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Approximate Bound State Solutions for Certain Molecular Potentials

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Abstract

We present solutions of the Schrodinger equation with superposition of Manning-Rosen plus inversely Mobius square plus quadratic Yukawa potentials using parametric Nikiforov Uvarov method along with an approximation to the centrifugal term. The bound state energy eigenvalues for any angular momentum quantum number l and the corresponding un-normalized wave functions are calculated. The mixed potential which in some particular cases gives the solutions for different potentials: the Manning-Rosen, the Mobius square, the inversely quadratic Yukawa and the Hulthén potentials along with their bound state energies are obtained.

Keywords

Schrödinger Equation, Mobius Potential, Manning-Rosen Potential, Quadratic Yukawa Potential, Hulthén Potential, Bound State Energies, Wave Functions

1. Introduction

The Schrödinger wave equation is primarily considered as one of the most commonly used differential equations in non-relativistic quantum mechanics [1] [2] [3]. However, since the early times of quantum mechanics, the exact solutions of the Schrödinger equation with some particular physical potentials are of much interest. Such solutions provide profound conceptual understanding to physical models and certainly lead to a strong judgment supporting the correctness of quantum theory. The exact solutions of central and non-central potentials find their applications in various branches of physics such as molecular, solid-state and chemical physics [4] and so forth. Our choice for the real poten-

tial gives the bound state energy eigenvalues and wave functions of the Schrödinger wave equation that might describe essentially the particle dynamics in non-relativistic quantum mechanics. Moreover, these solutions are used in checking and try to improve models under study and then also finding methods in solving complicated physical models.

Since the early times of quantum mechanics the number of exactly solvable physical problems is very limited. Several authors have paid many efforts toward studying the exactly solvable physical problems by which one can determine the whole energy spectrum analytically for wide range values of potential parameters [5]. Therefore, in most of these potentials, the quasi-exactly solvable potentials are the ones that provide a part of the energy spectrum [6].

Recently, various methods are introduced and employed in quantum mechanics in solving the wave equations with a particular given solvable potential. We mention few among the many methods: the group theoretical technique [7], the factorization method [8] [9], functional analysis approach (FAA) [10], supersymmetric (SUSY) quantum mechanics [11], shape invariance (SI) [12], the Nikiforov-Uvarov (NU) method [13], exact quantization rule [14] [15] and asymptotic iteration method (AIM) [16].

The Manning-Rosen, the quadratic Yukawa and the Mobius square potentials have been intensively considered and studied in non-relativistic and relativistic wave equations in recent years [17]-[27]. Therefore, the main motivation of the present work is to give approximate solution to the non-relativistic Schrödinger equation with the superposition of Manning-Rosen plus inversely Mobius square plus Yukawa potential models. Hence we need to treat the centrifugal term with Greene-Aldrich approximation to enable for analytical solution of the Schrödinger equation for any angular momentum quantum number l. This would provide us the bound state energy spectrum for any angular momentum quantum number l and the corresponding wave functions by simply applying the parametric Nikiforov-Uvarov (pNU) method.

The structure of the present work is as follows. In Section 2, we present the brief methodology. In Section 3, we apply this method to derive the bound state energy and wave functions for the Schrödinger equation with the present potential model. Section 4 presents our results and discussion. Finally, in Section 5, we give our conclusion.

2. Methodology

The Nikiforov-Uvarov (NU) [13] method is an efficient tool which is usually used to reduce the second-order differential equation into a general form of a hypergeometric type. In that sense, any second order differential equation, *i.e.* Schrödinger, Fienberg-Horodecki [28] [29] [30] [31], relativistic Dirac, Klien-Gordon equation, etc., can be transformed, using a suitable coordinate transformation $s = s(t)$, into the form:

$$\psi_n''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. The method is noted to be tiresome and time-consuming. Therefore, Tezcan and Sever [32] derived a parametric form of the NU method to popularize the method more. The parametric NU method is straightforward, simpler and more accurate for the determination of the energy eigenvalues and the corresponding eigenstates. To apply the parametric NU method, the differential equation must be set into the general form given by [33]

$$\psi_n''(s)+\frac{\alpha_1-\alpha_2 s}{s(1-\alpha_3 s)}\psi_n'(s)+\frac{-\gamma_1^2 s^2+\gamma_2 s-\gamma_3}{s^2(1-\alpha_3 s)^2}\psi_n(s)=0, \tag{2}$$

The conditions for the energy eigenvalues and the corresponding eigenstates are, respectively, given as

$$\begin{aligned}&(\alpha_2-\alpha_3)n+\alpha_3 n^2-(2n+1)\alpha_5+(2n+1)\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right)\\&+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0,\end{aligned} \tag{3}$$

$$\psi_n(s)=N_{nl}s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \tag{4}$$

where

$$\alpha_4=\frac{1}{2}(1-\alpha_1), \tag{5}$$

$$\alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3), \tag{6}$$

$$\alpha_6=\alpha_5^2+\gamma_1, \tag{7}$$

$$\alpha_7=2\alpha_4\alpha_5-\gamma_2, \tag{8}$$

$$\alpha_8=\alpha_4^2+\gamma_3,\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6, \tag{9}$$

$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}, \tag{10}$$

$$\alpha_{11}=\alpha_2-2\alpha_5+2\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right), \tag{11}$$

$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8}, \tag{12}$$

$$\alpha_{13}=\alpha_5-\left(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\right), \tag{13}$$

where N_{nl} is the normalisation constant and $P_n^{(\beta,\gamma)}$ is the orthogonal Jacobi polynomial.

3. Solution of the Schrödinger Equation with Two Molecular Potential Models

The Schrödinger equation in spherical coordinates is given as [33]

$$\begin{aligned}&-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)\\&=E\psi(r,\theta,\phi),\end{aligned} \tag{14}$$

where $\hbar$ is the reduced Plank constant, μ is the reduced mass, E is the ener-

gy eigenvalues, and ψ is the wave function of the particle. If we define the wave function as

$$\psi(r,\theta,\phi)=\frac{R_{nl}(r)}{r}Y_{lm}(\theta,\phi), \tag{15}$$

the radial part of Schrödinger equation is given by

$$\frac{\mathrm{d}^2R_{nl}(r)}{\mathrm{d}r^2}+\left[\frac{2\mu}{\hbar^2}\left(E-V(r)\right)-\frac{l(l+1)}{r^2}\right]R_{nl}(r)=0, \tag{16}$$

where n and l are the radial and the angular momentum quantum numbers, respectively.

We shall solve the Schrodinger equation for the following two molecular potential models:

3.1. Combination of Manning-Rosen Plus Mobius Square Plus Quadratic Yukawa Potentials

The general potential is given as [34] [35] [36]

$$V(r)=-\left(\frac{C\mathrm{e}^{-\alpha r}+D\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-\alpha r}\right)^2}\right)-V_0\left(\frac{A+B\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}\right)^2+\frac{V_1\mathrm{e}^{-\alpha r}}{r^2}, \tag{17}$$

where C,D,V_0,V_1 are potential parameters and α is the screening parameter.

It is obvious that Equation (16) cannot be solved analytically due to the quadratic Yukawa and the centrifugal terms. However, this can be addressed using the Green-Aldrich approximation [37]

$$\frac{1}{r^2}\approx\frac{\alpha^2\mathrm{e}^{-\alpha r}}{\left(1-\mathrm{e}^{-\alpha r}\right)^2}. \tag{18}$$

Substituting Equation (17) into Equation (16) and using Equation (18) with $s=\mathrm{e}^{-\alpha r}$ one obtains Equation (2), where

$$-\gamma_1^2=\frac{2\mu}{\hbar^2\alpha^2}\left(E+D+V_0B^2-\alpha^2V_1\right), \tag{19}$$

$$\gamma_2=\frac{2\mu}{\hbar^2\alpha^2}\left(C+2ABV_0-2E\right)-l(l+1), \tag{20}$$

$$\gamma_3=-\frac{2\mu}{\hbar^2\alpha^2}\left(E+V_0A^2\right). \tag{21}$$

Comparing Equation (38) with the parameters Equations (5) to (13), one gets

$$\alpha_1=\alpha_2=\alpha_3=1,\alpha_4=0,\alpha_5=-\frac{1}{2}, \tag{22}$$

$$\alpha_6=\frac{1}{4}-\frac{2\mu}{\hbar^2\alpha^2}\left(E+D+V_0B^2-\alpha^2V_1\right), \tag{23}$$

$$\alpha_7=-\frac{2\mu}{\hbar^2\alpha^2}\left(C+2ABV_0-2E\right)-l(l+1), \tag{24}$$

$$\alpha_8=-\frac{2\mu}{\hbar^2\alpha^2}\left(E+V_0A^2\right), \tag{25}$$

$$\alpha_9 = \frac{1}{4}(2l+1)^2 - \frac{2\mu}{\hbar^2\alpha^2}\left(C + D + (A+B)^2 V_0 - \alpha^2 V_1\right), \tag{26}$$

$$\alpha_{10} = 1 + 2\sqrt{-\frac{2\mu}{\hbar^2\alpha^2}\left(E + V_0 A^2\right)}, \tag{27}$$

$$\alpha_{11} = 2 + 2\left[\frac{1}{2}\sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left(C + D + (A+B)^2 V_0 - \alpha^2 V_1\right)} + \sqrt{-\frac{2\mu}{\hbar^2\alpha^2}\left(E + V_0 A^2\right)}\right], \tag{28}$$

$$\alpha_{12} = \sqrt{-\frac{2\mu}{\hbar^2\alpha^2}\left(E + V_0 A^2\right)}, \tag{29}$$

$$\alpha_{13} = -\frac{1}{2} - \left[\frac{1}{2}\sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left(C + D + (A+B)^2 V_0 - \alpha^2 V_1\right)} + \sqrt{-\frac{2\mu E}{\hbar^2\alpha^2}}\right]. \tag{30}$$

Substituting the values of the parametric constants Equations (22) to (30) into Equations (3) and (4), respectively, one gets the energy eigenvalues and the corresponding unnormalized radial eigenstates as

$$E_{nl} = -V_0 A^2 - \frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1) + l(l+1) + \frac{1}{2} + (n+1)T - \frac{2\mu}{\hbar^2\alpha^2}\left(C + 2V_0 A(A+B)\right)}{1 + 2n + T}\right]^2, \tag{31}$$

and

$$\psi_{nl}(r) = N_{nl} \mathrm{e}^{-\alpha\sqrt{\gamma_3} r}\left(1 - \mathrm{e}^{-\alpha r}\right)^{\beta} P_n^{\left(2\sqrt{\gamma_3}, T\right)}\left(1 - 2\mathrm{e}^{-\alpha r}\right), \tag{32}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left(C + D + (A+B)^2 V_0 - \alpha^2 V_1\right)}, \tag{33}$$

$$\gamma_3 = -\frac{2\mu}{\hbar^2\alpha^2}\left(E + V_0 A^2\right), \tag{34}$$

and $\beta = \frac{1+T}{2} - \sqrt{\gamma_3}$.

3.2. Combination of Manning-Rosen Plus Quadratic Yukawa Potentials

The Manning-Rosen plus quadratic Yukawa potential is given by [34] [35]

$$V(r) = -\left[\frac{C\mathrm{e}^{-\alpha r} + D\mathrm{e}^{-2\alpha r}}{\left(1 - \mathrm{e}^{-\alpha r}\right)^2}\right] + \frac{V_1 \mathrm{e}^{-\alpha r}}{r^2}, \tag{35}$$

where C, D, V_1 are potential parameters and α is the screening parameter.

Substituting Equation (35) into Equation (16) gives

$$\frac{\mathrm{d}^2 R_{nl}(r)}{\mathrm{d}r^2} + \left[\frac{2\mu}{\hbar^2}\left(E + \left(\frac{C\mathrm{e}^{-\alpha r} + D\mathrm{e}^{-2\alpha r}}{\left(1 - \mathrm{e}^{-\alpha r}\right)^2}\right) - \frac{V_1 \mathrm{e}^{-\alpha r}}{r^2}\right) - \frac{l(l+1)}{r^2}\right] R_{nl}(r) = 0. \tag{36}$$

Substituting Equation (18) in Equation (36) leads

$$\frac{\mathrm{d}^2 R_{nl}(r)}{\mathrm{d}r^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\left(\frac{C\mathrm{e}^{-\alpha r}+D\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-\alpha r}\right)^2}\right)-\frac{V_1\alpha^2\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-\alpha r}\right)^2}\right)-\frac{l(l+1)\alpha^2\mathrm{e}^{-\alpha r}}{\left(1-\mathrm{e}^{-\alpha r}\right)^2}\right]R_{nl}(r)=0. \tag{37}$$

Now, changing of variables using $s=\mathrm{e}^{-\alpha r}$ to transform the equation to the form of Equation (2), one obtains

$$R_{nl}''(s)+\frac{1-s}{s(1-s)}R_{nl}'(s)+\frac{-\gamma_1^2 s^2+\gamma_2 s-\gamma_3}{s^2(1-s)^2}R_{nl}(s)=0, \tag{38}$$

where

$$-\gamma_1^2=\frac{2\mu}{\hbar^2\alpha^2}\left(E+D-\alpha^2V_1\right), \tag{39}$$

$$\gamma_2=\frac{2\mu}{\hbar^2\alpha^2}(C-2E)-l(l+1), \tag{40}$$

$$\gamma_3=-\frac{2\mu E}{\hbar^2\alpha^2}. \tag{41}$$

Comparing Equation (38) with the parameters Equations (5) to (13), one gets

$$\alpha_1=\alpha_2=\alpha_3=1,\alpha_4=0,\alpha_5=-\frac{1}{2} \tag{42}$$

$$\alpha_6=\frac{1}{4}-\frac{2\mu}{\hbar^2\alpha^2}\left(E+D-\alpha^2V_1\right), \tag{43}$$

$$\alpha_7=-\frac{2\mu}{\hbar^2\alpha^2}(C-2E)-l(l+1), \tag{44}$$

$$\alpha_8=-\frac{2\mu E}{\hbar^2\alpha^2}, \tag{45}$$

$$\alpha_9=\frac{1}{4}(2l+1)^2-\frac{2\mu}{\hbar^2\alpha^2}\left(C+D-\alpha^2V_1\right), \tag{46}$$

$$\alpha_{10}=1+2\sqrt{-\frac{2\mu E}{\hbar^2\alpha^2}}, \tag{47}$$

$$\alpha_{11}=2+2\left[\frac{1}{2}\sqrt{(2l+1)^2-\frac{8\mu}{\hbar^2\alpha^2}\left(C+D-\alpha^2V_1\right)}+\sqrt{-\frac{2\mu E}{\hbar^2\alpha^2}}\right], \tag{48}$$

$$\alpha_{12}=\sqrt{-\frac{2\mu E}{\hbar^2\alpha^2}}, \tag{49}$$

$$\alpha_{13}=-\frac{1}{2}-\left[\frac{1}{2}\sqrt{(2l+1)^2-\frac{8\mu}{\hbar^2\alpha^2}\left(C+D-\alpha^2V_1\right)}+\sqrt{-\frac{2\mu E}{\hbar^2\alpha^2}}\right]. \tag{50}$$

Substituting the values of the parametric constants Equations (42) to (50) into Equations (3) and (4), respectively, one gets the energy eigenvalues and the corresponding unnormalized radial eigenstates as

$$E_{nl}=-\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1)+l(l+1)+\frac{1}{2}+(n+1)T-\frac{2\mu C}{\hbar^2\alpha^2}}{1+2n+T}\right]^2, \tag{51}$$

and

$$\psi_{nl}(r) = N_{nl} e^{-\alpha\sqrt{\gamma_3} r} \left(1 - e^{-\alpha r}\right)^{\beta} P_n^{\left(2\sqrt{\gamma_3}, T\right)} \left(1 - 2e^{-\alpha r}\right), \tag{52}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left(C + D - \alpha^2 V_1\right)}, \tag{53}$$

$$\gamma_3 = -\frac{2\mu E}{\hbar^2\alpha^2}, \tag{54}$$

and $\beta = \frac{1+T}{2} - \sqrt{\gamma_3}$.

3.3. Special Cases

To get special cases some parameters should be set to zero. The first case is Manning-Rosen plus Mobius square which can be obtained by setting V_1 to zero and the energy eigenvalues will be

$$E_{nl} = -V_0 A^2 - \frac{\alpha^2\hbar^2}{2\mu}\left[\frac{n(n+1) + l(l+1) + 0.5 + T(n+1) - \frac{2\mu}{\alpha^2\hbar^2}\left(C + 2AV_0(A+B)\right)}{1 + 2n + T}\right]^2, \tag{55}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left(C + D + (A+B)^2 V_0\right)}. \tag{56}$$

The second case is Maning-Rosen plus quadratic Yukawa potential which can be obtained by setting V_0 to zero. The eigenvalues obtained are as follows

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{n(n+1) + l(l+1) + 0.5 + T(n+1) - \frac{2\mu C}{\alpha^2\hbar^2}}{1 + 2n + T}\right]^2, \tag{57}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left(C + D - \alpha^2 V_1\right)}, \tag{58}$$

which is the same results as in (51) and (53).

The third case is Mobius square plus inversely quadratic Yukawa potential, which results from substituting $C = D = 0$ in (17). The eigenvalues resulting from substituting these parameters in (55) are given by [38]

$$E_{nl} = -V_0 A^2 - \frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1) + l(l+1) + \frac{1}{2} + (n+1)T - \frac{2\mu}{\hbar^2\alpha^2}\left(2V_0 A(A+B)\right)}{1 + 2n + T}\right]^2, \tag{59}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left((A+B)^2 V_0 - \alpha^2 V_1\right)}. \tag{60}$$

The fourth case is Manning-Rosen which can be obtained by substituting $V_1 = V_0 = 0$ and the resulting eigenvalues are given as

$$E_{nl} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1)+l(l+1)+\frac{1}{2}+(n+1)T-\frac{2\mu C}{\hbar^2\alpha^2}}{1+2n+T}\right]^2, \tag{61}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}(C+D)}, \tag{62}$$

which agrees with the results in [39] and [40].

The fifth case is Mobius square potential which can be obtained by substituting $C = D = V_1 = 0$. The eigenvalues resulting are given by [19]

$$E_{nl} = -V_0A^2 - \frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1)+l(l+1)+\frac{1}{2}+(n+1)T-\frac{2\mu}{\hbar^2\alpha^2}\left(2V_0A(A+B)\right)}{1+2n+T}\right]^2, \tag{63}$$

where

$$T = \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2\alpha^2}\left((A+B)^2V_0\right)}. \tag{64}$$

The sixth case is inversely quadratic Yukawa potential which can be obtained by setting $C = D = V_0 = 0$ and the eigenvalues produced are given by [41]

$$E_{nl} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1)+l(l+1)+\frac{1}{2}+(n+1)T}{1+2n+T}\right]^2, \tag{65}$$

where

$$T = \sqrt{(2l+1)^2 + \frac{8\mu V_1}{\hbar^2}} \tag{66}$$

The seventh case is the Hulthèn potential which can be obtained by substituting $D = -C$ in the Manning-Rosen Potential and the resulting potential will be [42]

$$V(r) = -\frac{Ce^{-\alpha r}}{1-e^{-\alpha r}} \tag{67}$$

and the eigenvalues obtained by substituting the parameters in (31) will be

$$E_{nl} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n(n+1)+l(l+1)+\frac{1}{2}+(n+1)(2l+1)-\frac{2\mu C}{\hbar^2\alpha^2}}{2(1+n+l)}\right]^2, \tag{68}$$

which is the same results as in [43] and [44].

4. Numerical Results and Discussion

In this work, we have studied the solution of the Schrödinger wave equation with two sets of potentials. Here we tend to explain our results by commenting on the plotted Figures. In Figure 1, we showed the variation in the vibrational energy levels against the screening parameter α. It is noted that as α increases, the energy levels of the system decrease monotonically from zero. It is equally seen that rotational energy levels of the system decrease as the screening parameter increases as shown in Figure 2. Figure 3 shows the energy states decrease as the principal quantum number increases for various values of the screening parameter.

Figure 4 indicates the variation of the vibrational energy levels against the strength parameter D. A decrease in the strength D results in an increase in the energy. Moreover, as the potential strength D decreases beyond some value it results in a sharp decrease in the energy.

In Figure 5, we examined the variation in the energy against the potential strength V_1. It is seen that the energy of the system decreases monotonically from zero as the potential strength increases for various values of n. A reverse case is shown in Figure 6 when energy is plotted against the potential strength C. It is obvious that when a particle is subjected to this system, the particle exhibits different features of V_1 and C for various values of screening parameters; namely, $\alpha = 0.1$, $\alpha = 0.2$. and $\alpha = 0.3$. However, when the strength parameter C gets a large value, the energy drops sharply for $\alpha = 0.1$.

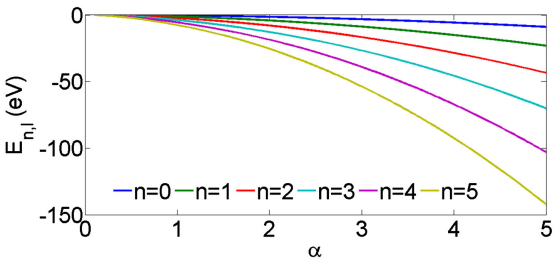

Figure 1. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs α for different values of n. $V_1 = 0.5$ eV, $D = -5$ eV, $C = 0.5$ eV and $l = 0$.

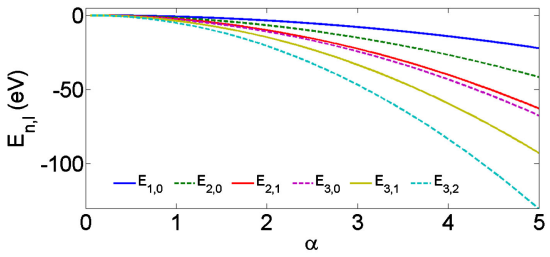

Figure 2. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs α. $V_1 = 0.5$ eV, $D = -5$ eV, and $C = 0.5$ eV for $n = 1$ ($l = 0$), $n = 2$ ($l = 0$, $l = 1$), and $n = 3$ ($l = 0$, $l = 1$, $l = 2$).

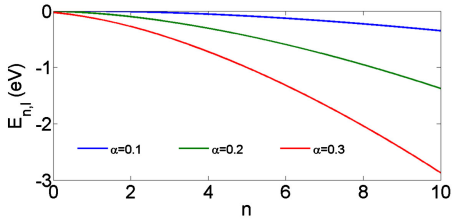

Figure 3. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs n. $V_1 = 0.5\,\text{eV}$, $D = -5\,\text{eV}$, $C = 0.5\,\text{eV}$ and $l = 0$ for different values of α.

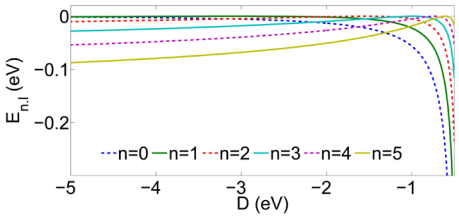

Figure 4. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs D. $V_1 = 0.5\,\text{eV}$, $\alpha = 0.1$, $C = 0.5\,\text{eV}$ and $l = 0$ for different values of n.

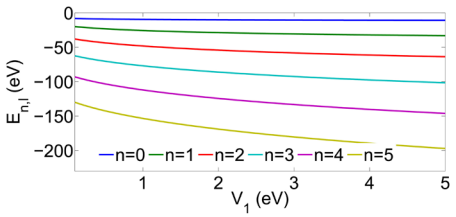

Figure 5. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs V_1. $D = -5\,\text{eV}$, $\alpha = 0.1$, $C = 0.5\,\text{eV}$ and $l = 0$ for different values of n.

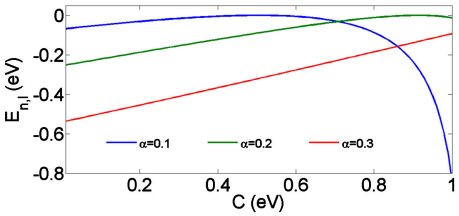

Figure 6. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs C. $D = -5\,\text{eV}$, $V_1 = 0.5\,\text{eV}$, $n = 0$ and $l = 0$ for different values of α.

Figure 7 shows a plot of the variation in the energy against the strength parameter C for various values of n. It is seen that the energy decreases as the strength parameter C increases. It is equally seen that the vibrational energy of the system decreases as the screening parameter increases for various values of n as shown in **Figure 8**. A similar behavior to **Figure 8** appears for the rotational energy levels when plotted against the screening parameters are shown in **Figure 9**.

Finally, Figure 10 shows the decrease of vibrational energy levels as the quantum number n increases for various values of screening parameter α.

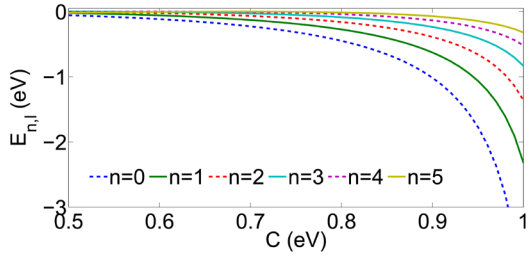

Figure 7. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen plus quadratic Yukawa potential vs C. $D=-5$ eV $V_1=0.5$ eV, $\alpha=0.1$ and $l=0$ for different values of n.

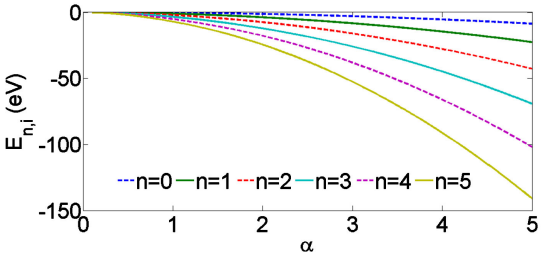

Figure 8. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen Mobius square plus quadratic Yukawa potential vs α. $D=-5$ eV, $V_1=0.5$ eV, $C=0.5$ eV and $l=0$ for different values of n.

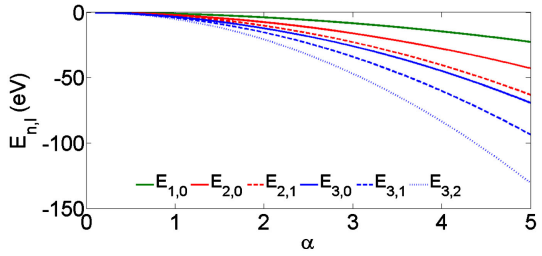

Figure 9. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen Mobius square plus quadratic Yukawa potential vs α. $D=-5$ eV, $V_1=0.5$ eV and $C=0.5$ eV for different values of n.

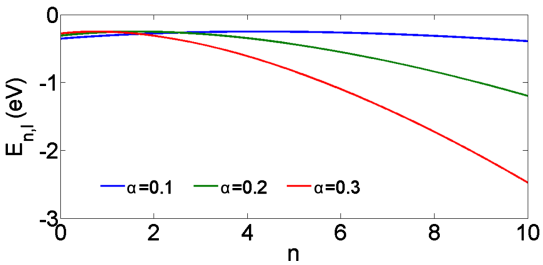

Figure 10. The quantized energy eigenvalues of Schrödinger equation with Manning-Rosen Mobius square plus quadratic Yukawa potential vs n. $D=-5$ eV, $V_1=0.5$ eV, $C=0.5$ eV and $l=0$ for different values of α.

5. Conclusion

In this work, we have analytically solved the non-relativistic wave equation with the combination of three important potentials via the parametric Nikiforov-Uvarov method. We have obtained the energy equation and the corresponding non-normalized wave functions of the combination set of Manning-Rosen plus Mobius square plus quadratic Yukawa potential and their subset of potentials. We have obtained in detail the energy eigenvalues and the corresponding wave function for subset of potentials. These results could find their applications in atomic as well as molecular physics. The effects of the strength parameters as well as screening parameter on the vibrational and rotational energy levels were also studied.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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