

Pseudo-Linear Mechanics with Pseudo-Translation Symmetry

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Abstract

In this paper, we will consider the simplest pseudo-addition where pseudo-multiplication and pseudo-division are the same as the ordinary ones and distributivity hold. Using this we consider a special type of the non-uniform discrete times with the pseudo-translation symmetry. Besides, we demand that the displacement should be invariant under the pseudo-translation. From these, we obtain the new definition of the instantaneous velocity which is pseudo-translation symmetry in both time and position and formulate the mechanics which are pseudo-linear and pseudo-translation invariant.

Keywords

Pseudo-Multiplication, Pseudo-Addition, Pseudo-Translation Symmetry, Pseudo-Linear

1. Introduction

Newton equation in one dimension is given by

$$Ma = M \frac{d}{dt} v = M \left(\frac{d}{dt} \right)^2 x = F, \quad (1)$$

where M denotes a particle mass. This gives the linear differential equation because $\frac{d}{dt}$ is linear. This derivative with respect to time comes from the Galilei's definition of the average velocity,

$$\bar{v} = \frac{x(t') - x(t)}{t' - t} \quad (2)$$

Indeed, taking the limit $t' \rightarrow t$ we get the Newton's instantaneous velocity from Equation (2). The definition (2) has the translational symmetry,

$$x \rightarrow x + \delta x, \quad t \rightarrow t + \delta t, \quad (3)$$

where the former is related to conservation of linear momentum while the latter is related to conservation of energy.

Recently, the relation between the average velocity and the instantaneous velocity was reexamined from the help of the discrete time. Use of the discrete time in physics has long history. First use was focused on the discrete calculus of variations [1] [2]. After this, the discrete Nöther theorem was found [3]. The possibility that time could be regarded as a discrete dynamical variable was examined through all phases of mechanics [4]. Some progress has been accomplished in this direction [5] [6] [7]. Principles to be used in the construction of discrete time classical mechanics were discussed in [7], where the authors introduced the concept of virtual path and constructed the system functions from classical Lagrangians and Cadzow's variational principle was applied to the action sum and Maeda-Noether and Logan invariants of the motion were obtained.

In the discrete mechanics, we introduce the discrete times

$$t_n = n\tau, \quad n = 0, 1, 2, \dots, \quad (4)$$

which obeys

$$t_{n+1} - t_n = \tau \quad (5)$$

In a discrete mechanic, the instantaneous velocity is not defined, instead the average one is well defined. We need two adjacent discrete times for the discrete velocity, which is defined as

$$v(t_n) = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} = \frac{x(t_n + \tau) - x(t_n)}{\tau}, \quad n = 0, 1, 2, \dots, \quad (6)$$

The discrete velocity is invariant under the transform $t_n \rightarrow t_n + L\tau, L \in \mathbb{Z}$, which is the discrete version of time translation symmetry. When we consider continuum limit from discrete time mechanics, we should fix the discrete time corresponding to finite continuous time t . Let such a discrete time be t_N . Then, the instantaneous velocity $v(t)$ is obtained by taking limit $\tau \rightarrow 0$ or $N \rightarrow \infty$ with $N\tau = t$ fixed; indeed, we have

$$\begin{aligned} v(t) &= \lim_{N \rightarrow \infty, \tau \rightarrow 0, N\tau = t} v(t_N) \\ &= \lim_{\tau \rightarrow 0} \frac{x(t_{N+1}) - x(t_N)}{\tau} \\ &= \lim_{\tau \rightarrow 0} \frac{x(t + \tau) - x(t)}{\tau} \\ &= \dot{x} \end{aligned} \quad (7)$$

The velocity (7) does not depend on choice of origin, which comes from the translation invariance of velocity. Indeed, the velocity remains unaltered under $x(t) \rightarrow x(t) + \xi$, which is the translation symmetry in space. The discrete velocity can be written as

$$v(t_n) = \Delta_t x(t_n), \quad (8)$$

where the difference operator is

$$\Delta_t F(t) = \frac{F(t+\tau) - F(t)}{\tau} \tag{9}$$

In the continuum limit, we have $\Delta_t \rightarrow \frac{d}{dt}$. The difference operator is linear,

$$\Delta_t (aF(t) + bG(t)) = a\Delta_t F(t) + b\Delta_t G(t) \tag{10}$$

The discrete velocity (8) is invariant under the translation in time and space and the difference operator is linear because we used the ordinary addition rule.

Similarly, the discrete acceleration is defined from three adjacent positions as

$$a(t_n) = \frac{x(t_{n+2}) - x(t_{n+1}) - (x(t_{n+1}) - x(t_n))}{\tau^2}, \quad n = 0, 1, 2, \dots \tag{11}$$

which is invariant under the transform $t_n \rightarrow t_n + L\tau, L \in \mathbb{Z}$ and $x(t) \rightarrow x(t) + \xi$. In the continuum limit, we have

$$\begin{aligned} a(t) &= \lim_{N \rightarrow \infty, \tau \rightarrow 0, N\tau = t} a(t_N) \\ &= \lim_{\tau \rightarrow 0} \frac{x(t_{N+2}) - 2x(t_{N+1}) + x(t_N)}{\tau^2} \\ &= \lim_{\tau \rightarrow 0} \frac{x(t + 2\tau) - 2x(t + \tau) + x(t)}{\tau^2} \\ &= \ddot{x} \end{aligned} \tag{12}$$

What will happen if we adopt deformed addition rule instead of the ordinary addition rule? The general deformed addition was introduced in the pseudo-analysis [8]. Pseudo-analysis first appeared by a mathematician Pap [8], where pseudo-addition, pseudo-subtraction, pseudo-multiplication and pseudo-division were defined by

$$\begin{aligned} x \oplus_g y &= g^{-1}(g(x) + g(y)) \\ x \ominus_g y &= g^{-1}(g(x) - g(y)) \\ x \odot_g y &= g^{-1}(g(x)g(y)) \\ x \oslash_g y &= g^{-1}(g(x)/g(y)) \end{aligned} \tag{13}$$

where g -map is bijective and monotonous with $g(0) = 0, g(1) = 1$. Then, the pseudo-additive identity is 0 and pseudo-multiplicative identity is 1. There can be many possible g 's. In this paper, we demand a constraint for g so that it may obey

$$g(x)g(y) = g(xy), \tag{14}$$

which is the homomorphism for the ordinary multiplication. Then, the pseudo-multiplication reduces to the ordinary multiplication. Inserting $y = 1/x$ in Equation (14) we have

$$g\left(\frac{1}{x}\right) = \frac{1}{g(x)}, \tag{15}$$

which renders the pseudo-division to be ordinary division. In this case the pseudo-addition is distributive,

$$k(x \oplus_g y) = (kx) \oplus_g (ky), \quad k \in \mathbb{R} \tag{16}$$

The general solution obeying Equation (15) and Equation (16) is

$$g(x) = |x|^{\alpha-1} x, \quad \alpha \in \mathbb{R} \quad (17)$$

Then, pseudo-addition and pseudo-subtraction are given by

$$a \oplus b = \left| |a|^{\alpha-1} + |b|^{\alpha-1} \right|^{1/\alpha-1} \left(|a|^{\alpha-1} + |b|^{\alpha-1} \right) \quad (18)$$

$$a \ominus b = \left| |a|^{\alpha-1} - |b|^{\alpha-1} \right|^{1/\alpha-1} \left(|a|^{\alpha-1} - |b|^{\alpha-1} \right) \quad (19)$$

From now on we call these kinds of pseudo-addition and pseudo-subtraction the α -addition and α -subtraction.

In this paper, we use the pseudo-addition to consider the non-uniform discrete times with the pseudo-translation symmetry. Besides, we demand that the displacement should be invariant under the pseudo-translation. From these, we formulate the mechanics which are pseudo-linear and pseudo-translation invariant. This paper is organized as follows: In Section 2, we discuss the α -discrete time and α -linear discrete mechanics. In Section 3, we discuss the α -Work and conservation of α -energy. In Section 4, we discuss the α -harmonic oscillator.

2. α -Discrete Time and α -Linear Discrete Mechanics

In this paper, we are to consider a special type of the non-uniform discrete times which obey the following relation

$$t_{n+1} \ominus t_n = \tau, \quad n = 0, 1, 2, \dots, \quad (20)$$

which gives

$$t_n = [n]_{\alpha} \tau, \quad (21)$$

where α -number is defined as

$$[z]_{\alpha} = |z|^{\frac{1}{\alpha}-1} z, \quad z \in \mathbb{C}, \quad (22)$$

From now on we will say that discrete times are α -uniform when they obey Equation (20). Besides, we refer to τ as the α -uniform time interval. We also refer to the α -uniform discrete times as shortly α -discrete times. We stress that α -uniform is not uniform unless $\alpha = 1$. The α -discrete times become the uniform discrete times when $\alpha = 1$ is chosen. Thus, $|\alpha - 1|$ denotes the degree of broken uniformity in discrete times.

2.1. α -Discrete Velocity

In a uniform time lattice the discrete velocity is written as

$$v(t_n) = \frac{\text{displacement}}{\text{uniform time interval}}, \quad (23)$$

where the uniform time interval is invariant under the translation $t_n \rightarrow t_n + L\tau$ and displacement is invariant under the translation $x \rightarrow x + \delta x$.

Now we will define the α -discrete velocity as

$$v(t_n) = \frac{\alpha\text{-displacement}}{\alpha\text{-uniform time interval}} \quad (24)$$

Here we know that α -uniform time interval is invariant under the discrete α -translation, $t_n \rightarrow t_n \oplus [L]_{\alpha} \tau$. Similarly, we define the α -displacement so that it may be invariant under the continuous α -translation $x \rightarrow x \oplus \delta x$. For two adjacent α -discrete times, the α -displacement is given by

$$\alpha\text{-displacement} = x(t_{n+1}) \ominus x(t_n) \quad (25)$$

Then, the α -discrete velocity is defined as

$$v_{\alpha}(t_n) = \frac{x(t_{n+1}) \ominus x(t_n)}{t_{n+1} \ominus t_n} = \frac{x(t_n \oplus \tau) \ominus x(t_n)}{\tau}, \quad n = 0, 1, 2, \dots, \quad (26)$$

The α -discrete velocity is invariant under the discrete α -translation. The α -discrete velocity can also be written as

$$v_{\alpha}(t_n) = \Delta_t^{\alpha} x(t_n), \quad (27)$$

where the α -difference operator is defined as

$$\Delta_t^{\alpha} F(t) = \frac{F(t \oplus \tau) \ominus F(t)}{\tau} \quad (28)$$

The α -difference operator is not linear but obeys the α -linearity,

$$\Delta_t^{\alpha} (aF(t) \oplus bG(t)) = a\Delta_t^{\alpha} F(t) \oplus b\Delta_t^{\alpha} G(t) \quad (29)$$

The α -discrete velocity remains unaltered under $x(t) \rightarrow x(t) \oplus \delta x(t)$, which is the α -translation symmetry in space.

2.2. α -Velocity

The continuum limit is obtained by taking the limit $\tau \rightarrow 0$ or $N \rightarrow \infty$ with $[N]_{\alpha} \tau = t$ fixed. Due to the α -uniformity in time, the α -discrete velocity does not give the ordinary Newton velocity in the continuum limit. Indeed we have

$$\begin{aligned} v_{\alpha}(t) &= \lim_{N \rightarrow \infty, \tau \rightarrow 0, [N]_{\alpha} \tau = t} v_{\alpha}(t_N) \\ &= \lim_{\tau \rightarrow 0} \frac{x(t_{N+1}) \ominus x(t_N)}{\tau} \\ &= \lim_{\tau \rightarrow 0} \frac{x(t \oplus \tau) \ominus x(t)}{\tau} \\ &= D_t^{\alpha} x(t), \end{aligned} \quad (30)$$

where the α -derivative is defined as

$$D_t^{\alpha} x(t) = |t|^{\frac{1}{\alpha}-1} |x(t)|^{1-\frac{1}{\alpha}} \left| \frac{dx}{dt} \right|^{\frac{1}{\alpha}-1} \frac{dx}{dt} \quad (31)$$

The derivation of Equation (31) is given in **Appendix A**. The α -derivative is α -linear in the sense that

$$D_t^{\alpha} (c_1 x(t) \oplus c_2 y(t)) = c_1 D_t^{\alpha} x(t) \oplus c_2 D_t^{\alpha} y(t) \quad (32)$$

For the α -derivative, the following Leibniz rule holds:

$$D_t^\alpha [F(t)G(t)] = (D_t^\alpha F(t))G(t) \oplus F(t)(D_t^\alpha G(t)) \tag{33}$$

We know that the α -velocity (30) remains invariant under α -translation in both time and space.

2.3. α -Discrete Acceleration and α -Discrete Newton Equation

The α -discrete acceleration is obtained like Equation (12) in the form

$$a_\alpha(t_n) = \frac{(x(t_{n+2}) \ominus x(t_{n+1})) \ominus (x(t_{n+1}) \ominus x(t_n))}{(t_{n+1} \ominus t_n)^2} \tag{34}$$

$$= \frac{x(t_{n+2}) \ominus [2]_\alpha x(t_{n+1}) \oplus x(t_n)}{\tau^2}, \quad n = 0, 1, 2, \dots,$$

The α -discrete acceleration can also be written as

$$a_\alpha(t_n) = (\Delta_t^\alpha)^2 x(t_n) = \Delta_t^\alpha v_\alpha(t_n) \tag{35}$$

The α -discrete acceleration remains unaltered under the transform $t_n \rightarrow t_n \oplus [L]_\alpha \tau, L \in \mathbb{Z}$ and $x(t) \rightarrow x(t) \oplus \delta x(t)$. The α -discrete Newton equation is then given by

$$Ma_\alpha(t_n) = MD_t^\alpha v_\alpha(t_n) = M(D_t^\alpha)^2 x(t_n) = F(t_n, x(t_n)) \tag{36}$$

2.4. α -Acceleration and α -Newton Equation

In the continuum limit, α -discrete acceleration reduces to the α -acceleration

$$a_\alpha(t) = D_t^\alpha v_\alpha(t) = (D_t^\alpha)^2 x(t), \tag{37}$$

which is invariant under α -translation in both time and space. The Newton equation with α -translation symmetry (or shortly α -Newton equation) is then given by

$$Ma_\alpha(t) = MD_t^\alpha v_\alpha(t) = M(D_t^\alpha)^2 x(t) = F \tag{38}$$

Because D_t^α is α -linear, we define the α -net force for forces, F_1, F_2, \dots , acted on the particle as

$$F_{\alpha,net} = F_1 \oplus F_2 \oplus \dots \tag{39}$$

3. α -Work and Conservation of α -Energy

Now let us introduce the α -work by the constant force F in the following definition,

$$W_\alpha = (\text{Force}) \times (\alpha\text{-displacement}) \tag{40}$$

If we consider that the particle moves from x_1 to x_2 , we can write the α -work by the constant force F ,

$$W_\alpha^{1 \rightarrow 2} = F(x_2 \ominus x_1) \tag{41}$$

Now let us consider the case that the force varies with the position and the

particle moves from x_0 to x_N in the discrete positions $\{x_0, x_1, \dots, x_N\}$. Let $\bar{F}_{i,i+1}$ be the α -average force when the particle moves from x_i to x_{i+1} . Then, the α -work is given by

$$W_\alpha^{i \rightarrow i+1} = \bar{F}_{i,i+1} (x_{i+1} \ominus x_i) \quad (42)$$

Then, the α -work from x_0 to x_N is

$$W_\alpha^{0 \rightarrow N} = \bigoplus_{i=0}^{N-1} \bar{F}_{i,i+1} (x_{i+1} \ominus x_i), \quad (43)$$

where α -sum is define as

$$\bigoplus_{i=0}^{N-1} A_i = A_0 \oplus A_1 \oplus \dots \oplus A_{N-1} \quad (44)$$

The α -sum can be written in terms of the ordinary sum as

$$\bigoplus_{i=0}^{N-1} A_i = \left| \sum_{i=0}^{N-1} |A_i|^{\alpha-1} A_i \right|^{\frac{1}{\alpha-1}} \sum_{i=0}^{N-1} |A_i|^{\alpha-1} A_i \quad (45)$$

Now let us the continuum version of Equation (45) which is the α -integral. The α -integral is defined as

$$\int_a^x d_\alpha \xi F(\xi) = \left| \int_a^x \alpha d\xi |\xi|^{\alpha-1} |F(\xi)|^{\alpha-1} F(\xi) \right|^{\frac{1}{\alpha-1}} \int_a^x \alpha d\xi |\xi|^{\alpha-1} |F(\xi)|^{\alpha-1} F(\xi) \quad (46)$$

Here we know

$$\int_a^x d_\alpha \xi (D_\xi^\alpha F(\xi)) = F(x) \ominus F(0) \quad (47)$$

and

$$D_x^\alpha \int_a^x d_\alpha \xi F(\xi) = F(x) \quad (48)$$

The derivation of Equation (47) and Equation (48) is given in **Appendix B**.

With a help of the α -integral, we define the α -work from the position 1 to 2 for the varying force $F(x)$ as

$$W_\alpha^{1 \rightarrow 2} = \int_1^2 d_\alpha x F(x) \quad (49)$$

Using the α -Newton equation, we have

$$W = K_{\alpha,2} \ominus K_{\alpha,1} \quad (50)$$

where $K_\alpha = \frac{1}{[2]_\alpha} Mv^2$ is a α -kinetic energy. The derivation is given in **Appendix C**.

If we introduce the α -potential energy $U_\alpha(x)$ as

$$F = -D_x^\alpha U_\alpha(x) \quad (51)$$

and inserting it into the definition of the α -work, we have

$$W = -(U_{\alpha,2} \ominus U_{\alpha,1}) \quad (52)$$

Thus, the conservation of α -mechanical energy reads

$$E_\alpha = K_\alpha \oplus U_\alpha = \text{const} \quad (53)$$

4. α -Harmonic Oscillator

As an example, we discuss the α -harmonic oscillator problem. In the α -discrete time mechanics, the harmonic oscillator is described by

$$M \left(\frac{x(t_{n+2}) \ominus [2]_{\alpha} x(t_{n+1}) \oplus x(t_n)}{\tau^2} \right) = -Mw^2 x(t_n), \quad n = 0, 1, 2, \dots, \quad (54)$$

or

$$x(t_{n+2}) \ominus [2]_{\alpha} x(t_{n+1}) \oplus (1 \oplus \tau^2 w^2) x(t_n) = 0 \quad (55)$$

The initial condition is

$$x(t_0) = A, \quad v_{\alpha}(t_0) = 0 \quad (56)$$

Equation (55) is written as

$$x(t_{n+2}) \ominus px(t_{n+1}) = q(x(t_{n+1}) \ominus px(t_n)), \quad (57)$$

where

$$p \oplus q = [2]_{\alpha}, \quad pq = 1 \oplus w^2 \tau^2 \quad (58)$$

Thus, we get

$$p = 1 \oplus iw\tau \quad (59)$$

$$q = 1 \ominus iw\tau, \quad (60)$$

From the relation

$$\begin{aligned} x \oplus iy &= \left(|x|^{\alpha-1} x + |iy|^{\alpha-1} iy \right)^{\frac{1}{\alpha}} \left(|x|^{\alpha-1} x + |iy|^{\alpha-1} iy \right) \\ &= \left(|x|^{\alpha-1} x + i|y|^{\alpha-1} y \right)^{\frac{1}{\alpha}} \left(|x|^{\alpha-1} x + i|y|^{\alpha-1} y \right) \\ &= \left(\sqrt{|x|^{2\alpha} + |y|^{2\alpha}} \right)^{\frac{1}{\alpha}} \left(|x|^{\alpha-1} x + i|y|^{\alpha-1} y \right), \end{aligned}$$

we have

$$p = \left[1 + (w\tau)^{2\alpha} \right]^{\frac{1-\alpha}{2\alpha}} \left(1 + i(w\tau)^{\alpha} \right) \quad (61)$$

$$q = \left[1 + (w\tau)^{2\alpha} \right]^{\frac{1-\alpha}{2\alpha}} \left(1 - i(w\tau)^{\alpha} \right) \quad (62)$$

Solving Equation (57) with initial condition, we get

$$x(t_n) = \frac{A}{p \ominus q} \left[(p^n \ominus q^n) \ominus pq(p^{n-1} \ominus q^{n-1}) \right] \quad (63)$$

One can set

$$p = Re^{i\theta}, \quad q = Re^{-i\theta}, \quad (64)$$

where

$$R = \left(1 + (w\tau)^{2\alpha} \right)^{\frac{1}{2\alpha}} \quad (65)$$

and

$$\theta = \tan^{-1}(w\tau)^\alpha \quad (66)$$

Then we get

$$p^n \ominus q^n = i[2]_\alpha R^n |\sin n\theta|^{\frac{1}{\alpha}-1} \sin n\theta \quad (67)$$

$$pq = R^2 \quad (68)$$

Thus, we have

$$x(t_n) = AR^n |\cos n\theta|^{\frac{1}{\alpha}-1} \cos n\theta, \quad (69)$$

The derivation of Equation (69) is given in **Appendix D**. In the continuum limit we have

$$x(t) = A |\cos(w^\alpha t^\alpha)|^{\frac{1}{\alpha}-1} \cos(w^\alpha t^\alpha) \quad (70)$$

Equation (70) is not periodic but it is α -periodic in the sense,

$$x(t \oplus T_\alpha) = x(t), \quad (71)$$

where we call T_α the α -period and it is given by

$$T_\alpha = \frac{[2\pi]_\alpha}{w} \quad (72)$$

5. Conclusion

The uniform discrete times are related to the definition of velocity proposed by Newton. This is given by the ordinary derivative of the position with respect to time. In this paper, we considered a special type of the non-uniform discrete times which are related to pseudo calculus. Here we considered the special pseudo calculus called α -calculus where α -multiplication and α -division are the same as the ordinary ones and distributivity holds. We considered the time pseudo lattice with the pseudo-translation symmetry described by the α -addition. Besides, we demand that the displacement should be invariant under the pseudo-translation. From these, we obtained the new definition of the instantaneous velocity called α -velocity which is pseudo-translation symmetry in both time and position. With the α -velocity we formulated the mechanics which are pseudo-linear and pseudo-translation invariant. As an example, we discussed the simple harmonic oscillator in the α -mechanics.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A

The α -addition and α -subtraction depend on sign of two numbers.

Case of $x > 0, y > 0$:

$$\begin{aligned}
 x \oplus y &= (x^\alpha + y^\alpha)^{1/\alpha} \\
 x \ominus y &= \begin{cases} (x^\alpha - y^\alpha)^{1/\alpha} & (x > y) \\ -(y^\alpha - x^\alpha)^{1/\alpha} & (x < y) \end{cases}
 \end{aligned} \tag{73}$$

Case of $x > 0, y < 0$:

$$\begin{aligned}
 x \oplus y &= \begin{cases} (x^\alpha - (-y)^\alpha)^{1/\alpha} & (x > -y) \\ -((-y)^\alpha - x^\alpha)^{1/\alpha} & (x < -y) \end{cases} \\
 x \ominus y &= (x^\alpha + (-y)^\alpha)^{1/\alpha}
 \end{aligned} \tag{74}$$

Case of $x < 0, y > 0$:

$$\begin{aligned}
 x \oplus y &= \begin{cases} (y^\alpha - (-x)^\alpha)^{1/\alpha} & (-x < y) \\ -((-x)^\alpha - y^\alpha)^{1/\alpha} & (-x > y) \end{cases} \\
 x \ominus y &= -(y^\alpha + (-x)^\alpha)^{1/\alpha}
 \end{aligned} \tag{75}$$

Case of $x < 0, y < 0$:

$$\begin{aligned}
 x \oplus y &= -((-x)^\alpha + (-y)^\alpha)^{1/\alpha} \\
 x \ominus y &= \begin{cases} ((-y)^\alpha - (-x)^\alpha)^{1/\alpha} & (x > y) \\ -((-x)^\alpha - (-y)^\alpha)^{1/\alpha} & (x < y) \end{cases}
 \end{aligned} \tag{76}$$

For a small τ_α , we have

$$t \ominus_\alpha \tau_\alpha = (t^\alpha - \tau_\alpha^\alpha)^{1/\alpha} = t \left(1 - \frac{\tau_\alpha^\alpha}{t^\alpha} \right)^{1/\alpha} \approx t - \frac{1}{\alpha} t^{1-\alpha} \tau_\alpha^\alpha \tag{77}$$

and

$$x(t \ominus_\alpha \tau_\alpha) \approx x(t) - \frac{1}{\alpha} t^{1-\alpha} \tau_\alpha^\alpha \dot{x}(t) \tag{78}$$

Now let us consider the following four cases.

Case of $x(t) > 0, \dot{x}(t) > 0$: In this case we have

$$x(t) > x(t \ominus_\alpha \tau_\alpha) > 0 \tag{79}$$

Hence we get

$$D_t x(t) = \lim_{\tau_\alpha \rightarrow 0} \frac{\left(x(t)^\alpha - \left(x(t) - \frac{1}{\alpha} t^{1-\alpha} \tau_\alpha^\alpha \dot{x}(t) \right)^\alpha \right)^{1/\alpha}}{\tau_\alpha} = t^{\frac{1}{\alpha}-1} x(t)^{1-\frac{1}{\alpha}} (\dot{x})^{\frac{1}{\alpha}} \tag{80}$$

Case of $x(t) > 0, \dot{x}(t) < 0$: In this case we have

$$x(t \ominus_{\alpha} \tau_{\alpha}) > x(t) > 0 \tag{81}$$

Hence we get

$$D_{\tau} x(t) = - \lim_{\tau_{\alpha} \rightarrow 0} \frac{\left(\left(x(t) - \frac{1}{\alpha} t^{1-\alpha} \tau_{\alpha}^{\alpha} \dot{x}(t) \right)^{\alpha} - x(t)^{\alpha} \right)^{1/\alpha}}{\tau_{\alpha}} = -t^{\frac{1}{\alpha}-1} x(t)^{1-\frac{1}{\alpha}} |\dot{x}|^{\frac{1}{\alpha}} \tag{82}$$

Case of $x(t) < 0, \dot{x}(t) > 0$: In this case we have

$$x(t \ominus_{\alpha} \tau_{\alpha}) < x(t) < 0 \tag{83}$$

Hence we get

$$D_{\tau} x(t) = \lim_{\tau_{\alpha} \rightarrow 0} \frac{\left(\left(|x(t)| + \frac{1}{\alpha} t^{1-\alpha} \tau_{\alpha}^{\alpha} \dot{x}(t) \right)^{\alpha} - |x(t)|^{\alpha} \right)^{1/\alpha}}{\tau_{\alpha}} = t^{\frac{1}{\alpha}-1} |x(t)|^{1-\frac{1}{\alpha}} (\dot{x})^{\frac{1}{\alpha}} \tag{84}$$

Case of $x(t) < 0, \dot{x}(t) < 0$: In this case we have

$$x(t) < x(t \ominus_{\alpha} \tau_{\alpha}) < 0 \tag{85}$$

Hence we get

$$D_{\tau} x(t) = - \lim_{\tau_{\alpha} \rightarrow 0} \frac{\left(|x(t)|^{\alpha} - \left(|x(t)| + \frac{1}{\alpha} t^{1-\alpha} \tau_{\alpha}^{\alpha} \dot{x}(t) \right)^{\alpha} \right)^{1/\alpha}}{\tau_{\alpha}} = -t^{\frac{1}{\alpha}-1} |x(t)|^{1-\frac{1}{\alpha}} |\dot{x}|^{\frac{1}{\alpha}} \tag{86}$$

Appendix B

We have

$$\begin{aligned} & \int_a^x d_{\alpha} \xi (D_{\xi}^{\alpha} F(\xi)) \\ &= \left| \int_a^x \alpha d\xi |\xi|^{\alpha-1} |D_{\xi}^{\alpha} F(\xi)|^{\alpha-1} D_{\xi}^{\alpha} F(\xi) \right|^{\frac{1}{\alpha-1}} \int_a^x \alpha d\xi |\xi|^{\alpha-1} |D_{\xi}^{\alpha} F(\xi)|^{\alpha-1} D_{\xi}^{\alpha} F(\xi) \\ &= \left| \int_a^x \alpha d\xi |F(\xi)|^{\alpha-1} F'(\xi) \right|^{\frac{1}{\alpha-1}} \int_a^x \alpha d\xi |F(\xi)|^{\alpha-1} F'(\xi) \\ &= \left| |F(x)|^{\alpha-1} F(x) - |F(a)|^{\alpha-1} F(a) \right|^{\frac{1}{\alpha-1}} \left(|F(x)|^{\alpha-1} F(x) - |F(a)|^{\alpha-1} F(a) \right) \\ &= F(x) \ominus F(a), \end{aligned} \tag{87}$$

where we used

$$\left| D_{\xi}^{\alpha} F(\xi) \right|^{\alpha-1} D_{\xi}^{\alpha} F(\xi) = |\xi|^{1-\alpha} |F(\xi)|^{\alpha-1} F'(\xi) \tag{88}$$

Appendix C

From the definition of α -work, we have

$$\begin{aligned} W_{\alpha}^{1 \rightarrow 2} &= \int_1^2 d_{\alpha} x F(x) \\ &= \left| \int_1^2 \alpha dx |x|^{\alpha-1} |F(x)|^{\alpha-1} F(x) \right|^{\frac{1}{\alpha-1}} \int_1^2 \alpha dx |x|^{\alpha-1} |F(x)|^{\alpha-1} F(x) \end{aligned} \tag{89}$$

Now let us set

$$A = \int_1^2 \alpha dx |x|^{\alpha-1} |F(x)|^{\alpha-1} F(x) \tag{90}$$

Using α -Newton equation, we have

$$\begin{aligned} A &= M^\alpha \int_1^2 \alpha dt \dot{x} |x|^{\alpha-1} |D_t^\alpha v|^{\alpha-1} D_t^\alpha v \\ &= M^\alpha \int_1^2 \alpha dt t^{1-\alpha} \dot{x} |x|^{\alpha-1} |v|^{\alpha-1} \dot{v} \\ &= M^\alpha \int_1^2 \alpha dt v |v|^{2\alpha-2} \dot{v} \end{aligned} \tag{91}$$

Thus we have

$$W_\alpha^{1 \rightarrow 2} = M \left| \int_1^2 \alpha dt v |v|^{2\alpha-2} \dot{v} \right|^{\frac{1}{\alpha-1}} \int_1^2 \alpha dt v |v|^{2\alpha-2} \dot{v} \tag{92}$$

Also, we have

$$\begin{aligned} \int_1^2 d_\alpha t D_t^\alpha \left(\frac{1}{[2]_\alpha} M v^2 \right) &= M \int_1^2 d_\alpha t v D_t^\alpha v \\ &= M \left| \int_1^2 \alpha dt t^{\alpha-1} |v|^{\alpha-1} v |D_t^\alpha v|^{\alpha-1} D_t^\alpha v \right|^{\frac{1}{\alpha-1}} \int_1^2 \alpha dt t^{\alpha-1} |v|^{\alpha-1} v |D_t^\alpha v|^{\alpha-1} D_t^\alpha v \\ &= M \left| \int_1^2 \alpha dt v |v|^{2\alpha-2} \dot{v} \right|^{\frac{1}{\alpha-1}} \int_1^2 \alpha dt v |v|^{2\alpha-2} \dot{v}, \end{aligned} \tag{93}$$

Appendix D

Inserting Equation (67) and Equation (68) into Equation (63), we get

$$x(t_n) = \frac{AR^n}{w\tau} \left(|\sin n\theta|^{\frac{1}{\alpha-1}} \sin n\theta \ominus R |\sin(n-1)\theta|^{\frac{1}{\alpha-1}} \sin(n-1)\theta \right) \tag{94}$$

Here we have

$$\begin{aligned} &|\sin n\theta|^{\frac{1}{\alpha-1}} \sin n\theta \ominus R |\sin(n-1)\theta|^{\frac{1}{\alpha-1}} \sin(n-1)\theta \\ &= |\sin n\theta - R^\alpha \sin(n-1)\theta|^{\frac{1}{\alpha-1}} (\sin n\theta - R^\alpha \sin(n-1)\theta) \\ &= w\tau |\cos n\theta|^{\frac{1}{\alpha-1}} \cos n\theta, \end{aligned} \tag{95}$$

where we used

$$R^\alpha = \sec \theta \tag{96}$$