

Decoherence of a Damped Anisotropic Harmonic Oscillator under Magnetic Field Effects in a Two-Dimensional Noncommutative Phase-Space

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Abstract

In this paper, decoherence of a damped anisotropic harmonic oscillator in the presence of a magnetic field is studied in the framework of the Lindblad theory of open quantum systems in noncommutative phase-space. General fundamental conditions that should follow our quantum mechanical diffusion coefficients appearing in the master equation are kindly derived. From the master equation, the expressions of density operator, the Wigner distribution function, the expectation and variance with respect to coordinates and momenta are obtained. Based on these quantities, the total energy of the system is evaluated and simulations show its dependency to phase-space structure and its improvement due to noncommutativity effects and the environmental temperature as well. In addition, we also evaluate the decoherence time scale and show that it increases with noncommutativity phase-space effects as compared to the commutative case. It turns out from simulations that this time scale is significantly improved under magnetic field effects.

Keywords

Decoherence, Noncommutative Phase-Space, Lindblad Theory, Wigner Distribution

1. Introduction

The theory of open quantum systems has become one of the main interests of physicists during the last few decades due to their strong applications in several

domains of physics, especially in solid state physics [1], quantum optics [2], nanotechnology [3], quantum measurement theory [4], etc. Indeed, various research works conducted in the field proved that perfect isolated quantum system does not exist, since any realistic system permanently interacts with its environment, which typically presents a large number of degrees of freedom [5]. A physical system interacting with the environment is referred to an open system [6]. Thus, every physical system is affected by the presence of its surroundings leading to dissipation, thermalization and in the quantum case, decoherence [6]. Many reasons lead scientists to pay particular attention to the above mentioned phenomena, here, we refer particularly to decoherence and dissipation, which are the main obstacle in the realization of quantum computers and other quantum devices on one hand [7] [8] [9]. On the other hand, several interesting proposals to exploit the interaction between quantum systems and their environment to dissipatedly engineer challenging state of matter have emerged very recently [6].

In fact, dissipation in an open system results from microscopic reversible interactions between the observable system and the environment. It implies irreversibility and therefore, a preferred direction in time. However, we are particularly interested in this work on the phenomenon of decoherence, which somewhat “regroups dissipation” and introduces the (partial) destruction of quantum coherence through the interaction of quantum mechanical system with its surrounding [10]. From the theoretical analysis point of view, decoherence can be described with the help of microscopic models based on the interaction of quantum mechanical systems with a collection of an infinite number of harmonic oscillators, representing the environment [11] [12]. The strength and nature of the coupling with environment play a crucial role in selecting the preferred states leading to a distance-dependent nature of typical interaction. Furthermore, decoherence simply characterizes the transition of a given quantum system to classical one due to interactions, thus, the particular time that a quantum system starts behaving classically is known as “decoherence time” [12] [13] [14]. Whereas, the decoherence time scale defines the time during which quantum interferences between quantum states disappear (*i.e.* loss of quantum properties) [13] [15]. Deep studies have been made to determine this particular time and to see whether or not it can be controlled due to its relevance in many interesting physical problems such as quantum computation and quantum information processing [16], heavy ion collisions [17], etc. In many cases, physicists are also interested in understanding the specific causes of quantum decoherence in order to prevent it from damaging quantum states and protect the information stored inside [18]. Thus, it was discovered that, for most of superposed quantum states, the associated decoherence time is often found to be smaller than the corresponding relaxation (damping) time. Several research works have been successfully investigated to determine this time in the literature, many of them found that the main tool for its investigation is the master equations (ME) dynamics approach, which describes the evolution in time of the density operator

of a given particle. The ME approach allows describing various physical phenomena such as entanglement [19], decoherence and quantum to classical transition [20]. The most general ME approach was developed by Lindblad [5] [19] [21].

Indeed, the Lindblad formalism ME approach has been used to solve many concrete theoretical two modes systems and applied to various physical phenomena, for instance, in treatments of damped collective modes, in deep inelastic ions collisions, in nuclear physics [22] [23], or in the description of quantum dissipation in one dimensional harmonic oscillator [16]. But almost all the above studies have been conducted considering a commutative phase-space (*i.e.* the space-time coordinates and momenta can be measured simultaneously according to the Heisenberg principle). However, it was demonstrated that, the above mentioned problems including decoherence, could be treated in a noncommuting phase-space with significant results. For this reason, Dragovich *et al.* [24] investigated the influence of noncommutativity (NCty) on the occurrence of the so-called decoherence effect with external magnetic field, and they found that the decoherence can be highly affected when the external control (magnetic field) is constrained by the noncommutative (NC) effects. Ghorashi *et al.* [25] discovered that, by modifying the master equation of two-dimensional harmonic oscillators under the Brownian motion, the NCty effects lead to increase the rate of decoherence. Moreover, Tchoffo *et al.* [26] in their investigation on kinematical Brownian motion of harmonic oscillator in NC space showed that the structure of Fokker Planck's equation is not modified (*i.e.* the factorization theorem is conserved in both commutative and NC space). In the same idea, Santos *et al.* [27] found that, NCty induces a non-vanishing correlation between both coordinates at different times. Very recently in our previous works on decoherence of quantum Brownian particle in NC space, we realized that the NCty effects intervene in the system as a disruption and therefore, increase the rate of decoherence in the system [28]. Although significant results were achieved in the above mentioned works, they usually considered only the space coordinates as NC and the question that emerges is to know what could happen if beside coordinates the momentum was also NC. An answer to this question is the main objective of this research paper, with the approach consisting of deriving the decoherence time scale of our system in NC phase-space by the Lindblad theory for open quantum systems in NC phase-space approach.

To achieve this objective, we structure the paper as follows. In Section 2, we present a general model of the system under magnetic field effects in NC phase-space. We review the Lindblad ME approach of the density operator of the considered open system in the same section. The dynamical equation of the covariance matrix associated to the system state is derived in Section 3. We determine in Section 4 the density matrix and the Wigner function of the considered system. Section 5 is devoted to our main results where therein the decoherence time scale and the total energy of the considered system in NC phase-space are

derived. Finally Section 6 presents some concluding remarks.

2. Model Hamiltonian

The model described in this work forms the standard pattern of a damped quantum harmonic oscillator in NC phase-space. Similar model was already introduced in the literature [29] [30] [31] [32] [33], but with different approach and for different motivations. For instance, in [30] it was used to analyze the Landau diamagnetism in NC space, while in [32] it was introduced to study the connection between dissipation and NCty. However, in this work our oscillator is assumed to be a non-relativistic charged particle of mass m moving in a two-dimensional NC phase-space under the effect of a homogeneous magnetic field, and confined by a damped anisotropic harmonic potential with frequencies ω_1 and ω_2 in the x and y directions respectively. The Hamiltonian of this system is given by:

$$H = \frac{1}{2m} \left(\hat{p}_1 - \frac{e}{c} \hat{A}_1 \right)^2 + \frac{1}{2m} \left(\hat{p}_2 - \frac{e}{c} \hat{A}_2 \right)^2 + \frac{m}{2} (\omega_1^2 \hat{q}_1^2 + \omega_2^2 \hat{q}_2^2) + k_{12} \hat{q}_1 \hat{q}_2, \quad (1)$$

where (\hat{A}_1, \hat{A}_2) are the components of the potential vector of the magnetic field, k_{12} the damping parameter, (\hat{q}_1, \hat{q}_2) and (\hat{p}_1, \hat{p}_2) are respectively the non-commutative coordinates and momenta of the quantum oscillator satisfying the following commutation relations [34] [35] [36] [37] [38] $[\hat{q}_k, \hat{q}_l] = i\theta\epsilon_{kl}$, $[\hat{p}_k, \hat{p}_l] = i\eta\epsilon_{kl}$, $[\hat{q}_k, \hat{p}_l] = i\tilde{\hbar}\delta_{kl}$, with $\tilde{\hbar} = \hbar \left(1 + \frac{\theta\eta}{4\hbar^2} \right)$ the effective Planck constant, θ and η the NC parameters with respect to the coordinates and momenta, respectively. Due to the fact that the NCty structure of the phase-space is a pure geometrical property, its physical effects are independent of the particle nature. Therefore, NCty between momenta arises naturally as a consequence of NC coordinates, since momenta are defined to be partial derivatives of the action with respect to the NC coordinates. The NC parameters θ and η have the dimensions of $(\text{length})^2$ and $(\text{momentum})^2$ respectively [35] [39]. The NC coordinates and momenta in terms of the commuting ones are defined as:

$$\begin{cases} \hat{q}_1 = q_1 - \frac{\theta}{2\hbar} p_2, \hat{q}_2 = q_2 + \frac{\theta}{2\hbar} p_1 \\ \hat{p}_1 = p_1 + \frac{\eta}{2\hbar} q_2, \hat{p}_2 = p_2 - \frac{\eta}{2\hbar} q_1 \end{cases} \quad (2)$$

where the coordinates q_k and momenta p_k obey the usual commutation relations $[q_k, q_l] = 0$, $[p_k, p_l] = 0$ and $[q_k, p_l] = i\hbar\delta_{kl}$, $k, l = 1, 2$. By the gauge symmetric relation, the vector potentials of the magnetic field \hat{A}_k can be defined as:

$$\hat{A} = \left(-\frac{B}{2} \hat{q}_2, \frac{B}{2} \hat{q}_1 \right) \quad (3)$$

Using the latest relation and Equation (2), the Hamiltonian (1) can be rewrit-

ten as follows:

$$\begin{aligned}
 H = & \frac{1}{2m_1} p_1^2 + \frac{1}{2m_2} p_2^2 + \frac{m}{2} (\varpi_1^2 q_1^2 + \varpi_2^2 q_2^2) \\
 & + k_{12} \left(q_1 q_2 + \frac{\theta}{2\hbar} (q_1 p_1 - q_2 p_2) - \frac{\theta^2}{2\hbar^2} p_1 p_2 \right) \\
 & - (\lambda_1 q_1 p_2 - \lambda_2 q_2 p_1),
 \end{aligned} \tag{4}$$

where

$$\begin{cases}
 \varpi_1 = \sqrt{\omega_1^2 + \frac{\omega_c^2}{4} + \frac{\omega_c \eta}{2m\hbar} + \frac{\eta^2}{4m^2 \hbar^2}}, \\
 \varpi_2 = \sqrt{\omega_2^2 + \frac{\omega_c^2}{4} + \frac{\omega_c \eta}{2m\hbar} + \frac{\eta^2}{4m^2 \hbar^2}}, \\
 \lambda_1 = \frac{m\theta}{2\hbar} \left(\omega_1^2 + \frac{\omega_c^2}{4} \right) + \frac{\omega_c}{2} \left(1 + \frac{\theta\eta}{4\hbar^2} \right) + \frac{\eta}{2m\hbar}, \\
 \lambda_2 = \frac{m\theta}{2\hbar} \left(\omega_2^2 + \frac{\omega_c^2}{4} \right) + \frac{\omega_c}{2} \left(1 + \frac{\theta\eta}{4\hbar^2} \right) + \frac{\eta}{2m\hbar},
 \end{cases} \tag{5}$$

with $m_1 = \frac{m}{\left(1 + \frac{m\omega_c\theta}{4\hbar}\right)^2 + \left(\frac{m\theta\omega_2}{2\hbar}\right)^2}$ and $m_2 = \frac{m}{\left(1 + \frac{m\omega_c\theta}{4\hbar}\right)^2 + \left(\frac{m\theta\omega_1}{2\hbar}\right)^2}$. It is

important to notice that the effective masses m_1 and m_2 are different due to the anisotropy of the system, and $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The last term appearing in the Hamiltonian (4) is due to the NCty effects. This term introduces some difficulties to solve the eigenvalue equation of the corresponding Hamiltonian, thus the whole Hamiltonian requires further transformations. For this reason, let's define a canonical transformation involving the set of coordinates-momenta (q_1, q_2, p_1, p_2) [31] [32] [40] by:

$$\begin{cases}
 q_1 = \chi \cos(\Phi) Q_1 + \frac{1}{\rho} \sin(\Phi) P_2, \\
 q_2 = \chi \cos(\Phi) Q_2 + \frac{1}{\rho} \sin(\Phi) P_1,
 \end{cases}
 \quad \text{and} \quad
 \begin{cases}
 p_1 = -\rho \sin(\Phi) Q_2 + \frac{1}{\chi} \cos(\Phi) P_1, \\
 p_2 = -\rho \sin(\Phi) Q_1 + \frac{1}{\chi} \cos(\Phi) P_2.
 \end{cases} \tag{6}$$

The new coordinates (Q_1, Q_2) and momenta (P_1, P_2) also satisfy the canonical commutation relations:

$$[Q_1, P_1] = [Q_2, P_2] = i\hbar, \tag{7}$$

while all other permutations vanish. Using the transformations of Equation (6), the Hamiltonian (4) becomes:

$$\begin{aligned}
 H = & \frac{1}{2M_1} P_1^2 + \frac{1}{2M_2} P_2^2 + \frac{M_1}{2} \Omega_1^2 Q_1^2 + \frac{M_2}{2} \Omega_2^2 Q_2^2 + c_{12} Q_1 Q_2 \\
 & + \mu_{11} Q_1 P_2 + \mu_{22} Q_2 P_1 + d_{12} P_1 P_2,
 \end{aligned} \tag{8}$$

where

$$M_1 = \frac{1}{2I_1^2}, M_2 = \frac{1}{2I_2^2}, \Omega_1 = 2I_1 K_1, \Omega_2 = 2I_2 K_2, \tag{9}$$

with

$$\begin{cases} I_1^2 = \frac{1}{2\rho^2} \left[\frac{\rho^2}{m_1\chi^2} \cos^2(\Phi) + m\varpi_2^2 \sin^2(\Phi) + \frac{\lambda_2\rho}{\chi} \sin(2\Phi) \right], \\ I_2^2 = \frac{1}{2\rho^2} \left[\frac{\rho^2}{m_2\chi^2} \cos^2(\Phi) + m\varpi_1^2 \sin^2(\Phi) + \frac{\lambda_1\rho}{\chi} \sin(2\Phi) \right], \\ K_1^2 = \frac{\chi^2}{2} \left[\frac{\rho^2}{m_2\chi^2} \sin^2(\Phi) + m\varpi_1^2 \cos^2(\Phi) + \frac{\lambda_1\rho}{\chi} \sin(2\Phi) \right], \\ K_2^2 = \frac{\chi}{2} \left[\frac{\rho^2}{m_1\chi^2} \sin^2(\Phi) + m\varpi_2^2 \cos^2(\Phi) + \frac{\lambda_2\rho}{\chi} \sin(2\Phi) \right], \end{cases} \quad (10)$$

and

$$\begin{cases} \mu_{11} = k_{12} \frac{\rho}{2\chi} \left[\left(\frac{\chi^2}{\rho^2} + \frac{\theta^2}{4\hbar^2} \right) \sin(2\Phi) + \frac{\theta\chi}{\hbar\rho} \right], \\ \mu_{22} = k_{12} \frac{\rho}{2\chi} \left[\left(\frac{\chi^2}{\rho^2} + \frac{\theta^2}{4\hbar^2} \right) \sin(2\Phi) - \frac{\theta\chi}{\hbar\rho} \right], \\ c_{12} = k_{12}\chi^2 \left[\cos^2(\Phi) - \frac{\theta^2\rho^2}{4\hbar^2\chi^2} \sin^2(\Phi) \right], \\ d_{12} = \frac{k_{12}}{\rho^2} \left[\sin^2(\Phi) - \frac{\theta^2\rho^2}{4\hbar^2\chi^2} \cos^2(\Phi) \right]. \end{cases} \quad (11)$$

The set (ρ, χ, Φ) satisfies the following relations:

$$\begin{cases} \frac{\rho}{\chi} = \sqrt{\frac{mm_1m_2(\lambda_2\varpi_1^2 + \lambda_1\varpi_2^2)}{m_2\lambda_1 + m_1\lambda_2}} \\ \tan(2\Phi) = \frac{2\sqrt{m_1m_2(m_1\lambda_2 + m_2\lambda_1)}(\lambda_2\varpi_1^2 + \lambda_1\varpi_2^2)}{\sqrt{m}(m_2\varpi_1^2 - m_1\varpi_2^2)} \end{cases} \quad (12)$$

Equation (8) is the Hamiltonian of a harmonic oscillator in two-dimensional NC phase-space, where the mass varies in terms of the NC parameters, which can provide simple solution to the eigenvalues equation. **Table 1** depicts the values of all the parameters and constants that appear in this Hamiltonian and in the entire work, and which will be necessary for numerical simulations in Section 5. In order to investigate the dynamics of two bosonic modes (Harmonic oscillators) in weak interaction with thermal reservoir in NC phase-space, we employ the axiomatic formalism based on the completely positive quantum dynamical semigroups [41].

3. Derivation of the Variance and Covariance Matrices

Generally, the simplest dynamics for an open system which describes an irreversible time evolution is given by the Gorini-Kossakowski-Linblad-Sudarshan (GKLS) quantum master equation [41] [42]. Indeed, the GKLS method is based on the non-Markovian approximation, meaning that there are no memory effects. The GKLS quantum non-Markovian master equation for the density operator $\rho(t)$ in the Schrodinger representation is given [42] [43] [44] by:

Table 1. Recapitulation of all the parameters and constants that are used in this work.

Parameter	Symbol	Value	Unit
Charge of electron	e	1	C
Magnetic field	B	$\propto 10^{-2}$	T
Reduced Planck constant	\hbar	1	j·s
Frequency in the x-direction	ω_1	1.5	s^{-1}
Frequency in the y-direction	ω_2	2.5	s^{-1}
Effective mass	m	1	kg
Light constant	c	1	$m \cdot s^{-1}$
Squeezing parameter	δ	0.5	s^{-1}
Friction coefficients	$\alpha_u, \beta_u, \lambda_u$	between 0 and 1	dimensionless
Boltzmann constant	k	1	$m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}$
Temperature	T	between 0 and 20	K
NC-parameter along the coordinates	θ	$\propto 10^{-38}$	m^2
NC-parameter along the momentum	η	$\propto 10^{-60}$	$kg^2 \cdot m^2 \cdot s^{-2}$

$$L(\rho) = \frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \frac{1}{2\hbar} \sum_j \left([V_j \rho, V_j] + [V_j, \rho V_j] \right), \tag{13}$$

where H denotes the open system’s Hamiltonian and the operators V_k and V_k^\dagger defined on the Hilbert space of H , describe the interaction of open system with the environment. Given that, we have an interest in the set of Gaussian states, we introduce such quantum dynamical semi group that preserves this set. Consequently, H is considered to be a polynomial of second degree in coordinates Q_1, Q_2 and momenta P_1, P_2 of the quantum oscillators. In this case, V_j and V_j^\dagger are taken as polynomials of first degree in these canonical observables. Then, in the linear space spanned by the coordinates and momenta, there exist only four linearly independent operators V_j ($j=1,2,3,4$) [16] [44]. Moreover, the operators are non-Hermitian and describe the dissipation and decoherence due to interaction between the system and its environment. Using the transformations (2) and (6), the Lindblad operators in the NC phase-space, are given by [42] [43]:

$$\begin{cases} V_j = A_{j1}P_1 + A_{j2}P_2 + B_{j1}Q_1 + B_{j2}Q_2 \\ V_j^\dagger = A_{j1}^*P_1 + A_{j2}^*P_2 + B_{j1}^*Q_1 + B_{j2}^*Q_2 \end{cases} \tag{14}$$

with

$$\begin{cases} A_{j1} = \frac{1}{\chi} \left(a_{j1} + b_{j2} \frac{\theta}{2\hbar} \right) \cos(\Phi) + \frac{1}{\rho} \left(b_{j2} + a_{j1} \frac{\eta}{2\hbar} \right) \sin(\Phi), \\ A_{j2} = \frac{1}{\chi} \left(a_{j1} - b_{j1} \frac{\theta}{2\hbar} \right) \cos(\Phi) + \frac{1}{\rho} \left(b_{j1} - a_{j2} \frac{\eta}{2\hbar} \right) \sin(\Phi), \\ B_{j1} = -\rho \left(a_{j2} - b_{j1} \frac{\theta}{2\hbar} \right) \sin(\Phi) + \chi \left(b_{j1} - a_{j2} \frac{\eta}{2\hbar} \right) \cos(\Phi), \\ B_{j2} = -\rho \left(a_{j1} + b_{j2} \frac{\theta}{2\hbar} \right) \sin(\Phi) + \chi \left(b_{j2} + a_{j1} \frac{\eta}{2\hbar} \right) \cos(\Phi), \end{cases} \tag{15}$$

where $a_{j1}, a_{j2}, b_{j1}, b_{j2}$ are complex numbers. The constant term is omitted because its contribution to the generator L is equivalent to the linear terms of the Hamiltonian in P_k and Q_k . Then, the harmonic oscillator's Hamiltonian (8) can be rewritten in the form:

$$H = H_0 + c_{12}Q_1Q_2 + \mu_{11}Q_1P_1 + \mu_{22}Q_2P_2 + d_{12}P_1P_2, \tag{16}$$

with

$$H_0 = \frac{1}{2M_1}P_1^2 + \frac{1}{2M_2}P_2^2 + \frac{M_1}{2}\Omega_1^2Q_1^2 + \frac{M_2}{2}\Omega_2^2Q_2^2. \tag{17}$$

Under the above conditions, Equation (13) becomes [5] [45]:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\mu_{11} + \lambda_{11})[Q_1, \rho P_1 + P_1\rho] \\ & - \frac{i}{2\hbar}(\mu_{22} + \lambda_{22})[Q_2, \rho P_2 + P_2\rho] - \frac{i}{2\hbar}\lambda_{12}([Q_1, \rho P_2 + P_2\rho] \\ & - [P_2, \rho Q_1 + Q_1\rho]) - \frac{i}{2\hbar}\lambda_{21}([Q_2, \rho P_1 + P_1\rho] - [P_1, \rho Q_2 + Q_2\rho]) \\ & - \frac{i}{2\hbar}(d_{12} + \alpha_{12})[P_2, \rho P_1 + P_1\rho] - \frac{i}{2\hbar}(c_{12} + \beta_{12})[Q_2, \rho Q_1 + Q_1\rho] \\ & + \frac{i}{2\hbar}(\lambda_{11} - \mu_{11})[P_1, \rho Q_1 + Q_1\rho] + \frac{i}{2\hbar}(\lambda_{22} - \mu_{22})[P_2, \rho Q_2 + Q_2\rho] \\ & + \frac{i}{2\hbar}(\alpha_{12} - d_{12})[P_1, \rho P_2 + P_2\rho] + \frac{i}{2\hbar}(\beta_{12} - c_{12})[Q_1, \rho Q_2 + Q_2\rho] \\ & - \frac{1}{\hbar^2} \sum_{k,l=1}^2 [D_{P_k P_l} [Q_k, [Q_l, \rho]] + D_{Q_k Q_l} [P_k, [P_l, \rho]] \\ & - D_{Q_k P_l} ([Q_k, [P_l, \rho]] + [P_k, [Q_l, \rho]])] \end{aligned} \tag{18}$$

where

$$\begin{aligned} D_{Q_l Q_l} &= \frac{\hbar}{2} \sum_j |A_{jl}|^2, \quad D_{P_l P_l} = \frac{\hbar}{2} \sum_j |B_{jl}|^2, \quad D_{Q_1 Q_2} = D_{Q_2 Q_1} = \frac{\hbar}{2} \operatorname{Re} \left[\sum_j A_{j1}^* A_{j2} \right], \\ D_{P_1 P_2} &= D_{P_2 P_1} = \frac{\hbar}{2} \operatorname{Re} \left[\sum_j B_{j1}^* B_{j2} \right], \quad D_{P_l Q_l} = D_{Q_l P_l} = -\frac{\hbar}{2} \operatorname{Re} \left[\sum_j A_{jl}^* B_{jl} \right], \\ D_{P_1 Q_2} &= D_{Q_2 P_1} = -\frac{\hbar}{2} \operatorname{Re} \left[\sum_j A_{j2}^* B_{j1} \right], \quad D_{P_2 Q_1} = D_{Q_1 P_2} = -\frac{\hbar}{2} \operatorname{Re} \left[\sum_j A_{j1}^* B_{j2} \right], \\ \alpha_{12} &= -\alpha_{21} = -\operatorname{Im} \left[\sum_j A_{j1}^* A_{j2} \right], \quad \beta_{12} = -\beta_{21} = -\operatorname{Im} \left[\sum_j B_{j1}^* B_{j2} \right], \\ \lambda_{11} &= -\operatorname{Im} \left[\sum_j A_{j1}^* B_{j1} \right], \quad \lambda_{22} = -\operatorname{Im} \left[\sum_j A_{j2}^* B_{j2} \right], \\ \lambda_{12} &= -\operatorname{Im} \left[\sum_j A_{j1}^* B_{j2} \right], \quad \lambda_{21} = -\operatorname{Im} \left[\sum_j A_{j2}^* B_{j1} \right]. \end{aligned}$$

The quantities $D_{P_k P_l}, D_{Q_k Q_l}, D_{P_k Q_l}$ and $D_{Q_k P_l}$ ($k = l = 1, 2$) are known as the diffusion coefficients, $\lambda_{kl}, \alpha_{12}, \alpha_{21}, \beta_{12}$ and β_{21} the frictions constants. From the Cauchy-Schwarz inequality, the above coefficients satisfy the following fundamental constraints [2]:

- The coefficients $D_{P_1P_1}$, $D_{Q_1Q_1}$, must be positive $\forall l$.
- $D_{P_1P_1}D_{Q_1Q_1} - D_{Q_1P_1}^2$ should be greater than the ratio $\frac{\lambda_{1l}^2 \hbar^2}{4} \forall l$.
- $D_{Q_1Q_1}D_{Q_2Q_2} - D_{Q_1Q_2}^2$ must be greater than $\frac{\alpha_{12}^2 \hbar^2}{4}$, and
- $D_{P_1P_1}D_{P_2P_2} - D_{P_1P_2}^2$ must be greater than $\frac{\beta_{12}^2 \hbar^2}{4}$.

In the particular case, we suppose that the asymptotic state of the considered

open system is a Gibbs state $\left(\rho_G(\infty) = \frac{e^{-\frac{H_0}{kT}}}{Tr \left[e^{-\frac{H_0}{kT}} \right]} \right)$, corresponding to a

quantum harmonic oscillator in thermal equilibrium at temperature T in two-dimensional phase-space. Under this assumption, the above quantum diffusion coefficients take the forms [45]:

$$\begin{cases} D_{P_1P_1} = \hbar M_l \Omega_l \frac{\lambda_{1l} + \mu_{1l}}{2} \coth\left(\frac{\hbar \Omega_l}{2kT}\right), \\ D_{Q_1Q_1} = \frac{\lambda_{1l} - \mu_{1l}}{2} \frac{\hbar}{M_l \Omega_l} \coth\left(\frac{\hbar \Omega_l}{2kT}\right), \end{cases} \tag{19}$$

other combinations being zero. In this case, the previous fundamental constraints are satisfied for example only if: $\lambda_{1l} > \mu_{1l}$ and $(\lambda_{1l} - \mu_{1l}) \coth\left(\frac{\hbar \Omega_l}{2kT}\right) \geq \lambda_{1l}^2$, $\forall l = 1, 2$.

Let $\sigma_{AA} = \langle A^2 \rangle - \langle A \rangle^2$ the dispersion (variance) of a given operator A , where $\langle A \rangle = Tr[\rho A]$ is the expectation value of the operator A , ρ the statistical operator (density matrix), and $\sigma_{AB} = \frac{\langle AB + BA \rangle}{2} - \langle A \rangle \langle B \rangle$ the correlation (covariance) of the operators A and B . It was demonstrated in [46] [47] that, for any Hermitian operators A and B , and for pure quantum states, the following generalized uncertainty relation holds:

$$\sigma_{AA} \sigma_{BB} - \sigma_{AB}^2 \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2. \tag{20}$$

From the master Equation (18), we obtain the following equation of motion for the expectation values of coordinates and momenta [22] [48]:

$$\frac{dn(t)}{dt} = Xn(t), \tag{21}$$

where $n(t) = [\sigma_{Q_1}, \sigma_{Q_2}, \sigma_{P_1}, \sigma_{P_2}]^T$ denotes the expectation vector and X a 4×4 matrix defined by:

$$X = \begin{pmatrix} -\lambda_{11} + \mu_{11} & -\lambda_{21} & \frac{1}{M_1} & -\alpha_{12} + d_{12} \\ -\lambda_{12} & -\lambda_{22} + \mu_{22} & \alpha_{12} + d_{12} & \frac{1}{M_2} \\ -M_1 \Omega_1^2 & \beta_{12} - c_{12} & -\lambda_{11} - \mu_{11} & -\lambda_{12} \\ -\beta_{12} - c_{12} & -M_2 \Omega_2^2 & -\lambda_{21} & -\lambda_{22} - \mu_{22} \end{pmatrix} \tag{22}$$

Solving Equation (21), we obtain

$$n(t) = N(t)n(0) = \exp(Xt)n(0), \tag{23}$$

where $n(0)$ denotes the initial conditions. The matrix $n(t)$ must fulfill the condition

$$\lim_{t \rightarrow \infty} n(t) = 0. \tag{24}$$

However, for this limit to exist, X must have only negative real eigenvalues parts. For this reason, we set $k_{12} = 0$, such that $\mu_{11} = d_{12} = c_{12} = 0$, which lead the system to uncoupled oscillators. Under these conditions, the matrix X becomes:

$$X = \begin{pmatrix} -\lambda_{11} & -\lambda_{21} & \frac{1}{M_1} & -\alpha_{12} \\ -\lambda_{12} & -\lambda_{22} & \alpha_{12} & \frac{1}{M_2} \\ -M_1\Omega_1^2 & \beta_{12} & -\lambda_{11} & -\lambda_{12} \\ -\beta_{12} & -M_2\Omega_2^2 & -\lambda_{21} & -\lambda_{22} \end{pmatrix} \tag{25}$$

In order to evaluate the exponential matrix $N(t)$, it is important to first diagonalize the matrix X by solving the corresponding secular equation *i.e.*

$$\det(X - z\mathbb{I}) = 0, \tag{26}$$

where z is the eigenvalues of X and \mathbb{I} the unit matrix. From Equation (26), one obtains an equation of fourth order with respect to the eigenvalues z , which can be easily solved. Let us for simplicity consider the particular case where $\beta_{12} = \alpha_{12} = \lambda_{12} = \lambda_{21} = 0$, then the secular equation obtained is:

$$\left[(\lambda_{11} + z)^2 + \Omega_1^2 \right] \left[(\lambda_{22} + z)^2 + \Omega_2^2 \right] = 0. \tag{27}$$

The resolution of this equation provides $z_1 = -\lambda_+ + i\Omega_+$, $z_2 = -\lambda_- + i\Omega_-$, $z_3 = -\lambda_+ - i\Omega_+$ and $z_4 = -\lambda_- - i\Omega_-$, where $\lambda_+ = \lambda_{11}$, $\lambda_- = \lambda_{22}$, $\Omega_+ = \Omega_1$, and $\Omega_- = \Omega_2$. To fulfill the condition of Equation (24), only the positive values of λ_+ and λ_- are considered. Thus, using the eigenvalues z_n of X , the time-dependent matrix $N(t)$ can be obtained as follows:

$$N_{ij}(t) = \sum_n M_{in} \exp(z_n t) M_{nj}^{-1}, \tag{28}$$

where the matrix M constitutes the eigenvectors of X :

$$\sum_j X_{ij} M_{jn} = z_n M_{in}. \tag{29}$$

Under the conditions $N_{ij}(t=0) = \delta_{ij}$ and $dN_{ij}(t)/dt|_{t=0} = X_{ij}$. The expressions of the matrix elements N_{ij} , can subsequently be obtained as:

$$N(t) = \begin{pmatrix} e^{-\lambda_{11}t} \cos \Omega_1 t & 0 & \frac{1}{M_1 \Omega_1} e^{-\lambda_{11}t} \sin \Omega_1 t & 0 \\ 0 & e^{-\lambda_{22}t} \cos \Omega_2 t & 0 & \frac{1}{M_2 \Omega_2} e^{-\lambda_{22}t} \sin \Omega_2 t \\ -M_1 \Omega_1 e^{-\lambda_{11}t} \sin \Omega_1 t & 0 & e^{-\lambda_{11}t} \cos \Omega_1 t & 0 \\ 0 & -M_2 \Omega_2 e^{-\lambda_{22}t} \sin \Omega_2 t & 0 & e^{-\lambda_{22}t} \cos \Omega_2 t \end{pmatrix}. \tag{30}$$

Considering Equation (23) and the initial conditions defined above, one get:

$$\begin{cases} \sigma_{Q_k}(t) = e^{-\lambda_{kk}t} \left[\sigma_{Q_k}(0) \cos(\Omega_k t) + \frac{1}{M_k \Omega_k} \sigma_{P_k}(0) \sin(\Omega_k t) \right], \\ \sigma_{P_k}(t) = e^{-\lambda_{kk}t} \left[-M_k \Omega_k \sigma_{Q_k}(0) \sin(\Omega_k t) + \sigma_{P_k}(0) \cos(\Omega_k t) \right], \end{cases} \quad k = 1, 2. \quad (31)$$

Thus, we can observe that, the expectation values of the coordinates and momenta decay quickly due to the exponential factors $e^{-\lambda_{11}t}$ and $e^{-\lambda_{22}t}$ and vanish when $t \rightarrow \infty$. Let us now determine the variance and covariance matrices. For this reason, let $\sigma(t)$ and D be respectively the covariance and the diffusion coefficients matrices where the elements are defined as follows:

$$D = \begin{pmatrix} D_{Q_1 Q_1} & D_{Q_1 Q_2} & D_{Q_1 P_1} & D_{Q_1 P_2} \\ D_{Q_2 Q_1} & D_{Q_2 Q_2} & D_{Q_2 P_1} & D_{Q_2 P_2} \\ D_{P_1 Q_1} & D_{P_1 Q_2} & D_{P_1 P_1} & D_{P_1 P_2} \\ D_{P_2 Q_1} & D_{P_2 Q_2} & D_{P_2 P_1} & D_{P_2 P_2} \end{pmatrix} \text{ and } \sigma(t) = \begin{pmatrix} \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_1 P_1} & \sigma_{Q_1 P_2} \\ \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} & \sigma_{Q_2 P_1} & \sigma_{Q_2 P_2} \\ \sigma_{P_1 Q_1} & \sigma_{P_1 Q_2} & \sigma_{P_1 P_1} & \sigma_{P_1 P_2} \\ \sigma_{P_2 Q_1} & \sigma_{P_2 Q_2} & \sigma_{P_2 P_1} & \sigma_{P_2 P_2} \end{pmatrix}, \quad (32)$$

Thus, from the master Equation (18), the equations of motion corresponding to the quantum correlations of canonical observables Q_1, Q_2 and P_1, P_2 are the following [48] [49]:

$$\frac{d\sigma}{dt} = X\sigma + \sigma X^T + 2D, \quad (33)$$

where the matrix X is defined in Equation (25), with X^T its transposed matrix. The time-dependent solution of (33) is given by:

$$\sigma(t) = N(t)(\sigma(0) - \sigma(\infty))N^T(t) + \sigma(\infty), \quad (34)$$

where $N(t)$ is defined in Equation (30). The matrix $\sigma(\infty)$ is time independent and solves the static problem Equation (33) (*i.e.* $\frac{d\sigma}{dt} = 0$). Assuming that the limit of $\sigma(t)$ exists as t goes to infinity (*i.e.* $\lim_{t \rightarrow \infty} \sigma(t)$ exists), then one has $\sigma(\infty) = \lim_{t \rightarrow \infty} \sigma(t)$. We can therefore, obtain the solutions of Equation (33) giving the elements of $\sigma(t)$. For illustration, the elements $\sigma_{Q_k Q_k}$, $\sigma_{P_k P_k}$ and $\sigma_{P_k Q_k}$ ($k = 1, 2$) are given by:

$$\begin{cases} \sigma_{Q_k Q_k}(t) = e^{-2\lambda_{kk}t} \left(\cos^2(\Omega_k t) (\sigma_{Q_k Q_k}(0) - \sigma_{Q_k Q_k}(\infty)) + \frac{1}{M_k \Omega_k} \sin(2\Omega_k t) (\sigma_{Q_k P_k}(0) - \sigma_{Q_k P_k}(\infty)) \right. \\ \quad \left. + \frac{1}{M_k^2 \Omega_k^2} (\sigma_{P_k P_k}(0) - \sigma_{P_k P_k}(\infty)) \sin^2(\Omega_k t) \right) + \sigma_{Q_k Q_k}(\infty); \\ \sigma_{P_k P_k}(t) = e^{-2\lambda_{kk}t} \left(M_k^2 \Omega_k^2 \sin^2(\Omega_k t) (\sigma_{Q_k Q_k}(0) - \sigma_{Q_k Q_k}(\infty)) - M_k \Omega_k \sin(2\Omega_k t) (\sigma_{P_k Q_k}(0) - \sigma_{P_k Q_k}(\infty)) \right. \\ \quad \left. + (\sigma_{P_k P_k}(0) - \sigma_{P_k P_k}(\infty)) \cos^2(\Omega_k t) \right) + \sigma_{P_k P_k}(\infty); \\ \sigma_{P_k Q_k}(t) = e^{-2\lambda_{kk}t} \left(-\frac{M_k \Omega_k}{2} \sin(2\Omega_k t) (\sigma_{Q_k Q_k}(0) - \sigma_{Q_k Q_k}(\infty)) + \cos(2\Omega_k t) (\sigma_{P_k Q_k}(0) - \sigma_{P_k Q_k}(\infty)) \right. \\ \quad \left. + \frac{1}{2M_k \Omega_k} (\sigma_{P_k P_k}(0) - \sigma_{P_k P_k}(\infty)) \sin(2\Omega_k t) \right) + \sigma_{P_k Q_k}(\infty), \end{cases} \quad (35)$$

where the remaining elements can be easily found following the same approach. It can be observed that the matrix elements σ_{ij} decay rapidly to zero due to the exponential terms $\exp(-2\lambda_{11}t)$, $\exp(-2\lambda_{22}t)$ and $\exp(-(\lambda_{11} + \lambda_{22})t)$. The matrix elements of $\sigma(\infty)$ depend on X and D , thus should be evaluated using the relation [22]:

$$\sigma(\infty) = 2 \int_0^\infty N(t') D N(t')^T dt'. \tag{36}$$

This relation is remarkably important, since it gives a very simple connection between the asymptotic values ($t \rightarrow \infty$) of $\sigma(t)$ and the diffusion coefficients D . One observe that the asymptotic values $\sigma_{Q_k Q_k}(\infty)$, $\sigma_{P_k P_k}(\infty)$ and $\sigma_{P_k Q_k}(\infty)$ do not depend on initial values $\sigma_{Q_k Q_k}(0)$, $\sigma_{P_k P_k}(0)$ and $\sigma_{P_k Q_k}(0)$. The initial state represents a correlated coherent state with variance and covariance of coordinates and momenta defined as [15]:

$$\sigma_{Q_k Q_k}(0) = \frac{\hbar \delta}{2M_k \Omega_k}, \sigma_{P_k P_k}(0) = \frac{\hbar M_k \Omega_k}{2\delta(1-r_k^2)} \text{ and } \sigma_{P_k Q_k}(0) = \frac{\hbar r_k}{2\delta\sqrt{1-r_k^2}}. \tag{37}$$

Here, δ represents the squeezing parameter and r_k the correlation coefficient at $t=0$ with $|r_k| < 1$, where $r_k = \frac{\sigma_{P_k Q_k}}{\sqrt{\sigma_{Q_k Q_k} \sigma_{P_k P_k}}}$. In the case of a thermal bath where $\mu_{ll} = 0$, with the coefficients (19) one can deduce that

$$\begin{aligned} \sigma_{Q_k Q_k}(\infty) &= \frac{\hbar}{2M_k \Omega_k} \coth\left(\frac{\hbar \Omega_k}{2kT}\right), \\ \sigma_{P_k P_k}(\infty) &= \frac{M_k \Omega_k \hbar}{2} \coth\left(\frac{\hbar \Omega_k}{2kT}\right) \text{ and } \sigma_{P_k Q_k}(\infty) = 0. \end{aligned} \tag{38}$$

The semigroup dynamics density operator which must hold for a quantum Markovian process is valid only for the weak-coupling regime, with the damping λ_{kk} obeying the inequality $\lambda_{kk} \ll \Omega_k$. Moreover, the diffusion coefficients $D_{Q_1 Q_2}$, $D_{P_2 Q_1}$, $D_{P_1 Q_2}$ and $D_{P_1 P_2}$ are generally zero for uncoupled oscillators interacting with an usual environment. Consequently, the expectation values $\sigma_{Q_1 Q_2}$, $\sigma_{P_2 Q_1}$, $\sigma_{P_1 Q_2}$ and $\sigma_{P_1 P_2}$ vanish when $t \rightarrow \infty$. It is a very interesting point that the general theory of Lindblad allows coupling via the environment between uncoupled oscillators ($d_{12} = 0$, $c_{12} = 0$, $\mu_{ll} = 0$, ($l = 1; 2$)). Furthermore, according to the definition of the parameters in terms of the vectors A_{jk} and B_{jk} , the diffusion coefficients above can be different from zero. In this case, a structure of the environment is reflected in the motion of the oscillators.

4. Density Matrix and Wigner Distribution Function

In this section, the NC anisotropic oscillator interacting with the usual environment is considered in the same idea presented in the previous section, but with $k_{12} = 0$, $\alpha_{12} = 0$, $\beta_{12} = 0$, $\lambda_{12} = 0$, $\lambda_{21} = 0$. Considering these conditions and following the method developed by Lampo et al. [6] in the one dimensional case, we derive the evolution in coordinate representation from Equation (18) as:

$$\begin{aligned}
\frac{d\rho}{dt} = & \sum_{k=1}^2 \left\{ \left[\frac{i\hbar}{M_k} \left(\frac{\partial^2}{\partial Q_k^2} - \frac{\partial^2}{\partial Q_k'^2} \right) - \frac{iM_k \Omega_k^2}{2\hbar} (Q_k^2 - Q_k'^2) \right] \rho \right. \\
& \left. - \frac{\lambda_{kk}}{2} (Q_k - Q_k') \left(\frac{\partial}{\partial Q_k} - \frac{\partial}{\partial Q_k'} \right) \rho \right\} \\
& + \sum_{k=1}^2 \left[\frac{\lambda_{kk}}{2} \left[(Q_k + Q_k') \left(\frac{\partial}{\partial Q_k} + \frac{\partial}{\partial Q_k'} \right) + 2 \right] \rho \right. \\
& \left. + D_{Q_k Q_k} \left(\frac{\partial}{\partial Q_k} + \frac{\partial}{\partial Q_k'} \right)^2 \rho - \frac{D_{P_k P_k}}{\hbar^2} (Q_k - Q_k')^2 \rho \right] \\
& + \sum_{k \neq l=1}^2 \left[-\frac{iD_{Q_k P_l}}{\hbar} (Q_k - Q_k') \left(\frac{\partial}{\partial Q_l} + \frac{\partial}{\partial Q_l'} \right) \rho \right. \\
& \left. + D_{Q_k Q_l} \left(\frac{\partial}{\partial Q_k} + \frac{\partial}{\partial Q_k'} \right) \left(\frac{\partial}{\partial Q_l} + \frac{\partial}{\partial Q_l'} \right) \rho \right] \\
& - \sum_{k \neq l=1}^2 \frac{D_{P_k P_l}}{\hbar^2} (Q_k - Q_k') (Q_l - Q_l') \rho \\
& - \sum_{k=1}^2 \frac{2iD_{Q_k P_k}}{\hbar} (Q_k - Q_k') \left(\frac{\partial}{\partial Q_k} + \frac{\partial}{\partial Q_k'} \right) \rho
\end{aligned} \tag{39}$$

Following [21], we can also transform the master Equation (18) for the density operator into the following Fokker-Planck-type equation satisfied by the Wigner distribution function $W(Q_1, Q_2, P_1, P_2, t)$:

$$\begin{aligned}
\frac{\partial W}{\partial t} = & \sum_{k=1}^2 \left\{ \left[-\frac{P_k}{M_k} \frac{\partial W}{\partial Q_k} + M_k \Omega_k^2 \frac{\partial W}{\partial P_k} \right] + \lambda_{kk} \left(\frac{\partial(P_k W)}{\partial P_k} + \frac{\partial(Q_k W)}{\partial Q_k} \right) \right\} \\
& + \sum_{k=1}^2 \left[D_{Q_k Q_k} \frac{\partial^2 W}{\partial^2 Q_k} + D_{P_k P_k} \frac{\partial^2 W}{\partial^2 P_k} + 2D_{Q_k P_k} \frac{\partial^2 W}{\partial Q_k \partial P_k} \right] \\
& + \sum_{k \neq l=1}^2 \left[2D_{Q_k Q_l} \frac{\partial^2 W}{\partial Q_k \partial Q_l} + 2D_{P_k P_l} \frac{\partial^2 W}{\partial P_k \partial P_l} + 2D_{Q_k P_l} \frac{\partial^2 W}{\partial Q_k \partial P_l} \right].
\end{aligned} \tag{40}$$

In Equation (40), the first four terms on the right-hand side give a purely unitary evolutions, the second four terms are the dissipative terms and have a damping effect (exchange of energy with environment), the last twelve terms are diffusive (noise) terms and produce fluctuation effects in the evaluation of the system. Moreover $D_{P_k P_k}$ promotes diffusion in momentum and generates decoherence in coordinate Q_k . It reduces the off-diagonal terms, responsible for correlations between spatially separated pieces of the wave packet. Analogically, $D_{Q_k Q_k}$ promotes diffusion in coordinate and generates decoherence in momentum P_k . The terms $D_{Q_k Q_l}$, $D_{Q_k P_l}$, $D_{Q_k P_k}$, and $D_{P_k P_l}$ ($k \neq l=1,2$) are the so-called “anomalous diffusion” terms. They promote diffusion in the variables $P_k P_l + P_l P_k$, $Q_k P_l + P_l Q_k$, $Q_k P_k + P_k Q_k$, and $Q_k Q_l + Q_l Q_k$ respectively. Just like the other diffusion terms but they do not generate decoherence [5]. To solve Equation (40), we have to first recall that the diffusion coefficients $D_{Q_1 Q_2}$, $D_{Q_1 P_2}$, $D_{Q_2 P_1}$, $D_{P_1 P_2}$ are well known and are generally zero for uncoupled oscillators. In addition, introducing the matrix notation $\langle Q_k Q_l \rangle = G_{kl}$, $\langle P_k P_l \rangle = E_{kl}$ and

$\langle Q_k P_l \rangle = F_{lk}$, which are the coefficients of the matrices G , E and F respectively, then for an initial Gaussian, the density matrix solution of Equation 40) and the Wigner function solution of Equation (40) are given by [21] [50] [51] [52]:

$$\begin{aligned} & \rho(Q_1, Q_2, Q'_1, Q'_2, t) \\ &= \frac{1}{2\pi\sqrt{\det G}} \exp \left\{ -\frac{1}{2} \left\{ \left((R - \bar{Q})^T, r^T \right) \begin{bmatrix} G^{-1} & -\frac{i}{\hbar} H \\ -\frac{i}{\hbar} H^T & \beta \end{bmatrix} \begin{pmatrix} R - \bar{Q} \\ r \end{pmatrix} \right\} + \frac{i}{\hbar} \sum_{k=1}^2 \sigma_{P_k} r_k \right\}, \end{aligned} \tag{41}$$

and

$$W(Q_1, Q_2, P_1, P_2, t) = \frac{1}{\sqrt{\det(2\pi\sigma(t))}} \exp \left[-\frac{1}{2} (y - n(t)) \sigma(t)^{-1} (y - n(t))^T \right], \tag{42}$$

where $R = \begin{pmatrix} \frac{Q_1 + Q'_1}{2} \\ \frac{Q_2 + Q'_2}{2} \end{pmatrix}$, $\bar{Q} = \begin{pmatrix} \sigma_{Q_1} \\ \sigma_{Q_2} \end{pmatrix}$, $r = \begin{pmatrix} Q_1 - Q'_1 \\ Q_2 - Q'_2 \end{pmatrix}$, $H = G^{-1} F^T$ and

$\beta = E - FG^{-1}F^T$, with $E = \begin{bmatrix} \sigma_{P_1 P_1} & 0 \\ 0 & \sigma_{P_2 P_2} \end{bmatrix}$, $F = \begin{bmatrix} \sigma_{Q_1 P_1} & 0 \\ 0 & \sigma_{Q_2 P_2} \end{bmatrix}$, and

$G = \begin{bmatrix} \sigma_{Q_1 Q_1} & 0 \\ 0 & \sigma_{Q_2 Q_2} \end{bmatrix}$, $y = (Q_1, Q_2, P_1, P_2)$. $n(t) = (\sigma_{Q_1}, \sigma_{Q_2}, \sigma_{P_1}, \sigma_{P_2})$ denotes the expectation vector, and $\sigma(t)$ is the dispersion (correlation) matrix defined by:

$$\sigma(t) = \begin{pmatrix} G & F \\ F & E \end{pmatrix}. \tag{43}$$

Thus, in the case of a thermal bath and when the asymptotic state is a Gibbs state and considering Equation (38), we obtain the following steady state solution:

$$\begin{aligned} & \langle Q_1 Q_2 | \rho(\infty) | Q'_1 Q'_2 \rangle = \rho_1 \rho_2 \\ &= \left[\sqrt{\frac{M_1 \Omega_1}{\pi \hbar \coth \varepsilon_1}} \exp \left\{ -\frac{M_1 \Omega_1}{4 \hbar} \left[\frac{(Q_1 + Q'_1)^2}{\coth \varepsilon_1} + \coth \varepsilon_1 (Q_1 - Q'_1)^2 \right] \right\} \right] \\ & \times \left[\sqrt{\frac{M_2 \Omega_2}{\pi \hbar \coth \varepsilon_2}} \exp \left\{ -\frac{M_2 \Omega_2}{4 \hbar} \left[\frac{(Q_2 + Q'_2)^2}{\coth \varepsilon_2} + \coth \varepsilon_2 (Q_2 - Q'_2)^2 \right] \right\} \right] \end{aligned} \tag{44}$$

and

$$\begin{aligned} & W_\infty = W_1 W_2 \\ &= \left[\frac{1}{\pi \hbar \coth \varepsilon_1} \exp \left\{ -\frac{1}{\hbar \coth \varepsilon_1} \left[M_1 \Omega_1 Q_1^2 + \frac{1}{M_1 \Omega_1} P_1^2 \right] \right\} \right] \\ & \times \left[\frac{1}{\pi \hbar \coth \varepsilon_2} \exp \left\{ -\frac{1}{\hbar \coth \varepsilon_2} \left[M_2 \Omega_2 Q_2^2 + \frac{1}{M_2 \Omega_2} P_2^2 \right] \right\} \right], \end{aligned} \tag{45}$$

with $\varepsilon_k = \frac{\hbar \Omega_k}{kT}$. We observe that, all stationary solutions to the evolution equa-

tions obtained in the long time limit are possible as a result of a balance between the wave packet spreading induced by the Hamiltonian and the localizing effect of the Lindblad operators. **Figure 1** and **Figure 2** depict the behavior of the density matrix and the Wigner distribution functions respectively. It can be observed that both exhibit a Gaussian behavior with the amplitudes which strongly depend on the NC parameters. The diagonal elements of the density matrix (*i.e.* $Q_k = Q_k^p$, $k = 1, 2$) represent the probability of finding the system in this position, while the off-diagonal elements introduce the correlation in the density matrix between the points Q_k and Q_k^p , $k = 1, 2$.

5. Results and Discussion

In this section, we mainly focus on the decoherence time scale and the total energy of a two-dimensional harmonic oscillator in an NC phase-space. For the reasons of simplicity, let $\theta = \alpha \times 10^{-38} \text{ m}^2$ and $\eta = \beta \times 10^{-60} \text{ kg}^2 \cdot \text{m}^2 \cdot \text{s}^{-2}$, where θ and η are the NC parameters, whose values are selected following the works of refs. [38] [39]. It is worth noting that α and β are dimensionless.

5.1. Decoherence Time Scale

As previously mentioned, the diffusion coefficients in momentum $D_{P_k P_k}$ produces decoherence along coordinates Q_k . Mathematically, classical limit is often associated with the size of Planck's constant. In the macroscopic limit, this

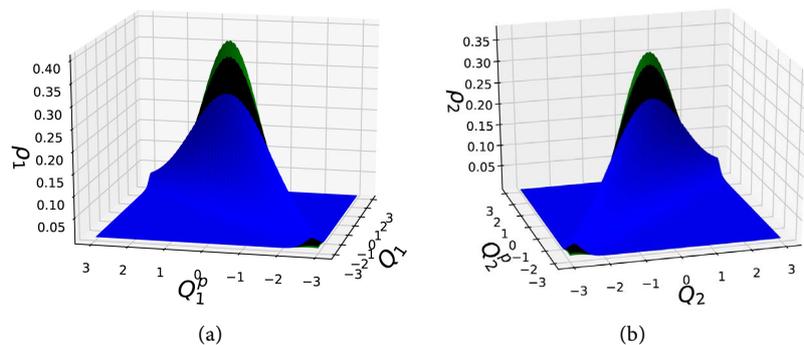


Figure 1. Density matrix ρ_∞ with respect to the coordinates in the Q_1 -direction (**Figure 1(a)**) and in the Q_2 -direction (**Figure 1(b)**) respectively.

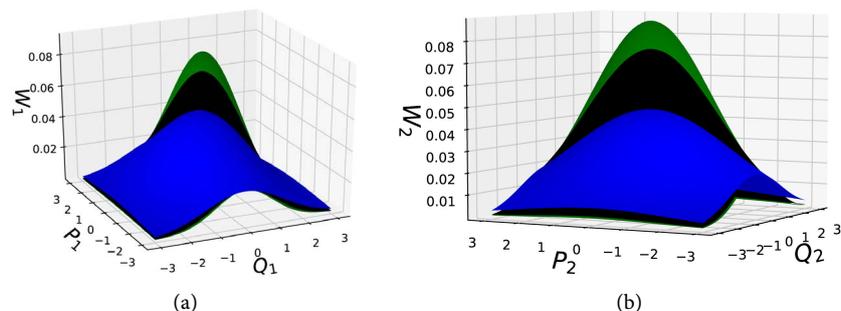


Figure 2. Wigner function W_∞ with respect to both the coordinates and momentum in the Q_1 -direction (**Figure 2(a)**) and in the Q_2 -direction (**Figure 2(b)**) respectively.

Planck constant \hbar is negligible compared to other quantities with dimension of action, such as $\sqrt{D_{P_1 P_1} \langle (Q_1 - Q'_1)^2 \rangle + D_{P_2 P_2} \langle (Q_2 - Q'_2)^2 \rangle}$. Thus, in this condition, the terms of Equation (39) containing the quantity $-\frac{D_{P_k P_k}}{\hbar^2}$ dominate on others.

To kindly appreciate its effects, let us rewrite an explicit solution of the master Equation (Equation (39)) approximating the right hand side by the dominating terms. In that case, the following time depend evolution of the density matrix is obtained:

$$\begin{aligned} & \langle Q_1 Q_2 | \rho(t) | Q'_1 Q'_2 \rangle \\ &= \langle Q_1 Q_2 | \rho(0) | Q'_1 Q'_2 \rangle \exp \left\{ - \left[\frac{D_{P_1 P_1}}{\hbar^2} (Q_1 - Q'_1)^2 + \frac{D_{P_2 P_2}}{\hbar^2} (Q_2 - Q'_2)^2 \right] t \right\} \quad (46) \\ &= \langle Q_1 Q_2 | \rho(0) | Q'_1 Q'_2 \rangle \exp \left\{ - \frac{1}{t_{deco}} t \right\}. \end{aligned}$$

We can thus, easily deduce the decoherence time scale as:

$$t_{deco} = \frac{\hbar^2}{D_{P_1 P_1} (Q_1 - Q'_1)^2 + D_{P_2 P_2} (Q_2 - Q'_2)^2}, \quad (47)$$

which reduces to:

$$t_{deco} = \frac{2\hbar}{\lambda_{11} M_1 \Omega_1 \sigma_{Q_1 Q_1}(0) \coth \varepsilon_1 + \lambda_{22} M_2 \Omega_2 \sigma_{Q_2 Q_2}(0) \coth \varepsilon_2}, \quad (48)$$

where the quantities $(Q_k - Q'_k)^2$ are in order of the initial dispersion in coordinates $\sigma_{Q_k Q_k}(0)$, in the particular case of thermal bath (*i.e.* at zero temperature). Considering Equation (37), Equation (48) becomes:

$$t_{deco} = \frac{4}{\lambda_{11} \delta \coth \varepsilon_1 + \lambda_{22} \delta \coth \varepsilon_2}, \quad (49)$$

with $\tau_1 = \frac{2kT}{\hbar\Omega_1} = \frac{1}{\varepsilon_1}$ and $\tau_2 = \frac{2kT}{\hbar\Omega_2} = \frac{1}{\varepsilon_2}$. The quantity t_{deco} defines the so-called

decoherence time scale which as we previously mentioned defines the particular time that the system initially quantum, starts behaving classically (*i.e.* the particular time that the system starts losing its quantum properties). Based on the above analytical expressions, we have numerically simulated the decoherence time scale as function of the temperature on one hand and as function of the cutting frequency (magnetic field) on the other hand. One can easily observe on **Figure 3(a)** that, the NCty effects present significant impacts on this time scale, and thus, considerably affect decoherence in the system in the sense that, the decoherence time scale increases as the NC parameters increase. However, this time scale decreases asymptotically in the case of NC phase-space as well as in the commutative phase-space case, and similar effects are also observed as in the case of total energy since the NCty effects are not more observable for high temperature. It is important to mention that, the NCty effects contribution to decoherence for zero temperature coincides with that of the commutative space since

the system receives very small feedback from the environment at this temperature. We can conclude that, even if the NCty effects increase decoherence in the system at high temperature, the system loses its coherence properties and becomes decoherent, because the decoherence time goes to zero for high temperature. Moreover, it is observed in **Figure 3(b)** that, the decoherence time scale increases as the magnetic field increases, in the case of NC phase-space as well as in the case of commutative phase-space, but very slowly in the latest case. Under lower magnetic field effects, the decoherence time scale is different from zero implying that even at this temperature, the system still receives feedback from its environment. In addition, the decoherence time scale increases asymptotically to a fixed value in the case of NC phase-space as well as in the case of commutative phase-space. This constant value defines the time during which the system loses its coherence properties and becomes decoherent (the system exhibits classical behaviour). The above results are confirmed by **Figure 4**, plotting the decoherence time scale versus simultaneously the NC parameters in the x- and y-directions, where we can easily observe that it increases with both parameters.

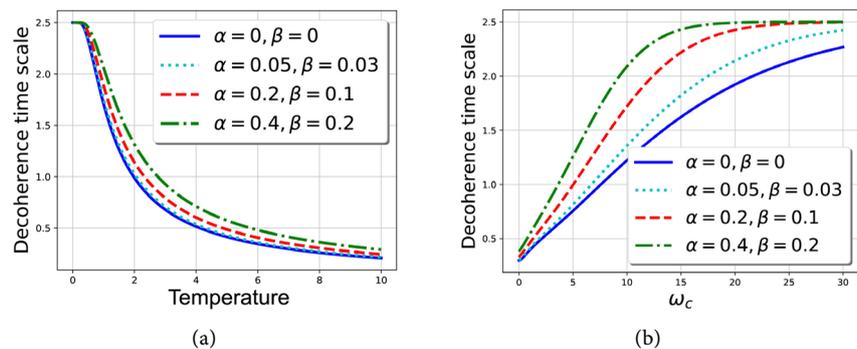


Figure 3. Decoherence time scale of an harmonic oscillator in two-dimensional phase-space with respect to the cutting frequency ω_c and with respect to the temperature T both as function of the NC phase-space parameters.

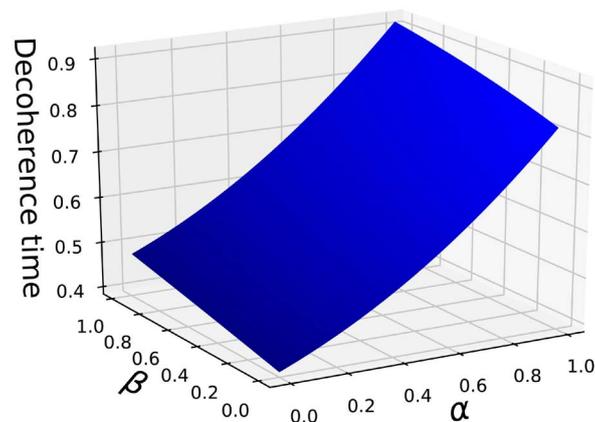


Figure 4. Decoherence time scale of an harmonic oscillator in two-dimensional phase-space with respect to both the NC parameters in the x- and y-directions, for a fixed temperature $T = 5$ K and a fixed magnetic field $\omega_c = 1.5$.

5.2. Total Energy of the System

The total energy of the anisotropic oscillator in a NC phase space can be derived from the mean value of the Hamiltonian as:

$$\begin{aligned}
 \langle H \rangle = & \frac{\sigma_{P_1 P_1}}{2M_1}(t) + \frac{\sigma_{P_2 P_2}}{2M_2}(t) + \frac{1}{2}M_1\Omega_1^2\sigma_{Q_1 Q_1}(t) + \frac{1}{2}M_2\Omega_2^2\sigma_{Q_2 Q_2}(t) \\
 & + \mu_{11}\sigma_{P_1 Q_1}(t) + \mu_{22}\sigma_{P_2 Q_2}(t) + d_{12}\sigma_{P_1 P_2}(t) + c_{12}\sigma_{Q_1 Q_2}(t) \\
 & + \frac{1}{2M_1}\sigma_{P_1}^2(t) + \frac{1}{2M_2}\sigma_{P_2}^2(t) + \frac{1}{2}M_1\Omega_1^2\sigma_{Q_1}^2(t) \\
 & + \frac{1}{2}M_2\Omega_2^2\sigma_{Q_2}^2(t) + \mu_{11}\sigma_{P_1}(t)\sigma_{Q_1}(t) + \mu_{22}\sigma_{P_2}(t)\sigma_{Q_2}(t) \\
 & + d_{12}\sigma_{P_1}(t)\sigma_{P_2}(t) + c_{12}\sigma_{Q_1}(t)\sigma_{Q_2}(t).
 \end{aligned} \tag{50}$$

Since the expectation and variances values of the coordinates and momenta decrease exponentially in time, the energy is dissipated to a minimum value. Assuming that the conditions (38) hold, the minimum value of the total energy of the system is given by the following expression:

$$E_{\min} = E(\infty) = \hbar \left(\frac{\Omega_1}{2} \coth \frac{\hbar\Omega_1}{2kT} + \frac{\Omega_2}{2} \coth \frac{\hbar\Omega_2}{2kT} \right). \tag{51}$$

At zero temperature, *i.e.* $T = 0$, the minimal energy becomes

$$\bar{E}_{\min} = \hbar \left(\frac{\Omega_1}{2} + \frac{\Omega_2}{2} \right),$$

which corresponds to the energy of the system in its

ground state. In the particular case where $\omega_1 = \omega_2$ implying $\Omega_1 = \Omega_2 = \Omega$ and $\theta = \eta = 0$, the minimum energy of the ground state can be reduced to $\bar{E}'_{\min} = \hbar\Omega$, and we recover the ground state energy of the usual harmonic oscillator in the commutative case. **Figure 5** depicts therefore, the energy with respect to the cutting frequency on one hand and with respect to the real temperature on the other hand both for different values of NC phase-space parameters.

It can be observed that, for fixed values of the temperature T (the thermal bath temperature), the total energy increases quickly with the magnetic field (cutting frequency), and that under lower magnetic field effects, the total energy for

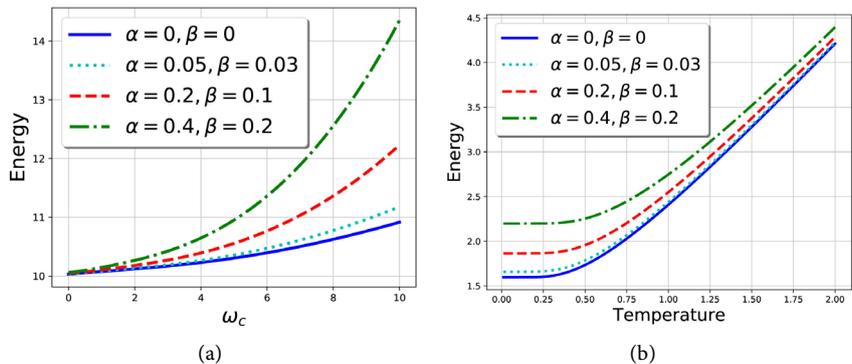


Figure 5. Energy of an harmonic oscillator in two-dimensional phase-space with respect to the cutting frequency ω_c and with respect to the temperature T both as function of the NC phase-space parameters.

different NC parameter values coincides. But under intense magnetic field effects, the NCty structure of the phase-space considerably affects the system since the energy increase with these effects. In addition, it is observed that, for extremely low temperature, the total energy of the system increase significantly with the NCty effects, however as the temperature increases, the energy increases too and for high temperature, the energy of the system in NC phase-space coincides with that of the system in commutative space (blue solid curve). This represents the fact that, when the temperature increases in the system, the coupling between the particle and the bath becomes considerably strong, the whole system takes a large dimension and then, the effects of quantum gravity manifest in NC phases-space are not more detectable. Similar results were found by Tchoffo *et al.* [26] in the case of Brownian particle in NC space. Analogically to the decoherence time scale, **Figure 6** plots the energy of the system with respect to simultaneously the NC parameters in the x- and y-directions, where we can easily observe that it increases with both parameters, confirming the above results.

6. Conclusions

In this paper, we studied decoherence of a damped anisotropic harmonic oscillator under magnetic field effects in two-dimensional NC phase-space. For this reason, the evolution of the system was studied within the framework of the Lindblad ME theory for open quantum systems, considering the general case of an environment consisting of a thermal bath at arbitrary temperature. Based on the above mentioned theory, the damping of the expectation values of coordinates and momenta was evaluated as functions of time. Then, from the ME of an NC damped anisotropic oscillator approach, the time evolution of the density matrix and the Wigner function were derived systematically. It turned out by solving these equations that:

- The solutions follow Gaussian distribution. Moreover, from the expectation values and the variances of coordinate and momenta, the total energy of the system was evaluated.

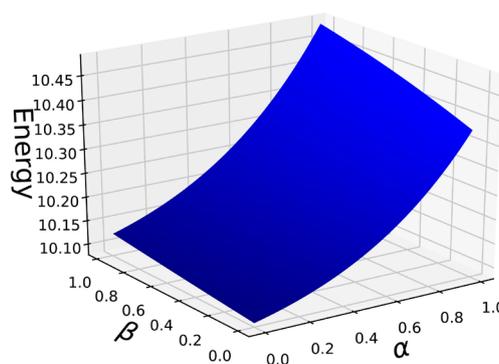


Figure 6. Energy of a harmonic oscillator in two-dimensional phase-space with respect to both the NC parameters in the x- and y-directions, for a fixed temperature $T = 5$ K and a fixed cutting frequency $\omega_c = 1.5$.

- The total energy increases significantly with the NCty effects, and as the temperature increases, the energy increases too. However, for high temperature, the energy of the system in NC phase-space coincides with that of the system in commutative space. Similar effects were observed with the magnetic field effects.
- In addition, the decoherence time scale was analytically derived, and its simulation proved that, the NCty effects present significant impact on this time scale, and thus, considerably affect decoherence in the system in the sense that, the decoherence time scale increases as the NC parameters increase. It turned out therefore that, the decoherence time scale was improved by NCty effects.
- The decoherence time was found to have a similar scale as the time after which statistic fluctuations become comparable with quantum fluctuations as expected, and the values of the scales become closer with the growth of temperature, magnetic field, and NCty effects.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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