_j'$ and so $F_i^{v'} \cup (F_i \times F_j')$ is a strong subdigraph of $D_1 \boxtimes F_j'$. Since for any $i \in [k]$, $V(C^v) \cap V(F_i^v) \neq \varnothing$, we may assume that for some vertex $u \in V(F_i)$, $(u,v) \in V(C^v) \cap V(F_i^v)$. As $v \in V(F_j')$, we have $(u,v) \in V(C^v) \cap V\left(F_i^{v'} \cup (F_i \times F_j')\right)$ and so $F_i^{v'} \cup (F_i \times F_j') \cup C^v$ is connected. Since $v \neq v'$, $A(C^v) \cap A\left(F_i^{v'} \cup (F_i \times F_j')\right) = \varnothing$, we conclude from the facts that C^v and $F_i \times F_j'$ are circulations (see Claim 1 (i)) that $F_i^{v'} \cup (F_i \times F_j') \cup C^v$ is eulerian. As $i \in [k]$ is arbitrary, we conclude that

$$H_{v,v';j} := \bigcup_{i=1}^{k} \left(F_i^{v'} \cup (F_i \times F_j')\right) \cup C^v$$

is an eulerian subdigraph with vertex set $V(H_{v,v';j}) = \bigcup_{i=1}^{k} (F_i \times F_j') = V(D_1 \boxtimes F_j')$. This proves Claim 2. ■

Claim 3 Let $u \in V(D_1)$ be an arbitrary vertex, F' be a circulation of D_2 such that D_2/F' has a cycle vertex cover $\mathcal{C}' = \{C_1', C_2', \cdots, C_m'\}$ with $m = f(D_2) \le |V(D_1)|$. Each of the following holds.

(i) F'^u is a circulation of D_2^u.

(ii) For any $j \in [m]$, $C_j'^u$ is a cycle of D_2^u / F'^u and $\{C_1'^u, C_2'^u, \cdots, C_m'^u\}$ is a cycle vertex cover of D_2^u / F'^u.

(iii) Let $u \in V(D_1)$ be a vertex, $h \in [m]$ be arbitrarily given. For any vertex $w_j' \in V(C_h')$, let $v(j), v'(j)$ be two distinct vertices in $V(F_j')$, and C_h be a lift of C_h' in D_2. Then

$$H_h^u = \left[\bigcup_{w_j' \in V(C_h')} H_{v(j),v'(j);j}\right] \cup C_h^u$$

is an eulerian digraph with $V(H_h^u) = \bigcup_{v_j \in V(C_h)} V(D_1^{v_j})$.

Proof. Each of (i) and (ii) follows from (3) and the definition of $\mathcal{C}'$. It remains to prove (iii). By Lemma 2.1, C_h' can be lifted to a cycle C_h in D_2. For any $w_j' \in V(C_h')$, pick two distinct vertices $v, v' \in V(F_j')$. By Claim 2, $H_{v,v';j}$ defined in Claim 2 is a spanning eulerian subdigraph $D_1 \boxtimes F_j'$. By Lemma 2.5, $C_h^u = D_1[\{u\}] \boxdot C_h$ is arc-disjoint from each $H_{v,v';j}$, and so by the facts that C_h^u is a directed cycle and $H_{v,v';j}$ is eulerian, it follows that H_h^u is a circulation. By Definition 3.1 (iii) and by Lemma 2.5, $w_j' \in V(C_h')$ if and only if $V(C_h^u) \cap V(F_j'^u) \neq \varnothing$. This is equivalent to saying that a vertex $w_j' \in V(C_h')$ if and only if for some vertex $v'' \in V(F'_j)$ with $(u,v'') \in V(C^u_h)$. Since C^u_h is a cycle, and since, for each $w'_j \in V(C'_h)$, there exists some vertex $v'' \in V(F'_j)$ with $(u,v'') \in V(C^u_h)$, we obtain that $V(H_{v,v';j}) \cap V(C^u_h)$ contains a vertex (u,v''), it follows that H^u_h must be connected. Hence H^u_h is a connected circulation, and so it must be eulerian. To complete the justification of Claim3 (iii), we note that by definition,

$$V(C^u_h) \subseteq \bigcup_{w'_j \in V(C'_h)} V(D_1 \boxtimes F'_j).$$

This, together with Claim 2, implies

$$V(H^u_h) = \bigcup_{w'_j \in V(C'_h)} (H_{v(j),v'(j);j}) \cup V(C^u_h) = \bigcup_{w'_j \in V(C'_h)} V(D_1 \boxtimes F'_j) = \bigcup_{v_j \in V(C_h)} V(D_1^{v_j}).$$

This completes the proof of Claim 3. ∎

Recall that $V(D_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ with $n_1 \ge m = f(D_2)$. We will complete the proof of Theorem 1.6 by proving that

$$H = \bigcup_{h=1}^{m} H^{u_h}_h$$

is a spanning eulerian subdigraph of $D_1 \boxtimes D_2$. By Claim 3 (iii), we conclude that

$$V(H) = \bigcup_{j=1}^{t} V(D_1 \boxtimes F'_j) = V(D_1 \boxtimes D_2).$$

As $u_1, u_2, \cdots, u_m$ are mutually distinct, and as $F'_1, F'_2, \cdots, F'_t$ are mutually vertex disjoint, we conclude that the $H^{u_h}_h$'s are mutually arc-disjoint. By Claim 3 (iii), each $H^{u_h}_h$ is eulerian, and so H is a circulation. It remains to show that H is connected. By Claim 3 (iii), H has a component H' that contains $H^{u_1}_1$. If $H = H'$, then done. Assume that $V(H) - V(H') \neq \emptyset$.

Since H' is a component, if some $H^{u_h}_h$ contains a vertex in H', then H' contains $H^{u_h}_h$ as subdigraph. Thus every $H^{u_h}_h$ is either contained in H' or totally disjoint from H'. Let $W = \{w'_j \in V(D_2/F') : H^{u_j}_j \text{ is contained in } H'\}$. Then as $H \neq H'$, $V(D_2/F') - W \neq \emptyset$. Since C' is a cycle vertex cover of D_2/F', it follows by Definition 1.3 (i-2) that there must be a cycle $C'_j \in \mathcal{C}'$ such that C'_j contains a vertex $w' \in W$ and a vertex $w'' \in (D_2/F') - W$. Since $w' \in W$, $H^{u_j}_j$ is contained in H'. Since $w', w'' \in V(C'_j)$, it follows that $w'' \in W$, contrary to the fact that $w'' \in (D_2/F') - W$. This contradiction indicates that we must have $H = H'$, and so H is a spanning eulerian subdigraph of $D_1 \boxtimes D_2$. ∎

4. Concluding Remark

This research provides new conditions to ensure digraph products to be supereulerian, and adds novel knowledge to the literature of supereulerian digraph theory. Analogues to Problem 6 proposed in [25], it would also be of interest to seek natural conditions to assure supereulerian products of digraphs. Current results in this direction in [17] and in the current research also involve certain cycle cover properties on the factor digraphs. It would be of interest to see if there exist sufficient conditions on supereulerian digraphs products that do not depend on cycle cover properties.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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