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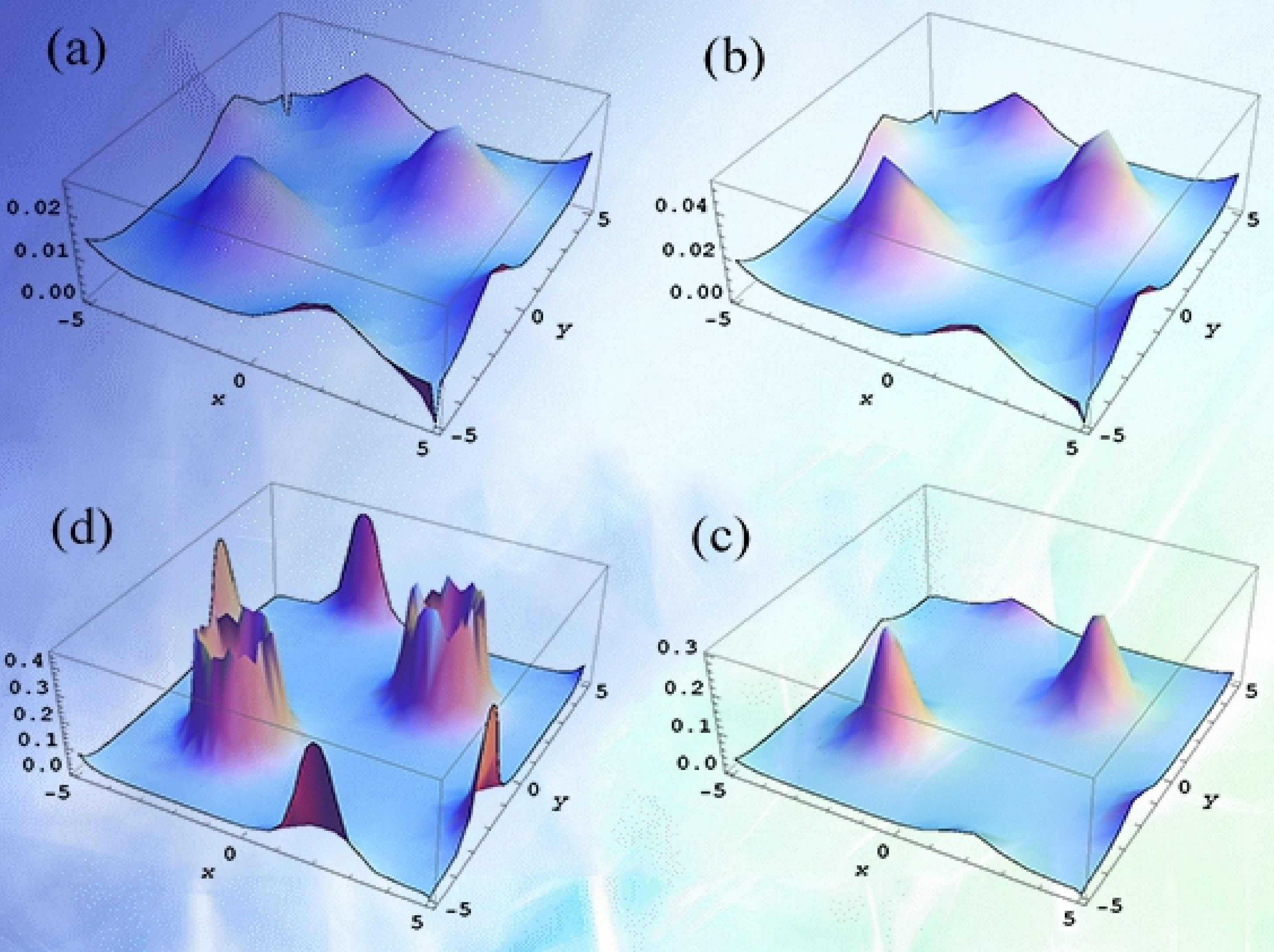

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Unified Field Theory

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Abstract

In the Einstein field equations, the geometry or the *curvature* of space-time defined as depended on the distribution of mass and energy principally resides on the left-hand side is set identical to a non-geometrical tensorial representation of matter on the right-hand side. In one or another form, general relativity accords a direct geometrical significance only to the gravitational field while the other physical fields are not of space time. They reside only in space time. Less well known, though of comparable importance is Einstein's dissatisfaction with the fundamental asymmetry between gravitational and non-gravitational fields and his contributions to develop a completely relativistic geometrical field theory of all fundamental interactions, *a unified field theory*. Of special note in this context and equally significant is Einstein's demand to replace the symmetrical tensor field by a non-symmetrical one and to drop the condition $g_{ik} = g_{ki}$ for the field components. Historically, many other attempts were made too, to extend the general theory of relativity's geometrization of gravitation to non-gravitational interactions, in particular, to electromagnetism. Still, progress has been very slow. It is the purpose of this publication to provide a unified field theory in which the gravitational field, the electromagnetic field and other fields are only different components or manifestations of the same unified field by mathematizing the relationship between cause and effect under conditions of general theory of relativity.

Keywords

Quantum Theory, Relativity Theory, Unified Field Theory, Causality

1. Introduction

The historical development of physics as such shows that formerly unrelated and separated parts of physics can be fused into one single conceptual formalism. Maxwell's theory unified the magnetic field and the electrical field once treated as fundamentally different. Einstein's special relativity theory provided a unification of the laws of Newton's mechanics and the laws of electromagnetism [1]. Thus far, the electromagnetic and weak nuclear forces have been unified together as an electroweak force. The unification with the strong interaction (chromodynamics) enabled the standard model of elementary particle physics. Meanwhile, the unification of gravitation with the other fundamental forces of nature is in the focus of much present research but still not in sight. A unification of all four fundamental interactions within one conceptual and formal framework has not yet met with success. Even Einstein himself spent years of his life on the unification [2] of the electromagnetic fields with the gravitational fields. In this context, Einstein's position concerning the unified field theory is strict and clear (**Figure 1**).

Despite of the many and different approaches of theorists worldwide spanning so many of years taken to develop a unified field theory, to describe and to understand the nature at the most fundamental (quantum) level progress has been very slow. Thus far, a unification of all four fundamental interactions within one conceptual and formal framework has not yet met with success. Excellent and very detailed reviews, some of them in an *highly and extraordinary satisfying way* [3], of the various aspects of the conceptually very different approaches of the unified field theories in the 20th century with a brief technical descriptions of the theories suggested and short biographical notes are far beyond the scope of this article and can be found in literature. The main focus of this article lies on the conceptual development of the geometrization of the electromagnetic field, by also paying attention to the unification of the electromagnetic and gravitational fields and the unified field theory as such. While the task to "geometrize" the electromagnetic field is not an easy one, a method how electromagnetic fields and gravitational fields can be joined into *a new hyper-field* [4], will be developed, and a new common representation of all four fundamental interactions will be presented. As will be seen, with regard to unified field theories, formerly unrelated parts of physics will be fused into one single conceptual formalism while following a deductive-hypothetical approach. We briefly define and describe the basic mathematical objects and tensor calculus rules needed to achieve unification. In this context, the point of departure for a unified field theory will be in accordance with general relativity theory from the beginning. Still, in order to decrease the amount of notation needed, we shall restrict ourselves as much as possible to *covariant* second rank tensors.

Past experience has shown that formerly published papers with new theoretical perspectives should be based on a clear terminology thus that contradictions and misunderstandings are prevented from the beginning. The views in this paper are different in some instances. Thus far, this paper is written such that physicists should be able to follow the technical aspects of the papers without any problems, while philosophers of science and other reader without prior knowledge of the mathematics or of tensor calculus and general relativity at least might gain an insight into the new concepts, methods, and scientific background involved. As will be seen, with regard to new insights and conclusions, the rest of the paper is organized as follows. In the section, we will give some basic definitions and a terminological distinction. View of the immense amount of literature known theories will be covered only as much as is necessary for a better understanding of this paper. I apologize for the shortcoming. In all the attempts to develop some basic fundamental insights, I will encounter a deductive-hypothetical methodological approach. Especially in section theorems, the starting point of the theorems is axiom +1 = +1 (lex identitatis) which serves as common ground for quantum and relativity theory. The section discussion examines some conditions and consequences of the theorems proved.

2. Material and Methods

2.1. Definitions

2.1.1. Definition: The Pythagorean Theorem

The Pythagorean (or Pythagoras') theorem is of far reaching and fundamental importance in Euclidean Geometry and in science as such. In physics, the Pythagorean (or Pythagoras') theorem serves especially as a basis for the definition of distance between two points. Historically, it is difficult to claim with a great degree of credibility that Pythagoras (~560 - ~480 B.C.) or someone else from his School was the first to discover this theorem.

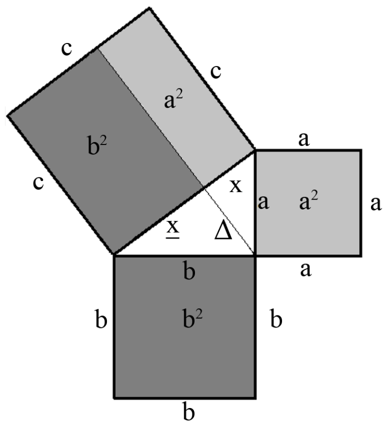
> "The theory we are looking for must therefore be a generalization of the theory of the gravitational field. The first question is: What is the natural generalization of the symmetrical tensor field? ... What generalization of the field is going to provide the most natural theoretical system? The answer ... is that the symmetrical tensor field must be replaced by a non-symmetrical one. This means that the condition $g_{ik} = g_{ki}$ for the field components must be dropped **[2]**".

Figure 1. Einstein and the problem of the unified field theory.

There is some evidence, that the Pythagorean (or Pythagoras') theorem was discovered on a Babylonian tablet [5] circa 1900 - 1600 B.C. Meanwhile, there are more than 100 published approaches proving this theorem, probably the most famous of all proofs of the Pythagorean proposition is the first of Euclid's two proofs (I.47), generally known as the Bride's Chair. The Pythagorean (or Pythagoras') theorem states that the sum of (the areas of) the two small squares equals (the area of) the big one square. In algebraic terms we obtain

$$a^2 + b^2 = c^2 \tag{1}$$

where c represents the length of the hypotenuse (the longest side within a right angled triangle) and a and b represents the lengths of the triangle's other two sides or legs (or *catheti*, singular: *cathetus*, greek: *káthetos*). Following Euclid (Elements Book I, Proposition 47) in right-angled triangles the sum of the squares on the sides containing the right angle equals the square on the side opposite the right angle.



2.1.2. Definition: The Normalization of the Pythagorean Theorem

The normalization of the Pythagorean theorem is defined as

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} = 1 \tag{2}$$

where c represents the length of the hypotenuse (the longest side within a right angled triangle) and a and b represents the lengths of the triangle's other two sides/legs.

2.1.3. Definition: The Negation Due to the Pythagorean Theorem

We define the negation of x, denoted as $n(\underline{x})$, as

$$n(\underline{x}) \equiv \frac{b^2}{c^2} \equiv 1 - \frac{a^2}{c^2} \equiv \frac{\underline{x}}{c} \tag{3}$$

We define the negation of anti x, denoted as $n(x)$, as

$$n(x) \equiv 1 - n(\underline{x}) \equiv \frac{a^2}{c^2} = 1 - \frac{b^2}{c^2} = 1 - \frac{\underline{x}}{c} \tag{4}$$

In general, it is

$$n(x) + n(\underline{x}) \equiv \frac{a^2}{c^2} + \frac{b^2}{c^2} \equiv 1 \tag{5}$$

2.1.4. Definition: The Determination of the Hypotenuse of a Right Angled Triangle

In general, we define

$$x + \underline{x} = c \tag{6}$$

where x and $\underline{x}$ denotes the segments on the hypotenuse c of a right angled triangle (c is the longest side within a right angled triangle).

Scholium.

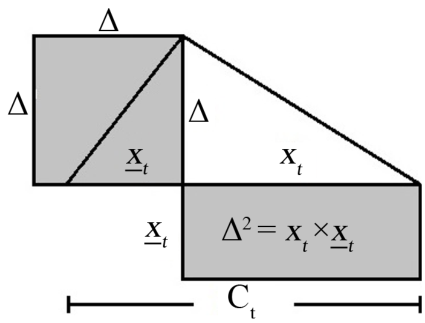
Form this follows that $(C \times X) + (C \times \underline{X}) = C^2$. Due to our definition above, it is $(C \times X) = a^2$ and $b^2 = (C \times \underline{X})$. The Pythagorean theorem is valid even if $\underline{x} = 1$ and $x = +\infty - 1$ while $c = +\infty$. Under these assumptions, the Pythagorean theorem is of use to prove the validity of the claim that $+1/+0 = +\infty$. In general, the normalized form of negation is denoted as $(1 -)$.

2.1.5. Definition: The Euclid's Theorem

According to Euclid's (ca. 360 - 280 BC) so called geometric mean theorem or *right triangle altitude theorem* or Euclid's theorem, published in Euclid's Elements in a corollary to proposition 8 in Book VI, used in proposition 14 of Book II [6] to square a rectangle too, it is

$$x \times \underline{x} \equiv \frac{a^2 \times b^2}{c^2} \equiv \Delta^2 \tag{7}$$

where Δ denotes the altitude in a right triangle and x and $\underline{x}$ denote the segments on the hypotenuse c of a right angled triangle.



Scholium.

The variance of a right angled triangle, denoted as $\sigma(x)^2$, can be defined as

$$\sigma(x)^2 \equiv \frac{x \times \underline{x}}{c^2 \times c^2} \equiv \frac{a^2 \times b^2}{c^2 \times c^2} \equiv \frac{\Delta^2}{c^2} \tag{8}$$

where Δ denotes the altitude in a right triangle and x and $\underline{x}$ denote the segments on the hypotenuse c of a right angled triangle.

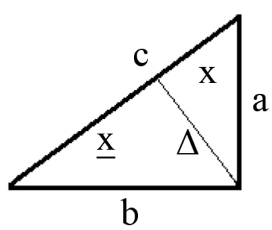
2.1.6. Definition: The Gradient

The gradient, denoted as *grad*(a, b), a measure of how steep a slope or a line is, is defined by dividing the vertical height a by the horizontal distance b of a right angled triangle. In other words, we obtain

$$\operatorname{grad}(a,b) \equiv \frac{a}{b} \equiv \frac{\text{Rise}}{\text{Run}} \tag{9}$$

Scholium.

The following picture of a right angled triangle may illustrate the background of a gradient



where b denotes the run; a denotes the rise and c denotes the slope length. The gradient has several meanings. In mathematics, the gradient is more or less something like a generalization of a derivative of a function in one dimension to a function in several dimensions. Consider a n-dimensional manifold with coordinates $_1x$, $_2x$, $_nx$. The gradient of a function $f(_1x, {}_2x, {}_nx)$ is defined as

$$(\nabla f)_\mu \equiv \frac{\partial f}{\partial_\mu x} \tag{10}$$

Due to our definition above it is equally $c^2 \times n(x) = a^2$. In this case c^2 is not identical to the speed of the light but with the hypotenuse, the longest side within a right angled triangle. Equally, it is $c^2 \times n(\underline{x}) = b^2$. In general, it is true that $a^2/b^2 = c^2 \times n(x)/c^2 \times n(\underline{x}) = n(x)/n(\underline{x})$. The *raise* can be calculated as $a/b = (n(x)/n(\underline{x})) = (n(x)/(1-n(x)))$. In other words, it is $a/b = y \times (x/c)$ or $a/b = ((c \times x)/(c \times \underline{x}))^{1/2} = ((x)/(\underline{x}))^{1/2}$.

2.2. Einstein's Special Theory of Relativity

2.2.1. Definition: The Relativistic Energy $_RE$ (of a System)

In general, it is

$$_RE = {}_Rm \times c^2 \tag{11}$$

where $_RE$ denotes the total ("relativistic") energy of a system; $_Rm$ denotes the "relativistic" mass and c denotes the speed of the light in vacuum.

Scholium.

Einstein *defined the matter/mass-energy equivalence* as follows:

"Gibt ein Körper die Energie L in Form von Strahlung ab, so verkleinert sich seine Masse um L/V^2 … Die *Masse* eines Körpers *ist* ein Maß für dessen *Energie*inhalt" [7].

In other words, due to Einstein, energy and mass are equivalent.

"Eines der wichtigsten Resultate der Relativitätstheorie ist die Erkenntnis, daß jegliche Energie E eine ihr proportionale Trägheit (E/c^2) besitzt" [8].

It was equally correct by Einstein to point out that *matter/mass and energy* are equivalent.

"Da *Masse und Energie* nach den Ergebnissen der speziellen Relativitätstheorie *das Gleiche* sind und die Energie formal durch den symmetrischen Energietensor ($T_{\mu\nu}$) beschrieben wird, so besagt dies, daß das G-Geld [gravitational field, *author*] durch den Energietensor der Materie bedingt und bestimmt ist" [9].

The term *relativistic mass* $_Rm$ was coined by Gilbert and Tolman [10].

2.2.2. Definition: Einstein's Mass-Energy Equivalence Relation

The Einsteinian *matter/mass-energy equivalence* [7] lies at the core of today physics. In general, due to Einstein's special theory of relativity it is

$$_0m = {}_Rm \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{12}$$

or equally

$$_0E \equiv {}_0m \times c^2 \equiv {}_Rm \times c^2 \times \sqrt[2]{1-\frac{v^2}{c^2}} \equiv {}_RE \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{13}$$

or equally

$$\frac{_0E}{_RE} = \frac{_0m \times c^2}{_Rm \times c^2} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{14}$$

where $_0E$ denotes the "rest" energy; $_0m$ denotes the "rest" mass; $_RE$ denotes the "relativistic" energy; $_Rm$ denotes the "relativistic" mass; v denotes the relative velocity between the two observers and c denotes the speed of light in vacuum.

2.2.3. Definition: Normalized Relativistic Energy-Momentum Relation

The normalized relativistic energy momentum relation [10], a probability theory consistent formulation of Einstein's energy momentum relation, is determined as

$$\frac{{}_{O}m^{2}}{{}_{R}m^{2}}+\frac{v^{2}}{c^{2}}=1 \tag{15}$$

while *the* "*particle-wave-dualism*" [10] is determined as

$$\frac{{}_{O}m^{2}}{{}_{R}m^{2}}+\frac{v^{2}}{c^{2}}\equiv\frac{{}_{O}m^{2}\times c^{2}\times c^{2}}{{}_{R}m^{2}\times c^{2}\times c^{2}}+\frac{v^{2}\times {}_{R}m^{2}\times c^{2}}{c^{2}\times {}_{R}m^{2}\times c^{2}}\equiv\frac{{}_{O}E^{2}}{{}_{R}E^{2}}+\frac{{}_{R}p^{2}\times c^{2}}{{}_{R}E^{2}}\equiv\frac{{}_{O}E^{2}}{{}_{R}E^{2}}+\frac{{}_{W}E^{2}}{{}_{R}E^{2}}\equiv 1 \tag{16}$$

where ${}_{W}E = ({}_{R}p \times c)$ denotes *the energy of an electro-magnetic wave* and ${}_{R}p$ denotes the "relativistic" momentum while c is the speed of the light in vacuum.

2.2.4. Definition: The Relativistic Potential Energy

Following Einstein in his path of thoughts, we define *the relativistic potential energy* ${}_{P}E$ [11] as

$${}_{P}E\equiv\frac{{}_{O}E\times {}_{O}E}{{}_{R}E}\equiv\frac{{}_{O}E}{{}_{R}E}\times {}_{O}E\equiv\sqrt[2]{1-\frac{v^{2}}{c^{2}}}\times {}_{O}E \tag{17}$$

Scholium.

The definition of the relativistic potential energy ${}_{P}E$ is supported by Einstein's publication in 1907. Einstein himself demands that there is something like a relativistic potential energy.

"Jeglicher Energie E kommt also im Gravitationsfelde eine Energie der Lage zu, die ebenso groß ist, wie die *Energie der Lage* einer 'ponderablen' Masse von der Größe E/c^2" [12].

Translated into English:

"Thus, to each energy E in the gravitational field there corresponds an energy of position that equals the potential energy of a 'ponderable' mass of magnitude E/c^2".

The relativistic potential energy ${}_{P}E$ can be viewed as the energy which is determined by an observer P which is at rest relative to the relativistic potential energy. The observer which is at rest relative to the relativistic potential energy will measure its own time, the relativistic potential time ${}_{P}t$.

2.2.5. Definition: The Relativistic Kinetic Energy (the "Vis Viva")

The relativistic kinetic energy ${}_{K}E$ is defined [10] in general as

$${}_{K}E\equiv\frac{{}_{W}E\times {}_{W}E}{{}_{R}E}\equiv\frac{{}_{R}m\times v\times c_{R}\times {}_{R}m\times v\times c}{{}_{R}m\times c^{2}}\equiv {}_{R}p\times v\equiv {}_{R}m\times v^{2} \tag{18}$$

where ${}_{R}m$ denotes the "relativistic mass" and v denotes the relative velocity. In general, it is

$${}_{R}E\equiv {}_{R}H\equiv {}_{P}E+{}_{K}E\equiv {}_{P}H+{}_{K}H \tag{19}$$

where ${}_{P}E$ denotes the relativistic potential energy; ${}_{K}E$ denotes the relativistic kinetic energy; ${}_{P}H$ denotes the Hamiltonian of the relativistic potential energy; ${}_{K}H$ denotes the Hamiltonian of the relativistic kinetic energy. Multiplying this equation by the wave function ${}_{R}\Psi$, we obtain relativity consistent form of Schrödinger's equation as

$${}_{R}E\times {}_{R}\Psi\equiv {}_{R}H\times {}_{R}\Psi\equiv\left({}_{P}E\times {}_{R}\Psi\right)+\left({}_{K}E\times {}_{R}\Psi\right)\equiv\left({}_{P}H\times {}_{R}\Psi\right)+\left({}_{K}H\times {}_{R}\Psi\right) \tag{20}$$

Scholium.

The historical background of *the relativistic kinetic energy* ${}_{K}E$ is back grounded by the long lasting and very famous dispute between Leibniz (1646-1716) and Newton (1642-1726). In fact, the core of this controversy was the dispute about the question, what is preserved through changes. Leibniz himself claimed, that "*vis viva*" [13], [14] or the relativistic kinetic energy ${}_{K}E = {}_{R}m \times v \times v$ was preserved through changes. In contrast to Leibniz, Newton was of the opinion that the momentum ${}_{R}p = {}_{R}m \times v$ was preserved through changes. The observer which is at rest relative to the relativistic kinetic energy will measure its own time, the relativistic kinetic time ${}_{k}t$.

2.2.6. Definition: Einstein's Relativistic Time Dilation Relation

An accurate clock in motion slows down with respect a stationary observer (observer at rest). The proper time ${}_{O}t$ of a clock moving at constant velocity v is related to a stationary observer's coordinate time ${}_{R}t$ by Einstein's relativistic time dilation [15] and defined as

$$ {}_{O}t = {}_{R}t \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{21} $$

where ${}_{O}t$ denotes the "proper" time; ${}_{R}t$ denotes the "relativistic" (*i.e.* stationary or coordinate) time; v denotes the relative velocity and c denotes the speed of light in vacuum.

Scholium.

Coordinate systems can be chosen freely, deepening upon circumstances. In many coordinate systems, an event can be specified by one time coordinate and three spatial coordinates. The time as specified by the time coordinate is denoted as coordinate time. Coordinate time is distinguished from proper time. The concept of proper time, introduced by Hermann Minkowski in 1908 and denoted as ${}_{O}t$, incorporates Einstein's time dilation effect. In principle, Einstein is defining time exclusively for every place where a watch, measuring this time, is located.

"… Definition … der … Zeit … für den Ort, an welchem sich die Uhr … befindet…" [15].

In general, a watch is treated as being at rest relative to the place, where the same watch is located.

"Es werde ferner mittels der *imruhenden System* befindlichen *ruhende*n *Uhr*en die Zeit t [*i.e.* ${}_{R}t$, author] des ruhenden Systems … bestimmt, ebenso werde die Zeit τ [${}_{O}t$, author] des *bewegte*n *System*s, in welchen sich relativ zu letzterem *ruhende Uhr*en befinden, bestimmt…" [15].

Due to Einstein, it is necessary to distinguish between clocks as such which are qualified to mark the time ${}_{R}t$ when at rest relatively to the stationary system R, and the time ${}_{O}t$ when at rest relatively to the moving system O.

"Wir denken uns ferner eine der Uhren, welche *relativ zum ruhenden System ruhend* die Zeit t [${}_{R}t$, author], *relativ zum bewegten System ruhend* die Zeit τ [${}_{O}t$, author] anzugeben befähigt sind…" [15].

In other words, we have to take into account that both clocks *i.e.* observers have at least one point in common, the stationary observer R and the moving observer O are at rest, but at rest relative to what? The stationary observer R is at rest relative to a stationary co-ordinate system R, the moving observer O is at rest relative to a moving co-ordinate system O. Both co-ordinate systems can but must not be at rest relative to each other. The time ${}_{R}t$ of the stationary system R is determined by clocks which are at rest relatively to that stationary system R. Similarly, the time ${}_{O}t$ of the moving system O is determined by clocks which are at rest relatively to that the moving system O. In last consequence, due to Einstein's theory of special relativity, a moving clock (${}_{O}t$) will measure a smaller elapsed time between two events than a non-moving (inertial) clock (${}_{R}t$) between the same two events.

2.2.7. Definition: The Normalized Relativistic Time Dilation Relation

As defined above, due to Einstein's special relativity, it is

$$ {}_{O}t = {}_{R}t \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{22} $$

where ${}_{O}t$ denotes the "proper" time, ${}_{R}t$ denotes the "relativistic" (*i.e.* stationary or coordinate) time, v denotes the relative velocity and c denotes the speed of light in vacuum. Equally, it is

$$ \frac{{}_{O}t}{{}_{R}t} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{23} $$

or

$$ \frac{{}_{O}t}{c^2} \times \frac{c^2}{{}_{R}t} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{24} $$

or

$$ \frac{{}_{O}t^2}{{}_{R}t^2} = 1-\frac{v^2}{c^2} \tag{25} $$

The normalized relativistic time dilation is defined as

$$ \frac{{}_{O}t^2}{{}_{R}t^2} + \frac{v^2}{c^2} = 1 \tag{26} $$

In general, under conditions of the special theory of relativity, we define

$$ {}_{R}S \equiv {}_{R}E + {}_{R}t \tag{27}$$

and

$$ {}_{0}C \equiv {}_{0}E + {}_{0}t \tag{28}$$

and

$$ {}_{0}\underline{C} \equiv {}_{0}\underline{E} + {}_{0}\underline{t} = \Delta E + \Delta t \tag{29}$$

Scholium.

The following 2 × 2 table may illustrate the relationships before (Table 1).

The causal relationship k [16] under conditions of special theory of relativity (*i.e.* the particle-production apparatus) follows as

$$ k\left({}_{0}C, {}_{R}E\right) = \frac{\left(\left({}_{R}S \times {}_{0}E\right) - \left({}_{0}C \times {}_{R}E\right)\right)}{\left(\sqrt[2]{{}_{0}C \times {}_{0}\underline{C} \times {}_{R}E \times {}_{R}t}\right)} \tag{30}$$

Under conditions [17] where

$$ {}_{R}E \times {}_{R}t = {}_{R}H \times {}_{R}\Psi \tag{31}$$

there is a relationship between the causal relationship k the Schrödinger equation in the form

$$ {}_{R}H \times {}_{R}\Psi = \frac{\left(\left({}_{R}S \times {}_{0}E\right) - \left({}_{0}C \times {}_{R}E\right)\right)^{2}}{\left({}_{0}C \times {}_{0}\underline{C} \times k\left({}_{0}C, {}_{R}E\right) \times k\left({}_{0}C, {}_{R}E\right)\right)} \tag{32}$$

2.3. Einstein's General Theory of Relativity

2.3.1. Definition: The General Kronecker Delta

The general Kronecker delta δ_{mn}, named after Leopold Kronecker, is +1 if the variables m and n are equal, and +0 otherwise.

Scholium.

For convenience, the restriction to positive integers is common, but not necessary. The general Kronecker delta, running from 1 to 4, denoted as δ_{mn} can be displayed in matrix form as

$$ \delta_{mn} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

The anti general Kronecker delta denoted as $\underline{\delta}_{mn}$ is defined as $\underline{\delta}_{mn} = 1_{mn} - \delta_{mn}$.

2.3.2. Definition: The Special Kroneker Delta

The special Kronecker delta $\delta(i, j)_{mn}$, named after Leopold Kronecker, is +1 if and only if $m = i$ and if $n = j$ and +0 otherwise.

Table 1. The unified field under conditions of the special theory of relativity.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	${}_{0}E$	${}_{0}\underline{E} = \Delta E$	${}_{R}E$
	no	${}_{0}t$	${}_{0}\underline{t} = \Delta t$	${}_{R}t$
		${}_{0}C$	${}_{0}\underline{C}$	${}_{R}S$
		The special theory of relativity.		

Scholium.

Example. The special Kronecker delta $\delta(i=1, j=1)_{mn}$ for $m=i=1$ and $n=j=1$, running from 1 to 4, can be displayed in matrix form as

$$\delta(i=1, j=1)_{mn} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

The *anti special* Kronecker delta denoted as $\underline{\delta}(i, j)_{mn}$ and defined as $\underline{\delta}(i, j)_{mn} = 1_{mn} - \delta(i, j)_{mn}$ for $m=i=1$ and $n=j=1$, running from 1 to 4, can be displayed as

$$\underline{\delta}(i=1, j=1)_{mn} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{35}$$

The special Kronecker delta is not grounded on the equality that $m = n$ but on the fact, the m equal to a certain value i and that n is equal to another certain value j. In other words, it is $m = i$ and $n = j$.

2.3.3. Definition: The Metric Tensor $g_{\mu\nu}$

In the following, let us define the following. Let

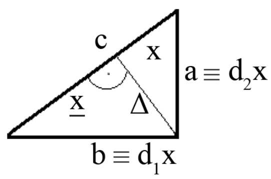
$$a^2 \equiv d_2 x \times d_2 x + \cdots + d_n x \times d_n x \tag{36}$$

and

$$b^2 \equiv d_1 x \times d_1 x \tag{37}$$

In Euclidean coordinates for an n-dimensional space the formula for the length ds^2 of an infinitesimal line segment due to the Pythagorean theorem follows as

$$c^2 \equiv ds^2 \equiv (d_1 x \times d_1 x) + (d_2 x \times d_2 x + \cdots + d_n x \times d_n x) \tag{38}$$



or

$$c^2 \equiv ds^2 \equiv \sum_{i=1}^{n} (d_i x)^2 \tag{39}$$

In general, a coordinate system can be changed from the Euclidean X's to some coordinate system of Y's then

$$d_m x \equiv \frac{\partial_m x}{\partial_r y} \times d_r y \tag{40}$$

and

$$d_n x \equiv \frac{\partial_n x}{\partial_s y} \times d_s y \tag{41}$$

The Pythagorean theorem is defined as

$$c^2 \equiv ds^2 \equiv \sum_m \sum_n d_m x \times d_n x \times \delta_{mn} \equiv \sum_m \sum_n \frac{\partial_m x}{\partial_r y} \times d_r y \times \frac{\partial_n x}{\partial_s y} \times d_s y \times \delta_{mn} \tag{42}$$

While using Einstein's summation convention, a (*i.e. position dependent*) metric tensor $g(x)_{\mu\nu}$ is defined as

$$g(x)_{\mu\nu} \equiv \delta_{mn} \times \frac{\partial_m x}{\partial_r y} \times \frac{\partial_n x}{\partial_s y} \tag{43}$$

and *a curved space compatible formulation of the Pythagorean theorem* follows as

$$c^2 \equiv ds^2 \equiv \delta_{mn} \times \frac{\partial_m x}{\partial_r y} \times \frac{\partial_n x}{\partial_s y} \times d_r y \times d_s y \equiv g(x)_{\mu\nu} \times d_r y \times d_s y \tag{44}$$

Scholium.

The metric tensor generalizes *the Pythagorean theorem of flat space in a manifold with curvature*. The metric tensor can be decomposed in many different ways. Let $g_{\mu\nu} = n_{\mu\nu} + \underline{n}_{\mu\nu}$ where $g_{\mu\nu}$ is the metric tensor of general relativity; $n_{\mu\nu}$ is the tensor of special relativity and $\underline{n}_{\mu\nu}$ is the anti tensor of general relativity. In general theory of relativity, the scalar Newtonian gravitational potential is replaced by the metric tensor. "In particular, in general realtivity, the gravitational potential is replaced by the metric tensor g_{ab}" [18]. In last consequence, the gravitational potential is something like a feature of the metric tensor. Following Renn *et al*., the metric tensor is "… the mathematical representation of the gravitation alpotential…" [19]. On this account it is necessary to make a distinction between a gravitational potential and a gravitational field. Due to Einstein, "… the introduction of independent gravitational fields is considered justified even though no masses generating the field are defined" [2]. The question is, can a gravitational potential exist even though no masses generating the gravitational potential are defined?

2.3.4. Definition: The Normalized Metric Tensor $n(X)_{\mu\nu}$

In the following, we define the normalized metric tensor $n_{\mu\nu}$, while using Einstein's summation convention, as

$$n(x)_{\mu\nu} \equiv \delta_{mn} \times \frac{\partial_m s}{\partial_r s} \times \frac{\partial_n x}{\partial_s s} \tag{45}$$

The line element follows in general as

$$c^2 \equiv ds^2 \equiv \delta_{mn} \times \frac{\partial_m s}{\partial_r s} \times \frac{\partial_n x}{\partial_s s} \times d_r s \times d_s s \equiv n(x)_{\mu\nu} \times d_r s \times d_s s \tag{46}$$

Scholium.

The normalized metric tensor is not based on the gradient. The metric tensor passes over into the normalized metric tensor and vice versa. We obtain

$$g(x)_{\mu\nu} \times d_r y \times d_s y \equiv n(x)_{\mu\nu} \times d_r s \times d_s s \tag{47}$$

or

$$n(x)_{\mu\nu} \equiv \frac{d_r y \times d_s y}{d_r s \times d_s s} \times g(x)_{\mu\nu} \tag{48}$$

2.3.5. Definition: Einstein's Field Equations

Einstein field equations (EFE), originally [20] published [21] without the extra "cosmological" term $\Lambda \times g_{\mu\nu}$ [22] may be written in the form

$$G_{\mu\nu} + \Lambda \times g_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu} + \Lambda \times g_{\mu\nu} = R_{\mu\nu} - \left(\frac{R}{2} \times g_{\mu\nu} - \Lambda \times g_{\mu\nu} \right) = \frac{4 \times 2 \times \pi \times \gamma}{c \times c \times c \times c} \times T_{\mu\nu} \tag{49}$$

where $G_{\mu\nu}$ is the Einsteinian tensor; $T_{\mu\nu}$ is the stress-energy tensor of matter (still a field devoid of any geometrical significance); $R_{\mu\nu}$ denotes the Ricci tensor (the curvature of space); R denotes the Ricci scalar (the trace of the Ricci tensor); Λ denotes the cosmological "constant" and $g_{\mu\nu}$ denotes the metric tensor (a 4 × 4 matrix) and where π is Archimedes' constant (π = 3.14159265358979323846264338327950288419716939937510582 09…) and γ is Newton's gravitational "constant" and the speed of light in vacuum is c = 299,792,458 [m/s] in S. I. units.

Scholium.

The stress-energy tensor $T_{\mu\nu}$, still a tensor devoid of any geometrical significance, contains all forms of energy and momentum which includes all matter present but of course any electromagnetic radiation too. Originally, Einstein's universe was spatially closed and finite. In 1917, Albert Einstein modified his own field equations and inserted the cosmological constant Λ (denoted by the Greek capital letter lambda) into his theory of general relativity in order to force his field equations to predict a stationary universe.

"Ich komme nämlich zu der Meinung, daß die von mir bisher vertretenen Feldgleichungen der Gravitation noch einer kleinen Modifikation bedürfen…" [22].

By the time, it became clear that the universe was expanding instead of being static and Einstein abandoned the cosmological constant Λ. "Historically the term containing the 'cosmological constant' λ was introduced into the field equations in order to enable us to account theoretically for the existence of a finite mean density in a static universe. It now appears that in the dynamical case this end can be reached without the introduction of λ" [23]. But lately, Einstein's cosmological constant is revived by scientists to explain a mysterious force counteracting gravity called dark energy. In this context it is important to note that Newton's gravitational "constant" big G is not [24] [25] a constant.

2.3.6. Definition: General Tensors

Independently of the tensors of the theory of general relativity, we introduce by definition the following covariant second rank tensors of yet unknown structure whose properties we leave undetermined as well. We define the following covariant second rank tensors of yet unknown structure as

$$A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu}, {}_RU_{\mu\nu}, {}_R\underline{U}_{\mu\nu}, {}_0W_{\mu\nu}, {}_0\underline{W}_{\mu\nu}, {}_RW_{\mu\nu} \tag{50}$$

Tensor can be decomposed (sometimes in many different ways). In the following of this publication we define the following relationships. It is

$$A_{\mu\nu} + B_{\mu\nu} \equiv {}_RU_{\mu\nu} \tag{51}$$

$$C_{\mu\nu} + D_{\mu\nu} \equiv {}_R\underline{U}_{\mu\nu} \tag{52}$$

$$A_{\mu\nu} + C_{\mu\nu} \equiv {}_0W_{\mu\nu} \tag{53}$$

$$B_{\mu\nu} + D_{\mu\nu} \equiv {}_0\underline{W}_{\mu\nu} \tag{54}$$

$$A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu} \equiv {}_RU_{\mu\nu} + {}_R\underline{U}_{\mu\nu} \equiv {}_0W_{\mu\nu} + {}_0\underline{W}_{\mu\nu} \equiv {}_RW_{\mu\nu} \tag{55}$$

Scholium.

The following 2 × 2 table may illustrate the relationships above (**Table 2**).

These tensors above may have different meanings depending upon circumstances. The unified field ${}_RW_{\mu\nu}$ can be decomposed into several (sub-) fields $A_{\mu\nu}$, $B_{\mu\nu}$, $C_{\mu\nu}$, $D_{\mu\nu}$. In order to achieve unification between general relativity theory and quantum (field) theory the (sub-) fields $A_{\mu\nu}$, $B_{\mu\nu}$, $C_{\mu\nu}$, $D_{\mu\nu}$ can denote *the four basic fields of nature*. The idea of quantum field theory is to describe a particle as a manifestation of an abstract field. In this context the particle a_i can be associated with the field $A_{\mu\nu}$, the particle b_i can be associated with the field $B_{\mu\nu}$, the particle c_i can be associated with the field $C_{\mu\nu}$, the particle d_i can be associated with the field $D_{\mu\nu}$. Thus far, we can define something like $A_{\mu\nu} = a_i \times {}_FA_{\mu\nu}$ and $B_{\mu\nu} = b_i \times {}_FB_{\mu\nu}$ and $C_{\mu\nu} = c_i \times {}_FC_{\mu\nu}$ and $D_{\mu\nu} = d_i \times {}_FD_{\mu\nu}$ where the subscript ${}_F$ can denote an individual particle field. Under conditions of general relativity, *Einstein field equation* can be rewritten (using the tensors above) as

Table 2. The unified field ${}_RW_{\mu\nu}$.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	$A_{\mu\nu}$	$B_{\mu\nu}$	${}_RU_{\mu\nu}$
	no	$C_{\mu\nu}$	$D_{\mu\nu}$	${}_R\underline{U}_{\mu\nu}$
		${}_0W_{\mu\nu}$	${}_0\underline{W}_{\mu\nu}$	${}_RW_{\mu\nu}$

The unified field.

$$ {}_{0}W_{\mu\nu} + \Lambda \times g_{\mu\nu} = {}_{R}U_{\mu\nu} \tag{56} $$

where ${}_{0}W_{\mu\nu} \equiv G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu}$ and ${}_{R}U_{\mu\nu} \equiv \frac{2 \times 4 \times \pi \times \gamma}{c \times c \times c \times c} \times T_{\mu\nu}$. From an epistemological point of view ${}_{R}U_{\mu\nu}$ is the tensor of the cause (in German: Ursache *U*) while ${}_{0}W_{\mu\nu}$ is the tensor of the effect (in German: Wirkung *W*). As we will see, from the definition ${}_{R}U_{\mu\nu} + {}_{R}\underline{U}_{\mu\nu} \equiv {}_{0}W_{\mu\nu} + {}_{0}\underline{W}_{\mu\nu} \equiv {}_{R}W_{\mu\nu}$ follows that $\Lambda \times g_{\mu\nu} = {}_{R}U_{\mu\nu} - {}_{0}W_{\mu\nu} \equiv {}_{0}\underline{W}_{\mu\nu} - {}_{R}\underline{U}_{\mu\nu} \equiv {}_{R}W_{\mu\nu} - {}_{0}W_{\mu\nu} - {}_{R}\underline{U}_{\mu\nu}$ even if Einstein's cosmological constant Λ cannot [26] be treated as a constant.

2.4. Unified Field Theory

2.4.1. Definition: The Tensor of Planck's Constant *h*

Planck defined in 1901 the constant of proportionality [27] as h. As long as Planck's constant *h* is a constant, a tensor form of this constant is not needed. We define the co-variant second rank tensor of Planck's constant ${}_{R}h_{\mu\nu}$ as

$$ {}_{R}h_{\mu\nu} = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{57} $$

2.4.2. Definition: The Tensor of Dirac's Constant

We define the co-variant second rank tensor of Dirac's constant as

$$ {}_{R}\hbar_{\mu\nu} = \begin{bmatrix} \hbar_{00} & \hbar_{01} & \hbar_{02} & \hbar_{03} \\ \hbar_{10} & \hbar_{11} & \hbar_{12} & \hbar_{13} \\ \hbar_{20} & \hbar_{21} & \hbar_{22} & \hbar_{23} \\ \hbar_{30} & \hbar_{31} & \hbar_{32} & \hbar_{33} \end{bmatrix} \tag{58} $$

Scholium.

In general it is known that

$$ {}_{R}h_{\mu\nu} \equiv 2_{\mu\nu} \cap {}_{R}\pi_{\mu\nu} \cap {}_{R}\hbar_{\mu\nu} \tag{59} $$

2.4.3. Definition: The Tensor of Speed of the Light ${}_{R}c_{\mu\nu}$

We define the co-variant second rank tensor of the speed of the light ${}_{R}c_{\mu\nu}$, denoted by small letter *c*, as

$$ {}_{R}c_{\mu\nu} = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} \\ c_{10} & c_{11} & c_{12} & c_{13} \\ c_{20} & c_{21} & c_{22} & c_{23} \\ c_{30} & c_{31} & c_{32} & c_{33} \end{bmatrix} = {}_{R}f_{\mu\nu} \cap {}_{R}\lambda_{\mu\nu} \tag{60} $$

where ${}_{R}f_{\mu\nu}$ denotes the stress energy tensor of frequency and ${}_{R}\lambda_{\mu\nu}$ denotes the wave-length tensor.

Scholium.

Following Einstein's own position, the constancy of the speed of the light *c* is something relative and nothing absolute. Theoretically, circumstances are possible where the speed of the light is not constant. Einstein himself linked the constancy of the speed of the light *c* to *a* constant gravitational potential.

"Dagegen bin ich der Ansicht, daß das Prinzip der *Konstanz der Lichtgeschwindigkeit* sich *nur insoweit* aufrecht erhalten läßt, als man sich auf raum-zeitliche Gebiete von *konstantem Gravitationspotential* beschränkt. Hier liegt nach meiner Meinung die Grenze der Gültigkeit … des Prinzips der Konstanz der Lichtgeschwindigkeit und damit unserer heutigen Relativitätstheorie" [8].

Thus far a tensor of the speed of the light is of use to face these theoretical possibilities.

2.4.4. Definition: The Tensor of Newton's Gravitational "Constant" ${}_{R}\gamma_{\mu\nu}$

We define the co-variant second rank tensor of Newton's gravitational constant ${}_{R}\gamma_{\mu\nu}$ as

$$
{}_R\gamma_{\mu\nu} = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{03} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{30} & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \tag{61}
$$

Scholium.

Newton's gravitational constant is not for sure a constant. Therefore, we prefer to use the same in the form of a tensor.

2.4.5. Definition: The Tensor of Archimedes "Constant" ${}_R\pi_{\mu\nu}$

We define the co-variant second rank tensor of Archimedes constant ${}_R\pi_{\mu\nu}$ as

$$
{}_R\pi_{\mu\nu} = \begin{bmatrix} \pi_{00} & \pi_{01} & \pi_{02} & \pi_{03} \\ \pi_{10} & \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{20} & \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{30} & \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} \tag{62}
$$

Scholium.

Archimedes of Syracuse (ca. 287 BC - ca. 212 BC) himself was able to find π, the circumference of a circle with diameter 1 commonly approximated as 3.14159, to 99.9% accuracy about 2000 years ago. Archimedes constant π is an irrational number, π never settles into a permanent repeating pattern, the decimal representation of Archimedes constant π never ends.

2.4.6. Definition: The Tensor of Imaginary Number $i_{\mu\nu}$

We define the co-variant second rank tensor of the imaginary number $i_{\mu\nu}$ as

$$
i_{\mu\nu} = \begin{bmatrix} i_{00} & i_{01} & i_{02} & i_{03} \\ i_{10} & i_{11} & i_{12} & i_{13} \\ i_{20} & i_{21} & i_{22} & i_{23} \\ i_{30} & i_{31} & i_{32} & i_{33} \end{bmatrix} \tag{63}
$$

2.4.7. Definition: The Tensor of Space

We define the second rank *tensor of space* of yet unknown structure as

$$
{}_RS_{\mu\nu} \equiv {}_RW_{\mu\nu} \equiv U_{\mu\nu} \cap \left({}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \right) \tag{64}
$$

Under conditions of general relativity, we define

$$
{}_RS_{\mu\nu} \equiv {}_RW_{\mu\nu} \equiv R_{\mu\nu} \tag{65}
$$

where $R_{\mu\nu}$ denotes the Ricci tensor, the tensor of the curvature of space. *Under conditions different form general relativity*, ${}_RS_{\mu\nu}$ can be determined in a different way. It is important to note that ${}_RU_{\mu\nu}$ *is not identical with* $U_{\mu\nu}$.

2.4.8. Definition: The Tensor of Energy

Similar to general theory of relativity, it is at present appropriate to introduce a corresponding energy tensor, a tensor which represents the amounts of energy, momentum, pressure, stress et cetera in the space, a tensor which describes the energy/matter/momentum et cetera distribution (at each event) in space. The energy tensor expressed mathematically by a symmetrical tensor of the second rank of yet unknown structure is defined as

$$
{}_RE_{\mu\nu} \equiv {}_RH_{\mu\nu} \equiv {}_RU_{\mu\nu} \tag{66}
$$

Ipso facto, the same tensor is determined by all matter present but of course any electromagnetic radiation too. *Under conditions of general relativity*, we define

$$
{}_RE_{\mu\nu} \equiv {}_RH_{\mu\nu} \equiv {}_RU_{\mu\nu} \equiv \frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap_R \pi_{\mu\nu} \cap_R \gamma_{\mu\nu}}{{}_Rc_{\mu\nu} \cap_R c_{\mu\nu} \cap_R c_{\mu\nu} \cap_R c_{\mu\nu}} \cap T_{\mu\nu} \tag{67}
$$

To assure compatibility with quantum theory, we define

$$i \cap_R \hbar \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \equiv i_{\mu\nu} \cap_R \hbar_{\mu\nu} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \equiv \frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap_R \pi_{\mu\nu} \cap_R \gamma_{\mu\nu}}{{}_Rc_{\mu\nu} \cap_R c_{\mu\nu} \cap_R c_{\mu\nu} \cap_R c_{\mu\nu}} \cap T_{\mu\nu} \equiv {}_RE_{\mu\nu} \equiv {}_RH_{\mu\nu} \equiv {}_RU_{\mu\nu} \tag{68}$$

Due to the definition before we obtain

$$\left(\frac{\partial}{\partial t}\right)_{\mu\nu} \equiv \frac{1}{i \cap {}_R\hbar} \times \frac{4 \times 2 \times {}_R\pi \times {}_R\gamma}{{}_Rc \times {}_Rc \times {}_Rc \times {}_Rc} \times T_{\mu\nu} \equiv \frac{1_{\mu\nu}}{i_{\mu\nu} \cap {}_R\hbar_{\mu\nu}} \times \frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap {}_R\pi_{\mu\nu} \cap {}_R\gamma_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \cap T_{\mu\nu} \tag{69}$$

The tensor of probability of energy follows as

$$p\left({}_RE_{\mu\nu}\right) \equiv p\left({}_RH_{\mu\nu}\right) \equiv p\left({}_RU_{\mu\nu}\right) \equiv \frac{\left(\dfrac{4_{\mu\nu} \cap 2_{\mu\nu} \cap {}_R\pi_{\mu\nu} \cap {}_R\gamma_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \cap T_{\mu\nu}\right)}{R_{\mu\nu}} \tag{70}$$

General relativity's geometry of space and time is one but not the only one geometry of space and time. Especially general relativity's stress-energy tensor as the source-term of Einstein's field equations is still a field devoid of any geometrical significance. A geometrical tensorial representation of the stress energy tensor of energy is possible as

$$\left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap {}_R\pi_{\mu\nu} \cap {}_R\gamma_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \cap T_{\mu\nu}\right) \equiv p\left({}_RE_{\mu\nu}\right) \cap R_{\mu\nu} \equiv p\left({}_RH_{\mu\nu}\right) \cap R_{\mu\nu} \equiv p\left({}_RU_{\mu\nu}\right) \cap R_{\mu\nu} \tag{71}$$

2.4.9. Definition: The Tensor of Frequency

In general, we define the covariant second rank tensor of frequency ${}_Rf_{\mu\nu}$ as

$${}_Rf_{\mu\nu} \equiv \frac{4 \times 2 \times \pi \times \gamma}{h \times c \times c \times c \times c} \times T_{\mu\nu} \equiv \frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap {}_R\pi_{\mu\nu} \cap {}_R\gamma_{\mu\nu}}{{}_Rh_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \cap T_{\mu\nu} \tag{72}$$

To assure compatibility with quantum theory, we define the inverse tensor ${}_R\tau_{\mu\nu}$ of the covariant second rank tensor of frequency ${}_Rf_{\mu\nu}$ as

$${}_R\tau_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_Rf_{\mu\nu}} \equiv \frac{h \times c \times c \times c \times c}{4 \times 2 \times \pi \times \gamma} \times \frac{1}{T_{\mu\nu}} \equiv \frac{h_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}}{4_{\mu\nu} \cap 2_{\mu\nu} \cap {}_R\pi_{\mu\nu} \cap {}_R\gamma_{\mu\nu} \cap T_{\mu\nu}} \tag{73}$$

Per definition it follows that

$${}_R\tau_{\mu\nu} \cap {}_Rf_{\mu\nu} \equiv 1_{\mu\nu} \tag{74}$$

2.4.10. Definition: The Tensor ${}_0\omega_{\mu\nu}$

In general, we define the covariant second rank tensor ${}_0\omega_{\mu\nu}$ as

$${}_0\omega_{\mu\nu} \equiv 2_{\mu\nu} \cap_R \pi_{\mu\nu} \cap {}_Rf_{\mu\nu} \tag{75}$$

Scholium.

The tensor of frequency ${}_Rf_{\mu\nu}$ and the ${}_0\omega_{\mu\nu}$ tensor are related. Under circumstances of general relativity, there are conditions where

$${}_0\omega_{\mu\nu} \equiv 2_{\mu\nu} \cap_R \pi_{\mu\nu} \cap {}_Rf_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_R\hbar_{\mu\nu}} \cap \left(G_{\mu\nu} - \Lambda \times g_{\mu\nu}\right) \tag{76}$$

2.4.11. Definition: The Tensor of Matter ${}_RM_{\mu\nu}$

The matter tensor expressed mathematically by a symmetrical tensor of the second rank of yet unknown structure is defined as

$${}_RM_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \cap {}_RE_{\mu\nu} \tag{77}$$

Under conditions of general relativity, we define

$$
{}_{R}M_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap {}_{R}E_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap {}_{R}H_{\mu\nu}
\equiv \frac{1_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap \frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap {}_{R}\pi_{\mu\nu} \cap {}_{R}\gamma_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap T_{\mu\nu} \tag{78}
$$

Scholium.

This definition is based on *the equivalence of mass/matter and energy* due to Einstein's special theory of relativity.

"Da *Masse und Energie* nach den Ergebnissen der speziellen Relativitätstheorie *das Gleiche* sind und die Energie formal durch den symmetrischen Energietensor ($T_{\mu\nu}$) beschrieben wird, so besagt dies, daß das G-Geld [*gravitational field*, *author*] durch den Energietensor der Materie bedingt und bestimmt ist" [9].

2.4.12. Definition: The Tensor of Ordinary Energy ${}_{0}E_{\mu\nu}$

We define the second rank *tensor of ordinary energy* ${}_{0}E_{\mu\nu}$ of yet unknown structure as

$$
{}_{O}E_{\mu\nu} \equiv A_{\mu\nu} \tag{79}
$$

Scholium.

Under some well defined circumstances, ${}_{0}E_{\mu\nu}$ can denote *the unity of strong interaction and weak interaction*. Under conditions of general relativity, it is

$$
{}_{O}E_{\mu\nu} \equiv {}_{R}E_{\mu\nu} - {}_{0}\underline{E}_{\mu\nu} \equiv \frac{4 \times 2 \times \pi \times \gamma}{c \times c \times c \times c} \times T_{\mu\nu} - \left(\left(\frac{1_{\mu\nu}}{4_{\mu\nu} \times \pi_{\mu\nu}} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F_{d\nu} \right) \right) \right) \tag{80}
$$

The associated probability tensor can be achieved as

$$
p\left({}_{O}E_{\mu\nu}\right) \equiv p\left(A_{\mu\nu}\right) \equiv \frac{\left({}_{R}E_{\mu\nu} - {}_{0}\underline{E}_{\mu\nu}\right)}{R_{\mu\nu}}
\equiv \frac{\frac{4 \times 2 \times \pi \times \gamma}{c \times c \times c \times c} \times T_{\mu\nu} - \left(\left(\frac{1_{\mu\nu}}{4 \times \pi_{\mu\nu}} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right) \right)}{R_{\mu\nu}} \tag{81}
$$

2.4.13. Definition: The Tensor of "Ordinary" Matter ${}_{0}M_{\mu\nu}$

The *tensor of ordinary mater* expressed mathematically as a covariant second rank of yet unknown structure is defined as

$$
{}_{0}M_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap {}_{0}E_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap {}_{0}H_{\mu\nu} \tag{82}
$$

2.4.14. Definition: The Anti Tensor of "Ordinary" Matter ${}_{0}\underline{E}_{\mu\nu}$

We define the second rank *anti tensor* ${}_{0}\underline{E}_{\mu\nu}$ of the tensor ${}_{0}E_{\mu\nu}$ as

$$
{}_{0}\underline{E}_{\mu\nu} \equiv {}_{0}\underline{H}_{\mu\nu} \equiv B_{\mu\nu} \tag{83}
$$

Under conditions of general relativity, where ${}_{0}E_{\mu\nu}$ is tensor of ordinary energy/matter, the electromagnetic field is an anti tensor of ordinary energy/matter. Under conditions of general relativity, the tensor of the electromagnetic field is determined by an anti-symmetric second-order tensor known as the electromagnetic field (Faraday) tensor F. In general, under conditions of general relativity, the second rank covariant tensor of the electromagnetic field in the absence of "ordinary" matter, which is different from the electromagnetic field tensor F, is defined by

$$
{}_{0}\underline{E}_{\mu\nu} \equiv {}_{0}\underline{H}_{\mu\nu} \equiv B_{\mu\nu} \equiv \left(\left(\frac{1_{\mu\nu}}{4 \times \pi_{\mu\nu}} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right) \right) \tag{84}
$$

where F is the electromagnetic field tensor and $g_{\mu\nu}$ is the metric tensor.

Scholium.

The associated probability tensor is determined as

$$p\left({}_0\underline{E}_{\mu\nu}\right) \equiv p\left({}_0\underline{H}_{\mu\nu}\right) \equiv p\left(B_{\mu\nu}\right) \equiv \frac{\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)}{R_{\mu\mu}} \tag{85}$$

The geometric formulation of the stress-energy tensor of the electromagnetic field follows as

$$\begin{aligned}&\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)\\ &\equiv p\left({}_0\underline{E}_{\mu\nu}\right)\cap R_{\mu\mu} \equiv p\left({}_0\underline{H}_{\mu\nu}\right)\cap R_{\mu\mu} \equiv p\left(B_{\mu\nu}\right)\cap R_{\mu\mu}\end{aligned} \tag{86}$$

2.4.15. Definition: The Tensor ${}_0\underline{M}_{\mu\nu}$

The *tensor* ${}_0\underline{M}_{\mu\nu}$ is defined as

$$\begin{aligned}{}_0\underline{M}_{\mu\nu} &\equiv \frac{{}_0\underline{E}_{\mu\nu}}{{}_Rc_{\mu\nu}\cap{}_Rc_{\mu\nu}} \equiv \frac{{}_0\underline{H}_{\mu\nu}}{{}_Rc_{\mu\nu}\cap{}_Rc_{\mu\nu}} \equiv \frac{B_{\mu\nu}}{{}_Rc_{\mu\nu}\cap{}_Rc_{\mu\nu}}\\ &\equiv\left(\frac{1_{\mu\nu}}{{}_Rc_{\mu\nu}\cap{}_Rc_{\mu\nu}}\right)\cap\left(\left(\frac{1_{\mu\nu}}{4_{\mu\nu}\times{}_R\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)\end{aligned} \tag{87}$$

2.4.16. Definition: The Decomposition of the Tensor of Energy

A portion of the tensor of energy is due to the tensor of the *electromagnetic field*, another portion of the tensor of energy is due to the tensor of *ordinary energy*. Before going on to discuss this topic in more detail, we define in general

$${}_RE_{\mu\nu} \equiv {}_OE_{\mu\nu} + {}_0\underline{E}_{\mu\nu} \equiv {}_OH_{\mu\nu} + {}_0\underline{H}_{\mu\nu} \equiv A_{\mu\nu} + B_{\mu\nu} \tag{88}$$

Under conditions of general relativity, we define

$${}_RE_{\mu\nu} \equiv {}_OE_{\mu\nu} + {}_0\underline{E}_{\mu\nu} \equiv {}_OH_{\mu\nu} + {}_0\underline{H}_{\mu\nu} \equiv \frac{4\times2\times\pi\times\gamma}{c\times c\times c\times c}\times T_{\mu\nu} \tag{89}$$

Scholium.

The stress-energy tensor of the electromagnetic field is equivalent to the portion of the stress-energy tensor of energy due to the electromagnetic field. In this approach, we are following *Vranceanu* in his position, that the energy tensor T_{kl} can be treated as the sum of two tensors one of which is due to the electromagnetic field.

"On peut aussi supposer que le tenseur d'énergie T_{kl} soit la somme de deux tenseurs dont un dû au champ électromagnétique…" [28].

In English:

"One can also assume that the energy tensor T_{kl} be the sum of two tensors one of which is due to the electromagnetic field".

Einstein himself demanded something similar.

"Wir unterscheiden im folgenden zwischen 'Gravitationsfeld' und 'Materie' in dem Sinne, daß alles außer dem Gravitationsfeld als 'Materie' bezeichnet wird, also nicht nur *die* '*Materie*' *im üblichen Sinne*, sondern auch *das elektromagnetische Feld*" [21].

2.4.17. Definition: The Tensor of Time ${}_Rt_{\mu\nu}$

We define the second rank *tensor of time* of yet unknown structure as

$${}_Rt_{\mu\nu} \equiv {}_R\underline{E}_{\mu\nu} \equiv {}_R\underline{U}_{\mu\nu} \equiv {}_RS_{\mu\nu} - {}_RE_{\mu\nu} \tag{90}$$

Scholium.

All but energy is time, there is no third between energy and time. Under conditions of general theory of relativity, the associated probability tensor follows as

$$p\left({}_{R}t_{\mu\nu}\right) \equiv p\left({}_{R}\underline{E}_{\mu\nu}\right) \equiv p\left({}_{R}\underline{U}_{\mu\nu}\right) \equiv \frac{R_{\mu\nu} - {}_{R}E_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_{R}t_{\mu\nu}}{R_{\mu\nu}} \tag{91}$$

2.4.18. Definition: The Tensor ${}_{R}g_{\mu\nu}$

We define the second rank tensor ${}_{R}g_{\mu\nu}$ as

$${}_{R}g_{\mu\nu} \equiv \frac{{}_{R}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \equiv \frac{{}_{R}\underline{E}_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \equiv \frac{{}_{R}\underline{U}_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \equiv \frac{{}_{R}S_{\mu\nu} - {}_{R}E_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{92}$$

Scholium.
The tensor

$${}_{R}g_{\mu\nu} \tag{93}$$

is not identical with the metric tensor of general relativity, defined as

$$g_{\mu\nu} \tag{94}$$

Still, circumstances may exist, where both tensors can be treated as being identical.

2.4.19. Definition: The Tensor ${}_{0}t_{\mu\nu}$

We define the second rank *tensor* ${}_{0}t_{\mu\nu}$ as

$${}_{0}t_{\mu\nu} \equiv C_{\mu\nu} \equiv {}_{R}t_{\mu\nu} - {}_{W}t_{\mu\nu} \equiv {}_{0}C_{\mu\nu} - A_{\mu\nu} \tag{95}$$

Scholium.
Under conditions of general theory of relativity, the associated probability tensor follows as

$$\begin{aligned} p\left({}_{0}t_{\mu\nu}\right) \equiv p\left(C_{\mu\nu}\right) &\equiv \frac{{}_{R}t_{\mu\nu} - {}_{W}t_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_{0}C_{\mu\nu} - A_{\mu\nu}}{R_{\mu\nu}} \\ &\equiv \frac{+\left(\frac{1}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{dv}\times F^{dv}\right)\right)-\Lambda\times g_{\mu\nu}}{R_{\mu\nu}} \end{aligned} \tag{96}$$

2.4.20. Definition: The Tensor ${}_{0}g_{\mu\nu}$

We define the second rank *tensor* ${}_{0}g_{\mu\nu}$ as

$${}_{0}g_{\mu\nu} \equiv \frac{{}_{0}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \equiv \frac{{}_{R}t_{\mu\nu} - {}_{W}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{97}$$

2.4.21. Definition: The Tensor ${}_{W}t_{\mu\nu}$

We define the second rank *tensor* ${}_{W}t_{\mu\nu}$ as

$${}_{W}t_{\mu\nu} \equiv D_{\mu\nu} \equiv {}_{R}t_{\mu\nu} - {}_{0}t_{\mu\nu} \tag{98}$$

Scholium.
Under conditions of general theory of relativity, the associated probability tensor follows as

$$p\left({}_{W}t_{\mu\nu}\right) \equiv p\left(D_{\mu\nu}\right) \equiv \frac{{}_{R}t_{\mu\nu} - {}_{0}t_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{\frac{R}{2}\times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4_{\mu\nu}\cap\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{dv}\times F^{dv}\right)\right)}{R_{\mu\nu}} \tag{99}$$

2.4.22. Definition: The Tensor ${}_{W}g_{\mu\nu}$

We define the second rank tensor ${}_{W}g_{\mu\nu}$ as

$${}_{W}g_{\mu\nu} \equiv \frac{{}_{W}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \equiv \frac{{}_{R}t_{\mu\nu} - {}_{0}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{100}$$

2.4.23. Definition: The Wave Function Tensor ${}_R\Psi_{\mu\nu}$

We define the covariant second rank wave function tensor as

$$ {}_R\Psi_{\mu\nu} \tag{101} $$

Under conditions of general relativity, we define

$$ {}_R\Psi_{\mu\nu} \equiv \left(\frac{R}{2} \cap g_{\mu\nu}\right) - \left(\Lambda \cap g_{\mu\nu}\right) \equiv \left(\frac{R}{2} - \Lambda\right) \cap g_{\mu\nu} \equiv \Psi \cap \frac{1}{\Psi} \cap \left(\frac{R}{2} - \Lambda\right) \cap g_{\mu\nu} \equiv \Psi \cap {}_\Psi g_{\mu\nu} \tag{102} $$

2.4.24. Definition: The Complex Conjugate Wave Function Tensor ${}_R^*\Psi_{\mu\nu}$

We define the covariant second rank complex conjugate wave function tensor of yet unknown structure as

$$ {}_R^*\Psi_{\mu\nu} \tag{103} $$

2.4.25. Definition: The Decomposition of the Tensor of Space

A portion of the tensor of space is due to the tensor of time, another portion of the tensor of space is determined by the tensor of energy. In general, we define

$$ {}_R S_{\mu\nu} \equiv {}_R E_{\mu\nu} + {}_R t_{\mu\nu} \equiv {}_R H_{\mu\nu} + {}_R\Psi_{\mu\nu} \tag{104} $$

The field equation of the unified field theory follows in general as

$$ {}_R S_{\mu\nu} - {}_R t_{\mu\nu} \equiv {}_R E_{\mu\nu} \tag{105} $$

where ${}_R S_{\mu\nu}$ denotes the tensor of space; ${}_R E_{\mu\nu}$ denotes the tensor of energy and ${}_R t_{\mu\nu}$ denotes the tensor of time.

2.4.26. Definition: The Normalization of the Tensor of Space

Let ${}_R Y_{\mu\nu}$ denote a covariant second rank tensor of preliminary unknown structure. In general, we define

$$ {}_R S_{\mu\nu} \cap {}_R Y_{\mu\nu} \equiv 1_{\mu\nu} \tag{106} $$

Scholium.

In general, the properties of the tensor ${}_R Y_{\mu\nu}$ are unknown. But one property of this tensor is known and this property assures the normalisation of the tensor of space as ${}_R S_{\mu\nu} \cap {}_R Y_{\mu\nu} = 1_{\mu\nu}$. Under conditions of the general theory of relativity, it is true that as ${}_R S_{\mu\nu} = R_{\mu\nu}$ and we do obtain $R_{\mu\nu} \cap {}_R Y_{\mu\nu} = 1_{\mu\nu}$.

2.4.27. Definition: The Probability Tensor

Let

$$ p\left({}_R X_{\mu\nu}\right) \tag{107} $$

denote a covariant second rank probability tensor of yet unknown structure as associated with a tensor ${}_R X_{\mu\nu}$. The probability tensor $p({}_R\Psi_{\mu\nu})$ of yet unknown structure as associated with the wave function tensor ${}_R\Psi_{\mu\nu}$ is defined as

$$ p\left({}_R\Psi_{\mu\nu}\right) \tag{108} $$

2.4.28. Definition: General Covariant form of Born's Rule

Under the assumption of the validity of *Born's rule even under conditions of accelerated frames of reference*, we define

$$ p\left({}_R\Psi_{\mu\nu}\right) \equiv {}_R\Psi_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_R\Psi_{\mu\nu} \cap {}_R Y_{\mu\nu} \tag{109} $$

where $p({}_R\Psi_{\mu\nu})$ denotes the probability tensor as associated *i.e.* with the wave function tensor ${}_R\Psi_{\mu\nu}$ and ${}_R^*\Psi_{\mu\nu}$ is the covariant second rank complex conjugate wave function tensor and $\cap$ denotes the *commutative* multiplication of tensors.

2.4.29. Definition: The Probability Tensor II

In general, we define

$$p\left({}_R\Psi_{\mu\nu}\right) \equiv {}_R\Psi_{\mu\nu} \cap {}_RY_{\mu\nu} \tag{110}$$

where $p({}_R\Psi_{\mu\nu})$ denotes the probability tensor as associated *i.e.* with the wave function tensor ${}_R\Psi_{\mu\nu}$ and ${}_RY_{\mu\nu}$ denote a covariant second rank tensor of preliminary unknown structure and $\cap$ denotes the *commutative* multiplication of tensors.

Scholium.

The properties of the tensor ${}_RY_{\mu\nu}$, as mentioned already before, are still unknown. Still, another second property of this tensor is the special relationship with the wave function tensor ${}_R\Psi_{\mu\nu}$. The interaction of the tensor ${}_RY_{\mu\nu}$ with the wave function tensor ${}_R\Psi_{\mu\nu}$ yields the probability tensor $p({}_R\Psi_{\mu\nu})$ as associated with the wave function tensor ${}_R\Psi_{\mu\nu}$. In general it is $p({}_R\Psi_{\mu\nu}) = {}_R\Psi_{\mu\nu} \cap {}_R\mathrm{Y}_{\mu\nu}$.

2.4.30. Definition: The Tensor $U_{\mu\nu}$

In general, we define the tensor $U_{\mu\nu}$ of yet unknown structure as

$$\begin{aligned} U_{\mu\nu} &\equiv \frac{1_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu}} \cap {}_RS_{\mu\nu} \equiv \left(\frac{1_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu}} \cap {}_RE_{\mu\nu}\right) + \left(\frac{1_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu}} \cap {}_Rt_{\mu\nu}\right) \\ &\equiv \left(\frac{1_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu}} \cap {}_RH_{\mu\nu}\right) + \left(\frac{1_{\mu\nu}}{c_{\mu\nu} \times c_{\mu\nu}} \cap {}_R\Psi_{\mu\nu}\right) \end{aligned} \tag{111}$$

2.4.31. Definition: The Decomposition of the Tensor $U_{\mu\nu}$

In general, we decompose the tensor $U_{\mu\nu}$ as

$$U_{\mu\nu} \equiv U_{\mu\nu} - {}_RM_{\mu\nu} + M_{\mu\nu} \equiv {}_RM_{\mu\nu} + \underline{M}_{\mu\nu} \equiv {}_RM_{\mu\nu} + {}_Rg_{\mu\nu} \tag{112}$$

Scholium.

By this definition we are following Einstein in his claim that something is determined by matter and the gravitational field. In other words, *there is no third between matter and gravitational field*, *i.e.* all but matter is gravitational field. To proceed further, in following Einstein, we make a strict distinction between matter and gravitational field too.

"Wir unterscheiden im folgenden zwischen 'Gravitationsfeld' und 'Materie' in dem Sinne, daß *alles außer dem Gravitationsfeld als* '*Materie*' *bezeichnet* wird, also nicht nur die 'Materie' im üblichen Sinne, sondern auch das elektromagnetische Feld" [21].

The tensor ${}_RU_{\mu\nu}$ is not identical with the tensor $U_{\mu\nu}$. In terms of set theory, we do obtain the following picture (**Table 3**).

2.4.32. Definition: The Tensor of Curvature ${}_0C_{\mu\nu}$

In general, we define the tensor of curvature as ${}_0C_{\mu\nu}$ of yet unknown structure as

$${}_0C_{\mu\nu} \equiv G_{\mu\nu} \equiv A_{\mu\nu} + C_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu} \tag{113}$$

where $G_{\mu\nu}$ is the Einsteinian tensor; $R_{\mu\nu}$ is the Ricci tensor; R is the Ricci scalar and $g_{\mu\nu}$ is the metric tensor of general relativity. Under conditions of the theory of general relativity it is ${}_0C_{\mu\nu} = G_{\mu\nu}$.

Scholium.

Table 3. The relationship between matter and gravitational field.

	${}_R\mathrm{M}_{\mu\nu}$	${}_R\mathrm{g}_{\mu\nu}$
		$\mathrm{U}_{\mu\nu}$

Under conditions of general theory of relativity, the associated probability tensor follows as

$$p\left({}_{0}C_{\mu\nu}\right) \equiv p\left(G_{\mu\nu}\right) \equiv \frac{A_{\mu\nu} + C_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{\left(R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu}\right)}{R_{\mu\nu}} \tag{114}$$

2.4.33. Definition: The Tensor of Anti-Curvature ${}_{0}\underline{C}_{\mu\nu}$

In general, we define the tensor of anti-curvature as ${}_{0}\underline{C}_{\mu\nu}$ of yet unknown structure as

$${}_{0}\underline{C}_{\mu\nu} \equiv {}_{R}S_{\mu\nu} - {}_{0}C_{\mu\nu} \tag{115}$$

where ${}_{R}S_{\mu\nu}$ is the tensor of space, ${}_{0}C_{\mu\nu}$ is the tensor of curvature. Under conditions of general relativity, the tensor of anti-curvature is equivalent with

$${}_{0}\underline{C}_{\mu\nu} \equiv B_{\mu\nu} + D_{\mu\nu} \equiv R_{\mu\nu} - G_{\mu\nu} \equiv R_{\mu\nu} - \left(R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu}\right) \equiv \frac{R}{2} \times g_{\mu\nu} \tag{116}$$

where $G_{\mu\nu}$ is the Einsteinian tensor; $R_{\mu\nu}$ is the Ricci tensor; R is the Ricci scalar and $g_{\mu\nu}$ is the metric tensor of general relativity.

Scholium.

Under conditions of general theory of relativity, the associated probability tensor follows as

$$p\left({}_{0}\underline{C}_{\mu\nu}\right) \equiv \frac{B_{\mu\nu} + D_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{R_{\mu\nu} - G_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{R_{\mu\nu} - \left(R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu}\right)}{R_{\mu\nu}} \equiv \frac{\frac{R}{2} \times g_{\mu\nu}}{R_{\mu\nu}} \tag{117}$$

2.5. Tensor Calculus

2.5.1. Definition: The Tensor of the Unified Field $1_{\mu\nu}$

In general, we define *the tensor of the unified field* $1_{\mu\nu}$, as

$$1_{\mu\nu} \tag{118}$$

Scholium.

Every component of *the tensor of the unified field* is equal to +1. The tensor of the unified field is of order two, its components can be displayed in 4 × 4 matrix form as

$$+1_{\mu\nu} \equiv \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 \end{bmatrix} \tag{119}$$

2.5.2. Definition: The Zero Tensor $0_{\mu\nu}$

In general, we define the zero tensor $0_{\mu\nu}$ as

$$0_{\mu\nu} \tag{120}$$

Scholium.

Every component of a zero tensor is equal to +0. The zero tensor is of order two, its components can be displayed in 4 × 4 matrix form too as

$$+0_{\mu\nu} \equiv \begin{bmatrix} +0 & +0 & +0 & +0 \\ +0 & +0 & +0 & +0 \\ +0 & +0 & +0 & +0 \\ +0 & +0 & +0 & +0 \end{bmatrix} \tag{121}$$

2.5.3. Definition: The Tensor of the Number $2_{\mu\nu}$

In general, we define tensor of any number, *i.e*. the number $2_{\mu\nu}$ as

$$2_{\mu\nu} \tag{122}$$

Scholium.

Every component of a tensor of the number +2 is equal to +2. The tensor of the number +2 can be displayed in 4 × 4 matrix form as

$$+2_{\mu\nu} \equiv \begin{bmatrix} +2 & +2 & +2 & +2 \\ +2 & +2 & +2 & +2 \\ +2 & +2 & +2 & +2 \\ +2 & +2 & +2 & +2 \end{bmatrix} \tag{123}$$

2.5.4. Definition: The Tensor of Infinity $\infty_{\mu\nu}$

In general, we define the tensor of infinity $\infty_{\mu\nu}$ as

$$\infty_{\mu\nu} \tag{124}$$

Scholium.

Every component of the tensor of infinity is equal to +∞. The tensor of infinity is of order two, its components can be displayed in 4 × 4 matrix form as

$$+\infty_{\mu\nu} \equiv \begin{bmatrix} +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty \end{bmatrix} \tag{125}$$

2.5.5. Definition: The Symmetrical Part of a Tensor $S({}_0X_{\mu\nu})$

Let ${}_0X_{\mu\nu}$ denote a second-tensor rank. The symmetric part of a tensor ${}_0X_{\mu\nu}$ is defined as

$$S\left({}_0X_{\mu\nu}\right) = \frac{1}{2} \times \left({}_0X_{\mu\nu} + {}_0X_{\nu\mu}\right) \tag{126}$$

and denoted using the capital letter S and the tensor itself within the parentheses.

2.5.6. Definition: The Anti-Symmetrical Part of a Tensor $\underline{S}({}_0X_{\mu\nu})$

Let ${}_0X_{\mu\nu}$ denote a second-tensor rank. The anti-symmetric part of a tensor ${}_0X_{\mu\nu}$ is defined as

$$\underline{S}({}_0X_{\mu\nu}) = \frac{1}{2} \times \left({}_0X_{\mu\nu} - {}_0X_{\nu\mu}\right) \tag{127}$$

and denoted using the capital letter $\underline{S}$ underscore and the tensor itself within the parentheses.

Scholium.

In general, the tensor ${}_0X_{\mu\nu}$ can be written as a sum of symmetric and antisymmetric parts as

$${}_0X_{\mu\nu} = S\left({}_0X_{\mu\nu}\right) + \underline{S}\left({}_0X_{\mu\nu}\right) = \frac{1}{2} \times \left({}_0X_{\mu\nu} + {}_0X_{\nu\mu}\right) + \frac{1}{2} \times \left({}_0X_{\mu\nu} - {}_0X_{\nu\mu}\right) \tag{128}$$

2.5.7. Definition: Tensor ${}_0X_{\mu\nu}$ and Anti Tensor ${}_0\underline{X}_{\mu\nu}$

In general, let

$${}_RC_{\mu\nu} = {}_0X_{\mu\nu} + {}_1X_{\mu\nu} + \cdots + {}_NX_{\mu\nu} \tag{129}$$

We define *the anti tensor* ${}_0\underline{X}_{\mu\nu}$ of the tensor ${}_0X_{\mu\nu}$ as

$${}_0\underline{X}_{\mu\nu} \equiv {}_RC_{\mu\nu} - {}_0X_{\mu\nu} \equiv +{}_1X_{\mu\nu} + \cdots + {}_NX_{\mu\nu} \tag{130}$$

Scholium.

There is no third tensor between a tensor and its own anti tensor, a third is not given, *tertium non datur* (Aristotle). An anti tensor is denoted by the name of the tensor with underscore. Theoretically, the distinction between an *anti-symmetrical tensor* and an *anti tensor* is necessary. The simplest nontrivial antisymmetric rank-2 tensor, written as a sum of symmetric and antisymmetric parts, satisfies the equation

$$ {}_{0}X_{\mu\nu} = -{}_{0}X_{\nu\mu} \equiv \frac{1}{2} \times \left({}_{0}X_{\mu\nu} + {}_{0}X_{\nu\mu} \right) + \frac{1}{2} \times \left({}_{0}X_{\mu\nu} - {}_{0}X_{\nu\mu} \right) \tag{131} $$

In general, the relationship between an *anti symmetrical tensor* and an *anti tensor* follows as

$$ {}_{0}X_{\mu\nu} = -{}_{0}X_{\nu\mu} \equiv {}_{R}C_{\mu\nu} - {}_{0}\underline{X}_{\mu\nu} \tag{132} $$

Only under conditions where ${}_{R}C_{\mu\nu} = 0$ we obtain

$$ -{}_{0}X_{\nu\mu} \equiv -{}_{0}\underline{X}_{\mu\nu} \tag{133} $$

but not in general. In this context it is

$$ 1_{\mu\nu} = 1_{\mu\nu} + 0_{\mu\nu} \equiv 1_{\mu\nu} + \underline{1}_{\mu\nu} \equiv \underline{0}_{\mu\nu} + 0_{\mu\nu} \tag{134} $$

The *anti tensor* $\underline{\delta}_{\mu\nu}$ *of the Kronecker delta* or Kronecker's delta $\delta_{\mu\nu}$, named after Leopold Kronecker (1823-1891), follows as

$$ \underline{\delta}_{\mu\nu} = 1_{\mu\nu} - \delta_{\mu\nu} \tag{135} $$

2.5.8. Definition: The Addition of Tensors

Tensors independent of any coordinate system or frame of reference as generalizations of scalars (magnitude, no direction associated with a scalar) which have no indices and other mathematical objects (vectors (single direction), matrices) to an arbitrary number of indices may be operated on by tensor operators or by other tensors. In general, tensors can be represented by uppercase Latin letters and the notation for a tensor is similar to that of a matrix even if a tensor may be determined by an arbitrary number of indices. A distinction between covariant and contravariant indices is made. A component of a second-rank tensor is indicated by two indices. Thus far, a component of any tensor of any tensor rank which vanishes in one particular coordinate system, will vanish in all coordinate systems too. As is known, two tensors X and $\underline{X}$ which have the same rank and the same covariant and/or contravariant indices can be added. The sum of two tensors of the same rank is also a tensor of the same rank. In general, it is

$$ {}_{R}C_{\mu\nu} = {}_{0}X_{\mu\nu} + {}_{0}\underline{X}_{\mu\nu} \tag{136} $$

or

$$ {}_{R}C^{\mu\nu} = {}_{0}X^{\mu\nu} + {}_{0}\underline{X}^{\mu\nu} \tag{137} $$

or

$$ {}_{R}C^{\mu}{}_{\nu} = {}_{0}X^{\mu}{}_{\nu} + {}_{0}\underline{X}^{\mu}{}_{\nu} \tag{138} $$

2.5.9. Definition: The Difference of Tensors

The difference of two tensors of the same rank is also a tensor of the same rank. In general, it is

$$ {}_{0}\underline{X}_{\mu\nu} = {}_{R}C_{\mu\nu} - {}_{0}X_{\mu\nu} \tag{139} $$

or

$$ {}_{0}\underline{X}^{\mu\nu} = {}_{R}C^{\mu\nu} - {}_{0}X^{\mu\nu} \tag{140} $$

or

$$ +{}_{0}\underline{X}^{\mu}{}_{\nu} = {}_{R}C^{\mu}{}_{\nu} - {}_{0}X^{\mu}{}_{\nu} \tag{141} $$

2.5.10. Definition: The Commutative Multiplication of Tensors

Let us display the individual components of a co-variant rank two tensor $X_{\mu\nu}$ in matrix form as

$$X_{\mu\nu} = \begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} \\ X_{10} & X_{11} & X_{12} & X_{13} \\ X_{20} & X_{21} & X_{22} & X_{23} \\ X_{30} & X_{31} & X_{32} & X_{33} \end{bmatrix} \tag{142}$$

Let us display the individual components of a co-variant rank two tensor $Y_{\mu\nu}$ in matrix form as

$$Y_{\mu\nu} = \begin{bmatrix} Y_{00} & Y_{01} & Y_{02} & Y_{03} \\ Y_{10} & Y_{11} & Y_{12} & Y_{13} \\ Y_{20} & Y_{21} & Y_{22} & Y_{23} \\ Y_{30} & Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \tag{143}$$

The *commutative* multiplication of tensors (*i.e.* matrices), which is different from the non-commutative multiplication of tensors (*i.e.* matrices), is an operation of *multiplying the corresponding elements of both tensors* by each other. We define *the commutative multiplication of tensors* in general as

$$X_{\mu\nu} \cap Y_{\mu\nu} \equiv Y_{\mu\nu} \cap X_{\mu\nu} \equiv \begin{bmatrix} X_{00}\times Y_{00} & X_{01}\times Y_{01} & X_{02}\times Y_{02} & X_{03}\times Y_{03} \\ X_{10}\times Y_{10} & X_{11}\times Y_{11} & X_{12}\times Y_{12} & X_{13}\times Y_{13} \\ X_{20}\times Y_{20} & X_{21}\times Y_{21} & X_{22}\times Y_{22} & X_{23}\times Y_{23} \\ X_{30}\times Y_{30} & X_{31}\times Y_{31} & X_{32}\times Y_{32} & X_{33}\times Y_{33} \end{bmatrix} \tag{144}$$

while the sign $\cap$ denotes *the commutative multiplication* of tensors which is equally related to the Hadamard [29] product. The *Hadamard product* (also known as the Schur product or the point wise product), due to Jacques Salomon Hadamard (1865-1963), is an operation of two matrices of the same dimensions which is commutative, associative and distributive.

2.5.11 Definition: The Tensor Raised to Power *n*

Let us introduce the notation of a co-variant rank two tensor $X_{\mu\nu}$ raised to power n as

$${}^{n}X_{\mu\nu} = \underbrace{X_{\mu\nu} \cap X_{\mu\nu} \cap \cdots \cap X_{\mu\nu}}_{n\text{-times}} \tag{145}$$

Each individual component of the tensor $X_{\mu\nu}$ is multiplied by itself n-times.

2.5.12. Definition: The Root of the Tensor Raised to Power 1/*n*

Let us introduce the notation of a co-variant rank two tensor $X_{\mu\nu}$ raised to power $1/n$ as

$${}^{1/n}X_{\mu\nu} = \sqrt[n]{\underbrace{X_{\mu\nu} \cap X_{\mu\nu} \cap \cdots \cap X_{\mu\nu}}_{n\text{-times}}} \tag{146}$$

Each individual component of the tensor $X_{\mu\nu}$ is raised to the power $1/n$.

2.5.13. Definition: The Commutative Division of Tensors

Let us once again display the individual components of a co-variant rank two tensor ${}_{R}X_{\mu\nu}$ in matrix form as

$$X_{\mu\nu} = \begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} \\ X_{10} & X_{11} & X_{12} & X_{13} \\ X_{20} & X_{21} & X_{22} & X_{23} \\ X_{30} & X_{31} & X_{32} & X_{33} \end{bmatrix} \tag{147}$$

The commutative division of tensors is defined by the division of the corresponding elements of both tensors by each other and displayed in matrix form as

$$X_{\mu\nu} : Y_{\mu\nu} = \begin{bmatrix} X_{00}/Y_{00} & X_{01}/Y_{01} & X_{02}/Y_{02} & X_{03}/Y_{03} \\ X_{10}/Y_{10} & X_{11}/Y_{11} & X_{12}/Y_{12} & X_{13}/Y_{13} \\ X_{20}/Y_{20} & X_{21}/Y_{21} & X_{22}/Y_{22} & X_{23}/Y_{23} \\ X_{30}/Y_{30} & X_{31}/Y_{31} & X_{32}/Y_{32} & X_{33}/Y_{33} \end{bmatrix} \tag{148}$$

while the sign: denotes *the commutative division of tensors*. The commutative division of tensors is displayed as

$$X_{\mu\nu} : Y_{\mu\nu} = \begin{bmatrix} X_{00}/Y_{00} & X_{01}/Y_{01} & X_{02}/Y_{02} & X_{03}/Y_{03} \\ X_{10}/Y_{10} & X_{11}/Y_{11} & X_{12}/Y_{12} & X_{13}/Y_{13} \\ X_{20}/Y_{20} & X_{21}/Y_{21} & X_{22}/Y_{22} & X_{23}/Y_{23} \\ X_{30}/Y_{30} & X_{31}/Y_{31} & X_{32}/Y_{32} & X_{33}/Y_{33} \end{bmatrix} = \frac{X_{\mu\nu}}{Y_{\mu\nu}} \tag{149}$$

too.

2.5.14. Definition: The Expectation Value of a Second Rank Tensor

Let $E(X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$. Let $p(X_{\mu\nu})$ denote the probability tensor of the second rank tensor $X_{\mu\nu}$. In general, we define

$$E\left(X_{\mu\nu}\right) \equiv p\left(X_{\mu\nu}\right) \cap X_{\mu\nu} \tag{150}$$

while the sign $\cap$ denotes the *commutative* multiplication of tensors.

2.5.15. Definition: The Expectation Value of a Second Rank Tensor Raised to Power 2

Let $E({}^2X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$ raised to the power 2. Let $p(X_{\mu\nu})$ denote the probability tensor of the second rank tensor $X_{\mu\nu}$. In general, we define

$$E\left({}^2X_{\mu\nu}\right) \equiv p\left(X_{\mu\nu}\right) \cap X_{\mu\nu} \cap X_{\mu\nu} \equiv p\left(X_{\mu\nu}\right) \cap {}^2X_{\mu\nu} \tag{151}$$

while the sign $\cap$ denotes the *commutative* multiplication of tensors.

2.5.16. Definition: The Variance of a Second Rank Tensor

Let $\sigma(X_{\mu\nu})^2$ denote the variance of the covariant second rank tensor $X_{\mu\nu}$. Let $E(X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$. Let $E({}^2X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$ raised to the power 2. Let $p(X_{\mu\nu})$ denote the probability tensor of the second rank tensor $X_{\mu\nu}$. In general, we define

$$\sigma\left(X_{\mu\nu}\right)^2 \equiv E\left({}^2X_{\mu\nu}\right) - \left(E\left(X_{\mu\nu}\right) \cap E\left(X_{\mu\nu}\right)\right) \tag{152}$$

which can be written as

$$\sigma\left(X_{\mu\nu}\right)^2 \equiv p\left(X_{\mu\nu}\right) \cap X_{\mu\nu} \cap X_{\mu\nu} - \left(\left(p\left(X_{\mu\nu}\right) \cap X_{\mu\nu}\right) \cap \left(p\left(X_{\mu\nu}\right) \cap X_{\mu\nu}\right)\right) \tag{153}$$

or as

$$\sigma\left(X_{\mu\nu}\right)^2 \equiv p\left(X_{\mu\nu}\right) \cap X_{\mu\nu} \cap X_{\mu\nu} - \left(X_{\mu\nu} \cap X_{\mu\nu} \cap p\left(X_{\mu\nu}\right) \cap p\left(X_{\mu\nu}\right)\right) \tag{154}$$

or as

$$\sigma\left(X_{\mu\nu}\right)^2 \equiv X_{\mu\nu} \cap X_{\mu\nu} \cap \left(p\left(X_{\mu\nu}\right) - p\left(X_{\mu\nu}\right) \cap p\left(X_{\mu\nu}\right)\right) \tag{155}$$

or as

$$\sigma\left(X_{\mu\nu}\right)^2 \equiv X_{\mu\nu} \cap X_{\mu\nu} \cap \left(p\left(X_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left(X_{\mu\nu}\right)\right)\right) \tag{156}$$

while the sign $\cap$ denotes the *commutative* multiplication of tensors and $1_{\mu\nu}$ is the tensor of the unified field.

2.5.17. Definition: The Standard Deviation of a Second Rank Tensor

Let $\sigma(X_{\mu\nu})$ denote the standard deviation of the covariant second rank tensor $X_{\mu\nu}$. Let $E(X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$. Let $E({}^2X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$ raised to the power 2. Let $p(X_{\mu\nu})$ denote the probability tensor of the second rank tensor $X_{\mu\nu}$. In general, we define

$$\sigma\left(X_{\mu\nu}\right) \equiv \sqrt[2]{E\left({}^2X_{\mu\nu}\right) - \left(E\left(X_{\mu\nu}\right) \cap E\left(X_{\mu\nu}\right)\right)} \tag{157}$$

which can be written as

$$\sigma\left(X_{\mu\nu}\right) \equiv X_{\mu\nu} \cap \sqrt[2]{\left(p\left(X_{\mu\nu}\right) - p\left(X_{\mu\nu}\right) \cap p\left(X_{\mu\nu}\right)\right)} \tag{158}$$

or as

$$\sigma\left(X_{\mu\nu}\right) \equiv X_{\mu\nu} \cap \sqrt[2]{\left(p\left(X_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left(X_{\mu\nu}\right)\right)\right)} \tag{159}$$

while the sign $\cap$ denotes the *commutative* multiplication of tensors and $1_{\mu\nu}$ is the tensor of the unified field. The covariant second rank tensor $X_{\mu\nu}$ follows as

$$X_{\mu\nu} \equiv \frac{\sigma\left(X_{\mu\nu}\right)}{\sqrt[2]{\left(p\left(X_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left(X_{\mu\nu}\right)\right)\right)}} \tag{160}$$

2.5.18. Definition: The Co-Variance of Two Second Rank Tensors

Let $\sigma(X_{\mu\nu}, Y_{\mu\nu})$ denote the co-variance of the two covariant second rank tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$. Let $E(X_{\mu\nu}, Y_{\mu\nu})$ denote the expectation value of the two covariant second rank tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$. Let $p(X_{\mu\nu}, Y_{\mu\nu})$ denote the probability tensor of the two covariant second rank tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$. Let $E(X_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $X_{\mu\nu}$. Let $p(X_{\mu\nu})$ denote the probability tensor of the second rank tensor $X_{\mu\nu}$. Let $E(Y_{\mu\nu})$ denote the expectation value of the covariant second rank tensor $Y_{\mu\nu}$. Let $p(Y_{\mu\nu})$ denote the probability tensor of the second rank tensor $Y_{\mu\nu}$. In general, we define

$$\sigma\left(X_{\mu\nu}, Y_{\mu\nu}\right) \equiv E\left(X_{\mu\nu}, Y_{\mu\nu}\right) - \left(E\left(X_{\mu\nu}\right) \cap E\left(Y_{\mu\nu}\right)\right) \tag{161}$$

which can be written as

$$\sigma\left(X_{\mu\nu}, Y_{\mu\nu}\right) \equiv p\left(X_{\mu\nu}, Y_{\mu\nu}\right) \cap X_{\mu\nu} \cap Y_{\mu\nu} - \left(p\left(X_{\mu\nu}\right) \cap X_{\mu\nu} \cap Y_{\mu\nu} \cap p\left(Y_{\mu\nu}\right)\right) \tag{162}$$

or as

$$\sigma\left(X_{\mu\nu}, Y_{\mu\nu}\right) \equiv X_{\mu\nu} \cap Y_{\mu\nu} \cap \left(p\left(X_{\mu\nu}, Y_{\mu\nu}\right) - p\left(X_{\mu\nu}\right) \cap p\left(Y_{\mu\nu}\right)\right) \tag{163}$$

while the sign $\cap$ denotes the *commutative* multiplication. In general, it is

$$X_{\mu\nu} \cap Y_{\mu\nu} \equiv \frac{\sigma\left(X_{\mu\nu}, Y_{\mu\nu}\right)}{\left(p\left(X_{\mu\nu}, Y_{\mu\nu}\right) - p\left(X_{\mu\nu}\right) \cap p\left(Y_{\mu\nu}\right)\right)} \tag{164}$$

2.5.19. Definition: Einstein's Weltformel

Let $\sigma({}_RU_{\mu\nu}, {}_0W_{\mu\nu})$ denote the co-variance of the two covariant second rank tensors ${}_RU_{\mu\nu}$ and ${}_0W_{\mu\nu}$. Let $\sigma({}_RU_{\mu\nu})$ denote the standard deviation of the covariant second rank tensor of the cause. Let $\sigma({}_0W_{\mu\nu})$ denote the standard deviation of the covariant second rank tensor of the effect ${}_0W_{\mu\nu}$. Let $k({}_RU_{\mu\nu}, {}_0W_{\mu\nu})$ denote the mathematical formula of the causal relationship in a general covariant form (*i.e.* Einstein's Weltformel). In general, we define

$$k\left({}_RU_{\mu\nu}, {}_0W_{\mu\nu}\right) \equiv \frac{\sigma\left({}_RU_{\mu\nu}, {}_0W_{\mu\nu}\right)}{\sigma\left({}_RU_{\mu\nu}\right) \cap \sigma\left({}_0W_{\mu\nu}\right)} \tag{165}$$

Scholium.

In this context, the above equation is able to bridge the gap between classical field theory and quantum theory since the same enables the existence elementary particles *i.e.* with unequal mass but with opposite though otherwise equal electric charge.

2.6. Axioms

2.6.1. Axiom I. (Lex Identitatis. Principium Identitatis. The Identity Law)

The foundation of all what may follow is the following axiom:

$$+1 \equiv +1. \tag{166}$$

Scholium.

From the standpoint of tensor calculus, it is

$$1_{\mu\nu} \equiv 1_{\mu\nu} \tag{167}$$

This article does not intend to give a review of the history of the identity law (*principium identitatis*). In the following it is useful to sketch, more or less chronologically, and by trailing the path to mathematics, the history of attempts of mathematizing the identity law. The identity law was used in *Plato*'s dialogue Theaetetus, in Aristotle's Metaphysics (Book IV, Part 4) and by many other authors too. Especially, Gottfried Wilhelm Leibniz (1646-1716) expressed the law of identity as *everything is that what it is*. "*Chaque chose est ce qu*'*elle est*. Et dans autant d'exemples qu'on voudra A est A, B est B" [30]. In *The problems of philosophy* (1912) Russell himself is writing about the identity law too.

Lex identitatis or the identity law or *principium identitatis* can be expressed mathematically in the very simple form as +1 = +1. Consequently, +1 is only itself, simple equality with itself, it is only self-related and unrelated to another, +1 is distinct from any relation to another, +1 contains nothing other, *no local hidden variable*, but only itself, +1. In this way, there does not appear to be any relation to another, any relation to another is removed, any relation to another has vanished. Consequently, +1 is just itself and thus somehow the absence of any other determination. +1 is in its own self only itself and nothing else. In this sense, +1 is identical only with itself, +1 is thus just the "pure" +1. Let us consider this in more detail, +1 is not the transition into its opposite, the negative of +1, denoted as −1, is not as necessary as the +1 itself, +1 is not confronted by its other, +1 is without any opposition or contradiction, is not against another, is not opposed to another, +1 is identical only with itself and has passed over into pure equality with itself. But lastly, identity as different from difference, contains within itself the difference itself. Thus, it is the same +1 which equally negates itself, +1 in the same respect is in its self-sameness different from itself and thus self-contradictory. It is true, that +1 = +1, but it is equally true that −1 = −1. It is the same 1 which is related to a +1 and a −1. It is the +1 which excludes at the same time the other out of itself, the −1, out of itself, +1 is +1 and nothing else, it is not −1, it is not +2, it is not … Especially +1 is at the same time not −1, +1 is thus far determined as non being at least as non-being of its own other. In excluding its own other out of itself, +1 is excluding itself in its own self. By excluding its own other, +1 makes itself into the other of what it excludes from itself, or +1 makes itself into its own opposite, +1 is thus simply the transition of itself into its opposite, +1 is therefore determined only in so far as it contains such a contradiction within itself. The non-being of its other (−1) is at the end the sublation of its other. This non-being is the non-being of itself, a non-being which has its non-being in its own self and not in another; each contains thus far a reference to its other. Not +1 (*i.e.* −1) is the pure other of +1. But at the same time, not +1 only shows itself in order to vanish; the other of +1 is not. In this context, +1 and not +1 are distinguished and at the same time both are related to one and the same 1, each is that what it is as distinct from its own other. Identity is thus far to some extent at the same time the vanishing of otherness. +1 is itself and its other, +1 has its determinateness not in another, but in its own self. +1 is thus far self-referred and the reference to its other is only a self-reference. On closer examination +1 therefore is, only in so far as its Not +1 is, +1 has within itself a relation to its other. In other words, +1 is in its own self at the same time different from something else or +1 is something. It is widely accepted that something is different from nothing, thus while +1 = +1 it is at the same time different from nothing or from non- +1. From this it is evident, that the other side of the identity +1 = +1 is the fact, that +1 cannot at the same time be +1 and −1 or not +1. In fact, if +1 = +1 then +1 is not at the same time not +1. What emerges from this consideration is, therefore, even if +1 = +1 it is a self-contained opposition, +1 is only in so far as +1 contains this contradiction within it, +1 is inherently self-contradictory, +1 is thus only as the other of the other. In so far, +1 includes within its own self its own non-being, a relation to something else different from its own self. Thus, +1 is at the same time the unity of identity with difference. +1 is itself and at the same time its other too, +1 is thus contradiction. Difference as such it unites sides which are, only in so far as they are at the same time not the same. +1 is only in so far as the other of +1, the non +1 is. +1 is thus far that what it is only through the other, through the non +1, through the non-being of itself. From the identity +1 = +1 follows that +1 − 1 = 0. +1 and −1 are negatively related to one another and both are indifferent to one another, +1 is separated in the same relation. +1 is itself and its other, it is self-referred, its reference to its other is thus a reference to itself; its non-being is thus only a moment in it. +1 is in its own self the opposite of itself, it has within itself the relation to its other; it is a simple and self-related negativity. Each of them are determined against the other, the other is in and for itself and not as the other of another. +1 is in its own self the negativity of itself. +1 therefore is, only in so far as its non-being is and vice versa. Non +1 therefore is, only in so far as its non-being is, both are through the non-being of its other, both as opposites cancel one another in their combination, it is +1 − 1 = 0.

2.6.2. Axiom II

$$+1 \equiv (+\infty) \times (+0). \tag{168}$$

Scholium.

From the standpoint of tensor calculus, it is

$$1_{\mu\nu} \equiv \infty_{\mu\nu} \cap 0_{\mu\nu} \tag{169}$$

2.6.3. Axiom III

$$\frac{+0}{+0} \equiv +1. \tag{170}$$

Scholium.

From the standpoint of tensor calculus, it is

$$\left(+0_{\mu\nu}\right) \equiv \left(\frac{+0_{\mu\nu}}{+0_{\mu\nu}}\right) \cap \left(+0_{\mu\nu}\right) \equiv \left(+1_{\mu\nu}\right) \cap \left(+0_{\mu\nu}\right). \tag{171}$$

The law of non-contradiction (LNC) is still one of the foremost among the principles of science and equally a fundamental principle of scientific inquiry too. Without the principle of non-contradiction we could not be able to distinguish between something true and something false. There are arguably many versions of the principle of non-contradiction which can be found in literature. The method of *reductio ad absurdum* itself is grounded on the validity of the principle of non-contradiction. To be consistent, a claim/a theorem/a proposition/a statement et cetera accepted as correct, cannot lead to a logical contradiction. In general, a claim/a theorem/a proposition/a statement et cetera which leads to the conclusion that +1 = +0 is refuted.

3. Results

3.1. Theorem. Einstein's Field Equation

Einstein's field equations can be derived from axiom I.

Claim. (Theorem. Proposition. Statement.)

In general, Einstein's field equations are derived as

$$G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) \tag{172}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{173}$$

Multiplying this equation by the stress-energy tensor of general relativity $\left(4 \times 2 \times \pi \times \gamma / \left(c^4\right)\right) \times T_{\mu\nu}$, it is

$$+1 \times \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) = +1 \times \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) \tag{174}$$

where γ is Newton's gravitational "constant" [25] [26]; c is the speed of light in vacuum and π, sometimes referred to as "Archimedes' constant", is the ratio of a circle's circumference to its diameter. Due to Einstein's general relativity, the equation before is equivalent with

$$R_{\mu\nu} - \left(\frac{R}{2} \times g_{\mu\nu}\right) + \left(\Lambda \times g_{\mu\nu}\right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) \tag{175}$$

$R_{\mu\nu}$ is the Ricci curvature tensor; R is the scalar curvature; $g_{\mu\nu}$ is the metric tensor; Λ is the cosmological constant and $T_{\mu\nu}$ is the stress – energy tensor. By defining the Einstein tensor as $G_{\mu\nu} = R_{\mu\nu} - (R/2) \times g_{\mu\nu}$, it is possible to write the Einstein field equations in a more compact as

$$G_{\mu\nu} + (\Lambda \times g_{\mu\nu}) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) \tag{176}$$

Quod erat demonstrandum.

3.2. Theorem. The Relationship between the Complex Tensor ${}_{R}Y_{\mu\nu}$ and the Tensor ${}_{R}S_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

In general, it is

$${}_{R}Y_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}S_{\mu\nu}} \tag{177}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{178}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{179}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{180}$$

Multiplying this equation by ${}_{R}S_{\mu\nu} \cap {}_{R}Y_{\mu\nu}$, we obtain

$${}_{R}S_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \cap 1_{\mu\nu} \equiv {}_{R}S_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \cap 1_{\mu\nu} \tag{181}$$

Due to our above definition the unknown tensor ${}_{R}Y_{\mu\nu}$ assures that ${}_{R}S_{\mu\nu} \cap {}_{R}Y_{\mu\nu} = 1_{\mu\nu}$. Consequently, equation before reduces too

$${}_{R}S_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv 1_{\mu\nu} \tag{182}$$

A *commutative* division yields

$${}_{R}Y_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}S_{\mu\nu}} \tag{183}$$

Quod erat demonstrandum.

3.3. Theorem. The Relationship between the Complex Conjugate Tensor ${}_{R}^{*}\Psi_{\mu\nu}$ and the Tensor ${}_{R}Y_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

In general, it is

$${}_{R}Y_{\mu\nu} \equiv {}_{R}^{*}\Psi_{\mu\nu} \tag{184}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{185}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{186}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{187}$$

Multiplying this equation by ${}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu}$, we obtain

$$ {}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \cap 1_{\mu\nu} \equiv {}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \cap 1_{\mu\nu} \tag{188} $$

Due to our above *definition*, it is ${}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{R}\Psi_{\mu\nu} \cap {}_{R}Y_{\mu\nu}$. Consequently, the equation before changes too

$$ {}_{R}\Psi_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \tag{189} $$

At the end, after a *commutative division*, we obtain

$$ {}_{R}Y_{\mu\nu} \equiv {}_{R}^{*}\Psi_{\mu\nu} \tag{190} $$

Quod erat demonstrandum.

3.4. Theorem. The Relationship between the Complex Conjugate Tensor ${}_{R}^{*}\Psi_{\mu\nu}$ and the Ricci Tensor $R_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

In general, it is

$$ {}_{R}^{*}\Psi_{\mu\nu} = \frac{1_{\mu\nu}}{R_{\mu\nu}} \tag{191} $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{192} $$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{193} $$

or

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{194} $$

Multiplying this equation by ${}_{R}Y_{\mu\nu}$, we obtain

$$ {}_{R}Y_{\mu\nu} \cap 1_{\mu\nu} = {}_{R}Y_{\mu\nu} \cap 1_{\mu\nu} \tag{195} $$

or

$$ {}_{R}Y_{\mu\nu} = {}_{R}Y_{\mu\nu} \tag{196} $$

Due to the theorem before, it is ${}_{R}Y_{\mu\nu} = {}_{R}^{*}\Psi_{\mu\nu}$. Consequently, substituting this equation into the equation before we obtain

$$ {}_{R}^{*}\Psi_{\mu\nu} = {}_{R}Y_{\mu\nu} \tag{197} $$

Due to another theorem before, it is ${}_{R}Y_{\mu\nu} = 1_{\mu\nu}: {}_{R}S_{\mu\nu}$. Consequently, substituting this equation into equation before, we obtain

$$ {}_{R}^{*}\Psi_{\mu\nu} = \frac{1_{\mu\nu}}{{}_{R}S_{\mu\nu}} \tag{198} $$

Under conditions of general relativity it is $R_{\mu\nu} = {}_{R}S_{\mu\nu}$ where $R_{\mu\nu}$ denotes the Ricci tensor. In general, under conditions of general relativity, we obtain

$$ {}_{R}^{*}\Psi_{\mu\nu} = \frac{1_{\mu\nu}}{R_{\mu\nu}} \tag{199} $$

Quod erat demonstrandum.

3.5. Theorem. The Probability Tensor $1_{\mu\nu} - p({}_{R}H_{\mu\nu})$ as Associated with the Energy Tensor ${}_{R}H_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

The probability $1_{\mu\nu} - p({}_{R}H_{\mu\nu})$ as associated with the energy tensor ${}_{R}H_{\mu\nu}$ is determined as

$$1_{\mu\nu} - p\left({}_{R}H_{\mu\nu}\right) = {}_{R}H_{\mu\nu} \times {}_{R}{}^{*}\Psi_{\mu\nu} \tag{200}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{201}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{202}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{203}$$

A commutative multiplication of this equation by the tensor ${}_{R}S_{\mu\nu}$ leads to

$${}_{R}S_{\mu\nu} \cap 1_{\mu\nu} = 1_{\mu\nu} \cap {}_{R}S_{\mu\nu} \tag{204}$$

or to

$${}_{R}S_{\mu\nu} = {}_{R}S_{\mu\nu} \tag{205}$$

Due to our definition above, we obtain

$${}_{R}H_{\mu\nu} + {}_{R}\Psi_{\mu\nu} = {}_{R}S_{\mu\nu} \tag{206}$$

A commutative multiplication of the equation before by the complex conjugate wave function tensor ${}_{R}^{*}\Psi_{\mu\nu}$, it is

$${}_{R}H_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} + {}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} = {}_{R}S_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \tag{207}$$

Due to the theorem before, it is ${}_{R}S_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} = 1_{\mu\nu}$. Thus far, equation before changes to

$${}_{R}H_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} + {}_{R}\Psi_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} = 1_{\mu\nu} \tag{208}$$

Following Born's rule, it is $p\left({}_{R}\Psi_{\mu\nu}\right) = {}_{R}\Psi_{\mu\nu} \times {}_{R}\Psi^{*}_{\mu\nu}$. We obtain

$${}_{R}H_{\mu\nu} \times {}_{R}^{*}\Psi_{\mu\nu} + {}_{R}p\left({}_{R}\Psi_{\mu\nu}\right) = 1_{\mu\nu} \tag{209}$$

At the end, it follows that

$$1_{\mu\nu} - p\left({}_{R}\Psi_{\mu\nu}\right) = {}_{R}H_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \tag{210}$$

Quod erat demonstrandum.

3.6. Theorem. The Normalization of the Relationship between Energy and Time

Claim. (Theorem. Proposition. Statement.)

The relationship between Energy ${}_{R}E_{\mu\nu}$ and time ${}_{R}t_{\mu\nu}$ can be normalized as

$$\frac{{}_{R}E_{\mu\nu}}{{}_{R}S_{\mu\nu}} + \frac{{}_{R}t_{\mu\nu}}{{}_{R}S_{\mu\nu}} = +1_{\mu\nu} \tag{211}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{212}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{213}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{214}$$

A commutative multiplication of this equation by the tensor ${}_{R}S_{\mu\nu}$ leads to

$$ {}_{R}S_{\mu\nu} \cap 1_{\mu\nu} = 1_{\mu\nu} \cap {}_{R}S_{\mu\nu} \tag{215} $$

or to

$$ {}_{R}S_{\mu\nu} = {}_{R}S_{\mu\nu} \tag{216} $$

Due to our definition above it is ${}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = {}_{R}S_{\mu\nu}$. The equation before changes to

$$ {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = {}_{R}S_{\mu\nu} \tag{217} $$

A *commutative* division of the equation before by the tensor ${}_{R}S_{\mu\nu}$ leads to

$$ \frac{{}_{R}E_{\mu\nu}}{{}_{R}S_{\mu\nu}} + \frac{{}_{R}t_{\mu\nu}}{{}_{R}S_{\mu\nu}} = 1_{\mu\nu} \tag{218} $$

Quod erat demonstrandum.

3.7. Theorem. The Normalization of the Relationship between Matter and Gravitational Field

Claim. (Theorem. Proposition. Statement.)

The relationship between the quantum mechanical operator of matter and the wave function of the gravitational field can be normalized as

$$ \frac{{}_{R}g_{\mu\nu}}{U_{\mu\nu}} + \frac{{}_{R}M_{\mu\nu}}{U_{\mu\nu}} = +1_{\mu\nu} \tag{219} $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{220} $$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{221} $$

or too

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{222} $$

A commutative multiplication by ${}_{R}M_{\mu\nu}$, leads to

$$ {}_{R}M_{\mu\nu} \cap 1_{\mu\nu} = {}_{R}M_{\mu\nu} \cap 1_{\mu\nu} \tag{223} $$

which is equivalent with

$$ {}_{R}M_{\mu\nu} = {}_{R}M_{\mu\nu} \tag{224} $$

and at the end with

$$ {}_{R}M_{\mu\nu} - {}_{R}M_{\mu\nu} = 0_{\mu\nu} \tag{225} $$

In our understanding ${}_{R}M_{\mu\nu}$ is a determining part of $U_{\mu\nu}$. We add $U_{\mu\nu}$, and do obtain

$$ {}_{R}M_{\mu\nu} + U_{\mu\nu} - {}_{R}M_{\mu\nu} = U_{\mu\nu} \tag{226} $$

Due to Einstein *all but matter is gravitational field*. Since ${}_{R}g_{\mu\nu} = U_{\mu\nu} - {}_{R}M_{\mu\nu}$, it follows that

$$ {}_{R}g_{\mu\nu} + {}_{R}M_{\mu\nu} = U_{\mu\nu} \tag{227} $$

A *commutative* division of the equation before by $U_{\mu\nu}$ leads to the normalization of matter and gravitational field as

$$ \frac{{}_{R}g_{\mu\nu}}{U_{\mu\nu}} + \frac{{}_{R}M_{\mu\nu}}{U_{\mu\nu}} = +1_{\mu\nu} \tag{228} $$

Quod erat demonstrandum.

3.8. Theorem. The Gravitational Field ${}_Rg_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

The gravitational field ${}_Rg_{\mu\nu}$ is determined as

$$ {}_Rg_{\mu\nu} \equiv \frac{{}_Rt_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \tag{229}$$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{230}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{231}$$

or too

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{232}$$

Due to a theorem before it is $\left({}_RE_{\mu\nu} : {}_RS_{\mu\nu}\right) + \left({}_Rt_{\mu\nu} : {}_RS_{\mu\nu}\right) = 1_{\mu\nu}$. The equation before changes too

$$ +1_{\mu\nu} = \frac{{}_RE_{\mu\nu}}{{}_RS_{\mu\nu}} + \frac{{}_Rt_{\mu\nu}}{{}_RS_{\mu\nu}} \tag{233}$$

Due to another theorem before it is $\left({}_RM_{\mu\nu} : U_{\mu\nu}\right) + \left({}_Rg_{\mu\nu} : U_{\mu\nu}\right) = 1_{\mu\nu}$. The equation before changes too

$$ \frac{{}_Rg_{\mu\nu}}{U_{\mu\nu}} + \frac{{}_RM_{\mu\nu}}{U_{\mu\nu}} = \frac{{}_RE_{\mu\nu}}{{}_RS_{\mu\nu}} + \frac{{}_Rt_{\mu\nu}}{{}_RS_{\mu\nu}} \tag{234}$$

A commutative multiplication by $U_{\mu\nu}$ leads to

$$ {}_Rg_{\mu\nu} + {}_RM_{\mu\nu} = \left(\frac{U_{\mu\nu}}{{}_RS_{\mu\nu}} \cap {}_RE\right) + \left(\frac{U_{\mu\nu}}{{}_RS_{\mu\nu}} \cap {}_Rt_{\mu\nu}\right) \tag{235}$$

According to our definition, it is ${}_RS_{\mu\nu} = \left({}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}\right) \cap U_{\mu\nu}$. Thus far, it is $\left(\left(1_{\mu}\right) / \left({}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}\right)\right) = U_{\mu\nu} / {}_RS_{\mu\nu}$. The equation before changes to

$$ {}_Rg_{\mu\nu} + {}_RM_{\mu\nu} = \frac{{}_RE_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} + \frac{{}_Rt_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \tag{236}$$

Due to our definition of matter as ${}_RM_{\mu\nu} = {}_RE_{\mu\nu} / \left({}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}\right)$. The equation changes to

$$ {}_Rg_{\mu\nu} + {}_RM_{\mu\nu} = {}_RM_{\mu\nu} + \frac{{}_Rt_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \tag{237}$$

The tensor of matter ${}_RM_{\mu\nu}$ drops out, and what is left is the tensor of the gravitational field ${}_Rg_{\mu\nu}$ as

$$ {}_Rg_{\mu\nu} = \frac{{}_Rt_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \tag{238}$$

Quod erat demonstrandum.

3.9. Theorem. The Normalization of the Relationship between the Tensor of Energy and the Wave Function Tensor

Claim. (Theorem. Proposition. Statement.)

The relationship between the Hamiltonian operator and the wave function can be normalized as

$$ \frac{{}_RH_{\mu\nu}}{{}_RS_{\mu\nu}} + \frac{{}_R\Psi_{\mu\nu}}{{}_RS_{\mu\nu}} = +1_{\mu\nu} \tag{239}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{240}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{241}$$

or too

$$1_{\mu\nu} = 1_{\mu\nu} \tag{242}$$

A commutative multiplication of this equation by the tensor ${}_R S_{\mu\nu}$ leads to

$${}_R S_{\mu\nu} \cap 1_{\mu\nu} = 1_{\mu\nu} \cap {}_R S_{\mu\nu} \tag{243}$$

or to

$${}_R S_{\mu\nu} = {}_R S_{\mu\nu} \tag{244}$$

Due to our definition above it is ${}_R H_{\mu\nu} + {}_R \Psi_{\mu\nu} = {}_R S_{\mu\nu}$. The equation before changes to

$${}_R H_{\mu\nu} + {}_R \Psi_{\mu\nu} = {}_R S_{\mu\nu} \tag{245}$$

After a commutative division of the equation before, the normalization of the relationship between the energy tensor ${}_R H_{\mu\nu}$ and the tensor of the wave function ${}_R \Psi_{\mu\nu}$ follows as

$$\frac{{}_R H_{\mu\nu}}{{}_R S_{\mu\nu}} + \frac{{}_R \Psi_{\mu\nu}}{{}_R S_{\mu\nu}} = +1_{\mu\nu} \tag{246}$$

Quod erat demonstrandum.

3.10. Theorem. The Relationship between the Wave Function Tensor ${}_R\Psi_{\mu\nu}$ and the Tensor of the Gravitational Field ${}_R g_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

In general, the tensor of the gravitational field ${}_R g_{\mu\nu}$ is determined as

$${}_R g_{\mu\nu} \equiv \frac{{}_R \Psi_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \tag{247}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{248}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{249}$$

or too

$$1_{\mu\nu} = 1_{\mu\nu} \tag{250}$$

Due to a theorem before it is $\left({}_R M_{\mu\nu} : {}_R S_{\mu\nu}\right) + \left({}_R Y_{\mu\nu} : {}_R S_{\mu\nu}\right) = 1_{\mu\nu}$. The equation before changes too

$$+1_{\mu\nu} = \frac{{}_R H_{\mu\nu}}{{}_R S_{\mu\nu}} + \frac{{}_R \Psi_{\mu\nu}}{{}_R S_{\mu\nu}} \tag{251}$$

Due to another theorem before it is $\left({}_R g_{\mu\nu} : U_{\mu\nu}\right) + \left({}_R M_{\mu\nu} : U_{\mu\nu}\right) = 1_{\mu\nu}$. The equation before changes too

$$\frac{{}_R g_{\mu\nu}}{U_{\mu\nu}} + \frac{{}_R M_{\mu\nu}}{U_{\mu\nu}} = \frac{{}_R H_{\mu\nu}}{{}_R S_{\mu\nu}} + \frac{{}_R \Psi_{\mu\nu}}{{}_R S_{\mu\nu}} \tag{252}$$

Multiplying this equation by $U_{\mu\nu}$, it is

$$ {}_{R}g_{\mu\nu} + {}_{R}M_{\mu\nu} = \left(\frac{U_{\mu\nu}}{{}_{R}S_{\mu\nu}} \cap {}_{R}H_{\mu\nu}\right) + \left(\frac{U_{\mu\nu}}{{}_{R}S_{\mu\nu}} \cap {}_{R}\Psi_{\mu\nu}\right) \tag{253} $$

According to our definition, it is ${}_{R}S_{\mu\nu} = \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}\right) \cap U_{\mu\nu}$. Thus far, it is $\left(1_{\mu\nu} / \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}\right)\right) = U_{\mu\nu} / {}_{R}S_{\mu\nu}$. The equation before changes to

$$ {}_{R}g_{\mu\nu} + {}_{R}M_{\mu\nu} = \frac{{}_{R}H_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} + \frac{{}_{R}\Psi_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{254} $$

Due to our definition of matter as ${}_{R}M_{\mu\nu} = {}_{R}H_{\mu\nu} / \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}\right)$, equation before changes to

$$ {}_{R}g_{\mu\nu} + {}_{R}M_{\mu\nu} = {}_{R}M_{\mu\nu} + \frac{{}_{R}\Psi_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{255} $$

Subtracting the tensor of matter ${}_{R}M_{\mu\nu}$ on both sides of the equation before, the tensor of the gravitational field ${}_{R}g_{\mu\nu}$ follows as

$$ {}_{R}g_{\mu\nu} = \frac{{}_{R}\Psi_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{256} $$

Quod erat demonstrandum.

3.11. Theorem. The Equivalence of the Tensor of Time ${}_{R}t_{\mu\nu}$ and the Tensor of the Wave Function ${}_{R}\Psi_{\mu\nu}$

Claim. (Theorem. Proposition. Statement.)

In general it is

$$ {}_{R}t_{\mu\nu} = {}_{R}\Psi_{\mu\nu} \tag{257} $$

Direct proof.

Starting with axiom I, we obtain

$$ +1 = +1 \tag{258} $$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{259} $$

or too

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{260} $$

A commutative multiplication by the tensor of the gravitational field ${}_{R}g_{\mu\nu}$, we obtain

$$ {}_{R}g_{\mu\nu} \cap 1_{\mu\nu} = {}_{R}g_{\mu\nu} \cap 1_{\mu\nu} \tag{261} $$

Due to a theorem before, it is ${}_{R}g_{\mu\nu} = {}_{R}t_{\mu\nu} / \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}\right)$. We obtain

$$ \frac{{}_{R}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} = {}_{R}g_{\mu\nu} \tag{262} $$

According to another theorem before, it is ${}_{R}g_{\mu\nu} = {}_{R}\Psi_{\mu\nu} / \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}\right)$. Rearranging equation, we obtain

$$ \frac{{}_{R}t_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} = \frac{{}_{R}\Psi_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \tag{263} $$

Rearranging equation yields

$$ {}_{R}t_{\mu\nu} = {}_{R}\Psi_{\mu\nu} \tag{264} $$

Quod erat demonstrandum.

3.12. Theorem. The Generally Covariant form of Schrödinger's Equation

Let us suppose that the classical Einstein's field equation holds at the fundamental level too. Under these circumstances, the Einstein's field equations can be rewritten explicitly as a wave equation. In order to geometrize the matter field in general, it is useful to bring Schrödinger's quantum mechanical "wave equation" into a generally covariant form.

Claim. (Theorem. Proposition. Statement.)

In general, the generally covariant form of Schrödinger's equation is determined by the equation

$$i_{\mu\nu} \cap {}_R\hbar_{\mu\nu} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap {}_R\Psi_{\mu\nu} \equiv {}_RH_{\mu\nu} \cap {}_R\Psi_{\mu\nu} \tag{265}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{266}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{267}$$

or too

$$1_{\mu\nu} = 1_{\mu\nu} \tag{268}$$

A *commutative* multiplication by ${}_RH_{\mu\nu} \cap {}_R\Psi_{\mu\nu}$ yields

$${}_RH_{\mu\nu} \cap {}_R\Psi_{\mu\nu} \equiv {}_RH_{\mu\nu} \cap {}_R\Psi_{\mu\nu} \tag{269}$$

Due to our definition it is ${}_RH_{\mu\nu} \equiv i_{\mu\nu} \cap {}_R\hbar_{\mu\nu} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu}$. Substituting this equation into the equation before, we obtain the generally covariant form of Schrödinger's equation as

$$i_{\mu\nu} \cap {}_R\hbar_{\mu\nu} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap {}_R\Psi_{\mu\nu} \equiv {}_RH_{\mu\nu} \cap {}_R\Psi_{\mu\nu} \tag{270}$$

Quod erat demonstrandum.

Scholium.

A methodological important point in the process of the establishment of field equations for the unified field theory is the relationship between quantum theory and (classical) field theory. The basic assumptions of quantum mechanics (QM) and general relativity (GR) contradict each other. Even general relativity (GR) is not free of inconsistencies. According to the singularity theorem of Hawking and Penrose (1970) near singularities the pure classical theory of general relativity becomes incomplete and inconsistent. Thus far, attempts to quantize gravity have encountered fundamental difficulties. In this context, with regard to the unified field theory, an extension of general relativity, this trial to bridge the gap between quantum theory and (classical) field theory yields the derivation of quantum theory as a consequence of the unified field theory. A satisfactory quantization of the gravitational field still remains to be achieved.

3.13. Theorem. The Quantization of the Gravitational Field

Claim. (Theorem. Proposition. Statement.)

In general, the quantization of the gravitational field is determined by the equation

$${}_RM_{\mu\nu} \cap {}_Rg_{\mu\nu} = \frac{i_{\mu\nu} \cap {}_R\hbar_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap \frac{{}_R\Psi_{\mu\nu}}{{}_Rc_{\mu\nu} \cap {}_Rc_{\mu\nu}} \tag{271}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{272}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{273}$$

or too

$$1_{\mu\nu} = 1_{\mu\nu} \tag{274}$$

A *commutative* multiplication by ${}_R H_{\mu\nu} \cap {}_R \Psi_{\mu\nu}$ yields

$${}_R H_{\mu\nu} \cap {}_R \Psi_{\mu\nu} \equiv {}_R H_{\mu\nu} \cap {}_R \Psi_{\mu\nu} \tag{275}$$

Due to a theorem before, this equation is equivalent with

$${}_R H_{\mu\nu} \cap {}_R \Psi_{\mu\nu} \equiv i_{\mu\nu} \cap {}_R \hbar_{\mu\nu} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap {}_R \Psi_{\mu\nu} \tag{276}$$

Dividing by the speed of the light squared, we obtain

$$\frac{{}_R H_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \cap \frac{{}_R \Psi_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} = \frac{i_{\mu\nu} \cap \hbar_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap \frac{{}_R \Psi_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \tag{277}$$

Due to our definition of matter it is ${}_R M_{\mu\nu} = {}_R H_{\mu\nu} / \left({}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}\right)$. The equation before changes to

$${}_R M_{\mu\nu} \cap \frac{{}_R \Psi_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} = \frac{i_{\mu\nu} \cap {}_R \hbar_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap \frac{{}_R \Psi_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \tag{278}$$

Due to a theorem before it is ${}_R g_{\mu\nu} = {}_R \Psi_{\mu\nu} / \left({}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}\right)$. The quantization of the gravitational field follows as

$${}_R M_{\mu\nu} \cap {}_R g_{\mu\nu} = \frac{i_{\mu\nu} \cap {}_R \hbar_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \cap \left(\frac{\partial}{\partial t}\right)_{\mu\nu} \cap \frac{{}_R \Psi_{\mu\nu}}{{}_R c_{\mu\nu} \cap {}_R c_{\mu\nu}} \tag{279}$$

Quod erat demonstrandum.

3.14. Theorem. The Tensor of Time ${}_R t_{\mu\nu}$

The tensor of time ${}_R t_{\mu\nu}$ under conditions of Einstein's general theory of relativity theory is determined by the equation

$${}_R t_{\mu\nu} = \left(\frac{R}{2} \cap g_{\mu\nu}\right) - \left(\Lambda \cap g_{\mu\nu}\right) \tag{280}$$

Claim.

In general, axiom I is determined as

$$+1 = +1 \tag{281}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{282}$$

or too

$$1_{\mu\nu} = 1_{\mu\nu} \tag{283}$$

A *commutative* multiplication of this equation by Einstein's stress energy tensor leads to

$$1_{\mu\nu} \cap {}_R E_{\mu\nu} = {}_R E_{\mu\nu} \cap 1_{\mu\nu} \tag{284}$$

or to

$${}_R E_{\mu\nu} = {}_R E_{\mu\nu} \tag{285}$$

which is equivalent with Einstein's field equation as

$$\left(R_{\mu\nu} - \frac{R}{2} \cap g_{\mu\nu}\right) + \left(\Lambda \cap g_{\mu\nu}\right) = {}_R E_{\mu\nu} \tag{286}$$

Rearranging equation, we obtain

$$R_{\mu\nu} = {}_R E_{\mu\nu} + \left(\frac{R}{2} \cap g_{\mu\nu}\right) - \left(\Lambda \cap g_{\mu\nu}\right) \tag{287}$$

Under conditions of general relativity, the tensor of space ${}_{R}S_{\mu\nu}$ is equivalent with the Ricci tensor $R_{\mu\nu}$. Thus far we equate ${}_{R}S_{\mu\nu} = R_{\mu\nu}$ and do obtain

$$S_{\mu\nu} = {}_{R}E_{\mu\nu} + \left(\frac{R}{2} \cap g_{\mu\nu}\right) - \left(\Lambda \cap g_{\mu\nu}\right) \tag{288}$$

In general, it is ${}_{R}S_{\mu\nu} = {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu}$. Rearranging equation before yields

$${}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = {}_{R}E_{\mu\nu} + \left(\frac{R}{2} \cap g_{\mu\nu}\right) - \left(\Lambda \cap g_{\mu\nu}\right) \tag{289}$$

In general, *under conditions of the theory of general theory*, the tensor of time ${}_{R}t_{\mu\nu}$ follows as

$${}_{R}t_{\mu\nu} = \left(\frac{R}{2} \cap g_{\mu\nu}\right) - \left(\Lambda \cap g_{\mu\nu}\right) \tag{290}$$

Quod erat demonstrandum.

3.15. Theorem. The Equivalence of Time and Gravitational Field

In general, the modification of our understanding of space and time undergone through Einstein's relativity theory is indeed a profound one. But even Einstein's relativity theory does not give satisfactory answers to a lot of questions. One of these questions is the problem of the "true" tensor of the gravitational field. The purpose of this publication is to provide some new and basic fundamental insights by the proof that the gravitational field and time is equivalent even under conditions of the general theory of relativity.

Einstein's successful geometrization of the gravitational field in his general theory of relativity does not include a geometrized theory of the electromagnetic field too. The theoretical physicists working in the field of the general theory of relativity were not able to succeed in finding a convincing geometrical formulation of the gravitational and electromagnetic field. Still, electromagnetic fields are not described by Riemannian metrics. More serious from the conceptual point of view, in order to achieve unification, with the development of quantum theory any conceptual unification of the gravitational and electromagnetic field should introduce a possibility that the fields can be quantized. In our striving toward unification of the foundations of physics a relativistic field theory we are looking for should therefore be an extension of the general theory of relativity and equally and of no less importance a generalization of the theory of the gravitational field. In the attempt to solve these problems one meets at least with another difficulty. Einstein was demanding that "the symmetrical tensor field must be replaced by a non-symmetrical one. This means that the condition $g_{ik} = g_{ki}$ for the field components must be dropped" [2].

Evidently, following up these trains of thoughts and in view of all these difficulties, the following theory is based on a (gravitational) field of more complex nature. Still, in our attempt to obtain a deeper knowledge of the foundations of physics the new and basic concepts are in accordance with general relativity theory from the beginning but with philosophy too. In general, energy, time and space are deeply related and interacting like the one with its own other and vice versa.

Claim.

The relationship between time and gravitational field is determined as

$${}_{R}t_{\mu\nu} = c^2 \times {}_{R}g_{\mu\nu} \tag{291}$$

Proof.

In general, axiom I is determined as

$$+1 = +1 \tag{292}$$

A *commutative* multiplication by the tensor of the unified field $1_{\mu\nu}$ leads to

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{293}$$

or too

$$1_{\mu\nu} = 1_{\mu\nu} \tag{294}$$

A *commutative* multiplication of this equation by ${}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu}$ yields

$$ {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} \tag{295}$$

Due to our definition, ${}_{R}S_{\mu\nu} \equiv {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu}$ it is ${}_{R}U_{\mu\nu} \equiv \frac{1_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap {}_{R}S_{\mu\nu} \equiv {}_{R}M_{\mu\nu} + {}_{R}g_{\mu\nu}$ and it follows that

$$ {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap \left({}_{R}M_{\mu\nu} + {}_{R}g_{\mu\nu} \right) = {}_{R}S_{\mu\nu} \tag{296}$$

Rearranging equation, it is as

$$ {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}M_{\mu\nu} \right) + \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}g_{\mu\nu} \right) \tag{297}$$

Due to the relationship ${}_{R}M_{\mu\nu} \equiv \frac{{}_{R}E_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \equiv \frac{4 \times 2 \times {}_{R}\pi_{\mu\nu} \times {}_{R}\gamma_{\mu\nu}}{\left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \right) \cap \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \right)} \cap T_{\mu\nu}$ it follows that

$$ {}_{R}E_{\mu\nu} + {}_{R}t_{\mu\nu} = {}_{R}E_{\mu\nu} + \left({}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}g_{\mu\nu} \right) \tag{298}$$

The equivalence of time and gravitational field follows in general as

$$ {}_{R}t_{\mu\nu} = {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}g_{\mu\nu} \tag{299}$$

Quod erat demonstrandum.

3.16. Theorem. The Generally Covariant form of Planck's-Einstein Relation

Claim.

In general, it is

$$ {}_{R}\hbar_{\mu\nu} \cap {}_{0}\omega_{\mu\nu} = {}_{R}h_{\mu\nu} \cap_{R} f_{\mu\nu} \tag{300}$$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{301}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{302}$$

or

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{303}$$

Multiplying this equation by the stress-energy tensor of general relativity $\left(4 \times 2 \times \pi \times \gamma / \left(c^4\right)\right) \times T_{\mu\nu}$, it is

$$ +1_{\mu\nu} \cap \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) = +1_{\mu\nu} \cap \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{304}$$

where γ is Newton's gravitational "constant"; c is the speed of light in vacuum and π, sometimes referred to as "Archimedes' constant", is the ratio of a circle's circumference to its diameter. Due to Einstein's general relativity, the equation before is equivalent with

$$ R_{\mu\nu} - \left(\frac{R}{2} \times g_{\mu\nu} \right) + \left(\Lambda \times g_{\mu\nu} \right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{305}$$

$R_{\mu\nu}$ is the Ricci curvature tensor; R is the scalar curvature; $g_{\mu\nu}$ is the metric tensor; Λ is the cosmological constant and $T_{\mu\nu}$ is the stress-energy tensor. By defining the Einstein tensor as $G_{\mu\nu} = R_{\mu\nu} - (R/2) \times g_{\mu\nu}$, it is possible to write the Einstein field equations in a more compact as

$$ G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu} \right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{306}$$

This equation can be rearranged as

$$\frac{{}_{R}\hbar_{\mu\nu}}{{}_{R}\hbar_{\mu\nu}} \cap \left(G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right)\right) = \frac{{}_{R}h_{\mu\nu}}{{}_{R}h_{\mu\nu}} \cap \left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap T_{\mu\nu}\right) \tag{307}$$

Simplifying equation we obtain

$${}_{R}\hbar_{\mu\nu} \cap \left(\frac{1_{\mu\nu}}{{}_{R}\hbar_{\mu\nu}} \cap \left(G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right)\right)\right) = {}_{R}h_{\mu\nu} \cap \left(\frac{1_{\mu\nu}}{{}_{R}h_{\mu\nu}} \cap \left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap T_{\mu\nu}\right)\right) \tag{308}$$

Due to our definitions before, the equation can be simplified as

$$\hbar_{\mu\nu} \cap {}_{0}\omega_{\mu\nu} = {}_{R}h_{\mu\nu} \cap \left(\frac{1_{\mu\nu}}{{}_{R}h_{\mu\nu}} \cap \left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{{}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu} \cap {}_{R}c_{\mu\nu}} \cap T_{\mu\nu}\right)\right) \tag{309}$$

and the generally covariant form of Planck's-Einstein relation follows as

$$\hbar_{\mu\nu} \cap {}_{0}\omega_{\mu\nu} = h_{\mu\nu} \cap_{R} f_{\mu\nu} \tag{310}$$

Quod erat demonstrandum.

3.17. Theorem. The Generally Covariant form of de Broglie Relationship

Claim.

The generally covariant form of de Broglie's relationship is determined as

$${}_{R}h_{\mu\nu} \equiv {}_{R}p_{\mu\nu} \cap {}_{R}\lambda_{\mu\nu} = \left(\frac{{}_{R}h_{\mu\nu}}{{}_{R}c_{\mu\nu}} \cap {}_{R}f_{\mu\nu}\right) \cap {}_{R}\lambda_{\mu\nu} \tag{311}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{312}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{313}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{314}$$

Multiplying this equation by ${}_{R}c_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap {}_{R}c_{\mu\nu} = 1_{\mu\nu} \cap {}_{R}c_{\mu\nu} \tag{315}$$

and at the end

$${}_{R}c_{\mu\nu} = {}_{R}c_{\mu\nu} \tag{316}$$

Due to our definition, it is

$${}_{R}c_{\mu\nu} = {}_{R}f_{\mu\nu} \cap {}_{R}\lambda_{\mu\nu} \tag{317}$$

This equation can be rearranged as

$$\frac{1_{\mu\nu}}{{}_{R}\lambda_{\mu\nu}} = \frac{{}_{R}f_{\mu\nu}}{{}_{R}c_{\mu\nu}} \tag{318}$$

Multiplying by ${}_{R}h_{\mu\nu}$, we obtain

$${}_{R}p_{\mu\nu} \equiv \frac{{}_{R}h_{\mu\nu}}{{}_{R}\lambda_{\mu\nu}} = \frac{{}_{R}h_{\mu\nu} \cap {}_{R}f_{\mu\nu}}{{}_{R}c_{\mu\nu}} = \frac{{}_{R}h_{\mu\nu}}{{}_{R}c_{\mu\nu}} \cap {}_{R}f_{\mu\nu} \tag{319}$$

where ${}_{R}p_{\mu\nu}$ denotes the tensor of the momentum. The generally covariant form of de Broglie's relationship follows as

$$ {}_{R}h_{\mu\nu} \equiv {}_{R}p_{\mu\nu} \cap {}_{R}\lambda_{\mu\nu} = \left(\frac{{}_{R}h_{\mu\nu}}{{}_{R}c_{\mu\nu}} \cap {}_{R}f_{\mu\nu} \right) \cap {}_{R}\lambda_{\mu\nu} \tag{320} $$

Quod erat demonstrandum.

3.18. Theorem. The Tensor of "Ordinary" Matter ${}_{0}E_{\mu\nu}$

Claim.

In general, *the tensor of ordinary matter* ${}_{0}E_{\mu\nu}$ follows as

$$ A_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \equiv \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) - \left(\frac{1_{\mu\nu}}{4 \times \pi} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right) \tag{321} $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{322} $$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{323} $$

or

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{324} $$

Multiplying this equation by the stress-energy tensor of general relativity $\left(4 \times 2 \times \pi \times \gamma / \left(c^4 \right) \right) \times T_{\mu\nu}$, it is

$$ +1_{\mu\nu} \cap \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) = +1_{\mu\nu} \cap \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{325} $$

or

$$ \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{326} $$

Due our definition this is equivalent with

$$ A_{\mu\nu} + B_{\mu\nu} = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{327} $$

and at the end

$$ A_{\mu\nu} = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) - B_{\mu\nu} \tag{328} $$

Due to our definition it is $B_{\mu\nu} = \left(1/(4 \times \pi) \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left((1/4) \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right)$. The equation changes to

$$ A_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \equiv \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) - \left(\frac{1_{\mu\nu}}{4 \times \pi} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right) \tag{329} $$

Quod erat demonstrandum.

Scholium.

Under conditions of general theory of relativity, the associated probability tensor, the "joint distribution" tensor between the tensor of energy ${}_{R}E_{\mu\nu}$ and Einstein's tensor $G_{\mu\nu}$, follows as

$$ \begin{aligned} & p\left(A_{\mu\nu} \right) \equiv p\left({}_{0}E_{\mu\nu} \right) \equiv p\left({}_{R}E_{\mu\nu}, G_{\mu\nu} \right) \\ & \equiv \frac{\left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) - \left(\frac{1_{\mu\nu}}{4 \times \pi} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right)}{R_{\mu\nu}} \equiv \frac{{}_{0}E_{\mu\nu}}{R_{\mu\nu}} \end{aligned} \tag{330} $$

The tensor of ordinary matter ${}_{0}M_{\mu\nu}$ is determined as

$$ {}_{0}M_{\mu\nu} \equiv \frac{A_{\mu\nu}}{c^2} \equiv \frac{{}_{0}E_{\mu\nu}}{c^2} \equiv \left(\frac{1}{c^2}\right) \times \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) - \left(\frac{1}{c^2}\right) \times \left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) \quad (331) $$

3.19. Theorem. The Probability Tensor Associated with "Ordinary" Matter ${}_{0}E_{\mu\nu}$

Claim.

In general, of ordinary matter follows as

$$ p\left(A_{\mu\nu}\right) \equiv p\left({}_{0}E_{\mu\nu}\right) \equiv A_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_{0}E_{\mu\nu}}{R_{\mu\nu}} \quad (332) $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \quad (333) $$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \quad (334) $$

or

$$ 1_{\mu\nu} = 1_{\mu\nu} \quad (335) $$

Multiplying this equation by $A_{\mu\nu}$, it is

$$ A_{\mu\nu} \equiv A_{\mu\nu} \quad (336) $$

or in general to

$$ A_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \quad (337) $$

Multiplying by the tensor ${}_{R}Y_{\mu\nu}$ it is

$$ A_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \quad (338) $$

The commutative multiplication with the tensor ${}_{R}Y_{\mu\nu}$ yields the probability tensor as associated with the tensor $A_{\mu\nu}$.

$$ p\left(A_{\mu\nu}\right) \equiv A_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \quad (339) $$

Due to our theorem before, it is ${}_{R}Y_{\mu\nu} = {}_{R}^{*}\Psi_{\mu\nu} = \left(1_{\mu\nu} / R_{\mu\nu}\right)$. The equation before simplifies as

$$ p\left(A_{\mu\nu}\right) \equiv p\left({}_{0}E_{\mu\nu}\right) \equiv A_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}E_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv_{0} E_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_{0}E_{\mu\nu}}{R_{\mu\nu}} \quad (340) $$

Quod erat demonstrandum.

3.20. Theorem. The Stress-Energy Tensor of the Electromagnetic Field $B_{\mu\nu}$

Claim.

In general, it is

$$ B_{\mu\nu} \equiv {}_{0}\underline{E}_{\mu\nu} \equiv \left(\left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)\right) \quad (341) $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \quad (342) $$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \quad (343)$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \quad (344)$$

Multiplying this equation by $B_{\mu\nu}$, it is

$$B_{\mu\nu} \equiv B_{\mu\nu} \quad (345)$$

We defined $B_{\mu\nu} = (1/(4\times\pi))\times((F_{\mu c}\times F_{\nu}^{\ c}) - ((1/4)\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}))$ where denotes the stress energy tensor of the electromagnetic field. In general, we obtain

$$B_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \equiv \left(\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right) \quad (346)$$

Quod erat demonstrandum.

3.21. Theorem. The Probability Tensor as Associated with the Electromagnetic Field ${}_0\underline{E}_{\mu\nu}$

Claim.

In general, it is

$$p(B) \equiv p\left({}_0\underline{E}_{\mu\nu}\right) \equiv B_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_0\underline{E}_{\mu\nu}}{R_{\mu\nu}} \quad (347)$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \quad (348)$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \quad (349)$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \quad (350)$$

Multiplying this equation by $B_{\mu\nu}$, it is

$$B_{\mu\nu} \equiv B_{\mu\nu} \quad (351)$$

or in general to

$$B_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \equiv \left(\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right) \quad (352)$$

where $B_{\mu\nu} = (1/(4\times\pi))\times((F_{\mu c}\times F_{\nu}^{\ c}) - ((1/4)\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}))$ denotes the stress energy tensor of the electromagnetic field. Multiplying by the tensor ${}_R Y_{\mu\nu}$ it is

$$B_{\mu\nu} \cap {}_R Y_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \cap {}_R Y_{\mu\nu} \quad (353)$$

The *commutative multiplication* with the tensor ${}_R Y_{\mu\nu}$ yields the probability tensor as associated with the tensor $B_{\mu\nu}$.

$$p\left(B_{\mu\nu}\right) \equiv B_{\mu\nu} \cap {}_R Y_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \cap {}_R Y_{\mu\nu} \quad (354)$$

Due to our theorem before, it is ${}_R Y_{\mu\nu} = {}_R^*\Psi_{\mu\nu} = \left(1_{\mu\nu}/R_{\mu\nu}\right)$. The equation before simplifies as

$$p(B) \equiv p\left({}_0\underline{E}_{\mu\nu}\right) \equiv B_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0\underline{E}_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_0\underline{E}_{\mu\nu}}{R_{\mu\nu}} \quad (355)$$

Quod erat demonstrandum.

Scholium.

Due to Einstein's theory of gravitation the stress-energy tensor of the electromagnetic field is a field devoid of any geometrical significance. An additional task of this approach to the unified field theory is the possibility to "geometrize" the electromagnetic field. A geometrical tensorial representation of the electro-magnetic field under conditions of the general theory of relativity within the framework of a "unified field theory" follows as

$$\left(\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)\equiv p\left(B\right)\cap R_{\mu\nu}\equiv p\left({}_{0}\underline{E}_{\mu\nu}\right)\cap R_{\mu\nu} \tag{356}$$

3.22. Theorem. The Relationship between the Gravitational and the Electromagnetic Field

Claim.

In general, it is

$$C_{\mu\nu}+\left(\Lambda\times g_{\mu\nu}\right)=B_{\mu\nu} \tag{357}$$

Direct proof.

In general, axiom I is determined as

$$+1=+1 \tag{358}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu}\cap 1=1_{\mu\nu}\cap 1 \tag{359}$$

or

$$1_{\mu\nu}=1_{\mu\nu} \tag{360}$$

Multiplying this equation by the stress-energy tensor of general relativity $\left(4\times 2\times\pi\times\gamma/\left(c^{4}\right)\right)\times T_{\mu\nu}$, it is

$$+1_{\mu\nu}\cap\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right)=+1_{\mu\nu}\cap\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right) \tag{361}$$

Due to Einstein's general relativity, the equation before is equivalent with

$$R_{\mu\nu}-\left(\frac{R}{2}\times g_{\mu\nu}\right)+\left(\Lambda\times g_{\mu\nu}\right)=\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right) \tag{362}$$

By defining the Einstein tensor as $G_{\mu\nu}=R_{\mu\nu}-\left(R/2\right)\times g_{\mu\nu}$, it is possible to write the Einstein field equations in a more compact as

$$G_{\mu\nu}+\left(\Lambda\times g_{\mu\nu}\right)=\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right) \tag{363}$$

According to our definition, under conditions of general relativity it is $A_{\mu\nu}+C_{\mu\nu}\equiv{}_{0}W_{\mu\nu}\equiv G_{\mu\nu}$
Substituting this relationship into Einstein's field equation, we obtain

$$A_{\mu\nu}+C_{\mu\nu}+\left(\Lambda\times g_{\mu\nu}\right)=\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right) \tag{364}$$

Under conditions of general relativity it is $A_{\mu\nu}+B_{\mu\nu}\equiv\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right)$

Substituting this relationship into Einstein's field equation, we obtain

$$A_{\mu\nu}+C_{\mu\nu}+\left(\Lambda\times g_{\mu\nu}\right)=A_{\mu\nu}+B_{\mu\nu} \tag{365}$$

We defined $B_{\mu\nu}$ as the second rank covariant stress-energy tensor of the electromagnetic field in the absence of "ordinary" matter and $C_{\mu\nu}$ as the tensor of time (*i.e.* gravitational field) as associated with the tensor $A_{\mu\nu}$. This equation before can be rearranged as

$$C_{\mu\nu}+\left(\Lambda\times g_{\mu\nu}\right)=B_{\mu\nu} \tag{366}$$

Quod erat demonstrandum.

Scholium.

The following 2 × 2 table may illustrate the equation before (**Table 4**).

Table 4. The unified field $_{R}W_{\mu\nu}$.

		Curvature		
		yes	no	
Energy/momentum	yes	$A_{\mu\nu}$	$B_{\mu\nu}$	$_{R}U_{\mu\nu}$
	no	$C_{\mu\nu}$	$D_{\mu\nu}$	$_{R}\underline{U}_{\mu\nu}$
		$_{0}W_{\mu\nu}$	$_{0}\underline{W}_{\mu\nu}$	$_{R}W_{\mu\nu}$

3.23. Theorem. The Tensor of Time $_{0}t_{\mu\nu}$ as Associated with Ordinary Energy Tensor $_{0}E_{\mu\nu}$

Claim.

In general, the tensor of time $_{0}t_{\mu\nu}$ as associated with ordinary energy $_{0}E_{\mu\nu}$ follows as

$$C_{\mu\nu} \equiv {}_{0}t_{\mu\nu} \equiv +\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)-\Lambda\times g_{\mu\nu} \tag{367}$$

Direct proof.

In general, axiom I is determined as

$$+1=+1 \tag{368}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu}\cap 1=1_{\mu\nu}\cap 1 \tag{369}$$

or

$$1_{\mu\nu}=1_{\mu\nu} \tag{370}$$

Multiplying this equation by $C_{\mu\nu}$ we obtain

$$C_{\mu\nu}=C_{\mu\nu} \tag{371}$$

Due to our definition, we rearrange this equation to

$$C_{\mu\nu}\equiv G_{\mu\nu}-A_{\mu\nu} \tag{372}$$

We define $A_{\mu\nu}=\left(4\times 2\times\pi\times\gamma/\left(c^{4}\right)\right)\times T_{\mu\nu}-B_{\mu\nu}$. The equation before changes too

$$C_{\mu\nu}\equiv G_{\mu\nu}-A_{\mu\nu}\equiv G_{\mu\nu}-\left(\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right)-\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right) \tag{373}$$

or to

$$C_{\mu\nu}\equiv G_{\mu\nu}-\left(\frac{4\times 2\times\pi\times\gamma}{c^{4}}\times T_{\mu\nu}\right)+\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right) \tag{374}$$

The tensor of time $_{0}t_{\mu\nu}$ as associated with ordinary energy $_{0}E_{\mu\nu}$ follows as

$$C_{\mu\nu} \equiv {}_{0}t_{\mu\nu} \equiv +\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)-\Lambda\times g_{\mu\nu} \tag{375}$$

Quod erat demonstrandum.

3.24. Theorem. The Probability Tensor as Associated with the Tensor $_{0}t_{\mu\nu}$

Claim.

In general, it is

$$p(C) \equiv p({}_0t_{\mu\nu}) \equiv C_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0t_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0t_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_0t_{\mu\nu}}{R_{\mu\nu}} \tag{376}$$

Direct proof.
In general, axiom I is determined as

$$+1 = +1 \tag{377}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{378}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{379}$$

Multiplying this equation by $C_{\mu\nu}$, it is

$$C_{\mu\nu} \equiv C_{\mu\nu} \tag{380}$$

or in general to

$$C_{\mu\nu} \equiv {}_0t_{\mu\nu} \equiv \left(\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)-\Lambda\times g_{\mu\nu} \tag{381}$$

where $B_{\mu\nu} = (1/(4\times\pi))\times((F_{\mu c}\times F_{\nu}{}^{c})-((1/4)\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}))$ denotes the stress energy tensor of the electromagnetic field and $\Lambda \times g_{\mu\nu}$ denotes the cosmological "constant" Λ times the metric $g_{\mu\nu}$ term. Multiplying by the tensor ${}_RY_{\mu\nu}$ it is

$$C_{\mu\nu} \cap {}_RY_{\mu\nu} \equiv {}_0t_{\mu\nu} \cap {}_RY_{\mu\nu} \tag{382}$$

The *commutative multiplication* with the tensor ${}_RY_{\mu\nu}$ yields the probability tensor as associated with the tensor $C_{\mu\nu}$.

$$p(C_{\mu\nu}) \equiv C_{\mu\nu} \cap {}_RY_{\mu\nu} \equiv {}_0t_{\mu\nu} \cap {}_RY_{\mu\nu} \tag{383}$$

Due to our theorem before, it is ${}_RY_{\mu\nu} = {}_R^*\Psi_{\mu\nu} = (1_{\mu\nu}/R_{\mu\nu})$. The equation before simplifies as

$$p(C) \equiv p({}_0t_{\mu\nu}) \equiv C_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0t_{\mu\nu} \cap {}_R^*\Psi_{\mu\nu} \equiv {}_0t_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_0t_{\mu\nu}}{R_{\mu\nu}} \tag{384}$$

Quod erat demonstrandum.
Scholium.
Under conditions of general theory of relativity, the associated probability tensor follows as

$$p(C_{\mu\nu}) \equiv p({}_0t_{\mu\nu}) \equiv \frac{+\left(\frac{1_{\mu\nu}}{4\times\pi}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)-\Lambda\times g_{\mu\nu}}{R_{\mu\nu}} \equiv \left(\frac{{}_0t_{\mu\nu}}{R_{\mu\nu}}\right) \tag{385}$$

3.25. Theorem. The Tensor of the Gravitational and the Electromagnetic Hyper-Field

Einstein himself spent decades of his life on the unification of the electromagnetic with the gravitational and other physical fields. Even from Einstein's and other failed attempts at unification the hunt for progress for reaching a common representation of all four fundamental interactions in the framework of "unified field theory" is justified. In all the attempts at unification we encounter that electromagnetic fields and gravitational are to be joined into a new field. Tonnelat points out: "a theory joining the gravitational and the electromagnetic field into one single hyper field whose equations represent the conditions imposed on the geometrical structure of the universe" [4].

Claim.
In general, the tensor of the gravitational and the electromagnetic hyper-field is determined as

$$2 \times C_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = C_{\mu\nu} + B_{\mu\nu} \tag{386}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{387}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{388}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{389}$$

Multiplying this equation by the tensor $B_{\mu\nu}$ we obtain

$$1_{\mu\nu} \cap B_{\mu\nu} = 1_{\mu\nu} \cap B_{\mu\nu} \tag{390}$$

or

$$B_{\mu\nu} = B_{\mu\nu} \tag{391}$$

Due to the theorem before, the equation before changes to

$$C_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = B_{\mu\nu} \tag{392}$$

Adding $C_{\mu\nu}$, the tensor of time (*i.e.* gravitational field) as associated with the tensor $A_{\mu\nu}$, we obtain

$$C_{\mu\nu} + C_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = C_{\mu\nu} + B_{\mu\nu} \tag{393}$$

or at the end the tensor of the gravitational and the electromagnetic hyper-field

$$2 \times C_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = C_{\mu\nu} + B_{\mu\nu} \tag{394}$$

Quod erat demonstrandum.

Scholium.

Under conditions of general theory of relativity, the gravitational and the electromagnetic hyper-field is determined as

$$C_{\mu\nu} + B_{\mu\nu} \equiv 2 \times C_{\mu\nu} + \Lambda \times g_{\mu\nu} \equiv 2 \times \left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) - \Lambda \times g_{\mu\nu} \tag{395}$$

It is very easy to get lost in the many different attempts by Weyl, Kaluza, Eddington, Bach, Einstein and other to include the electromagnetic field into a geometric setting. The point of departure to "geometrize" the electromagnetic field was general relativity. In view of the immense amount of material, neither a brief technical descriptions of the various unified field theories nor all the contributions from the various scientific schools to unify the electromagnetic and gravitational field can be discussed with the same intensity. The joining of previously separated electromagnetic and gravitational field within one conceptual and formal second rank tensor is based on a deductive-hypothetical methodological approach. Einstein himself spent decades of his life on the unification of the electromagnetic with the gravitational field. Mie, Hilbert, Ishiwara, Nordström and others joined Einstein in his unsuccessful hunt for progress on this matter. In contrast to Kaluza's geometrization of the electromagnetic and gravitational fields within a five-dimensional space, this approach is based completely within the conceptual and formal framework of general relativity. Under conditions of general theory of relativity, the associated probability tensor follows as

$$p\left(C_{\mu\nu} + B_{\mu\nu}\right) \equiv \frac{\left(2_{\mu\nu} \cap C_{\mu\nu}\right) + \Lambda \times g_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{2 \times \left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) - \Lambda \times g_{\mu\nu}}{R_{\mu\nu}} \tag{396}$$

3.26. Theorem. The Tensor ${}_{0}\underline{t}_{\mu\nu}$

Claim.

In general, the tensor $D_{\mu\nu} = {}_0\underline{t}_{\mu\nu}$ as associated with the stress energy tensor of the electromagnetic field $B_{\mu\nu} = {}_0\underline{E}_{\mu\nu}$ follows as

$$D_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \equiv \left(\frac{R}{2}\right) \times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) \tag{397}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{398}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{399}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{400}$$

Multiplying this equation by the Ricci tensor $R_{\mu\nu}$ we obtain

$$1_{\mu\nu} \cap R_{\mu\nu} = 1_{\mu\nu} \cap R_{\mu\nu} \tag{401}$$

or

$$R_{\mu\nu} = R_{\mu\nu} \tag{402}$$

Adding $0_{\mu\nu}$, it is

$$R_{\mu\nu} = R_{\mu\nu} + 0_{\mu\nu} \tag{403}$$

The zero tensor is equivalent to $0_{\mu\nu} = +\left((R/2) \times g_{\mu\nu}\right) - \left((R/2) \times g_{\mu\nu}\right)$. We rearrange the equation before as

$$R_{\mu\nu} = R_{\mu\nu} - \left(\frac{R}{2}\right) \times g_{\mu\nu} + \left(\frac{R}{2}\right) \times g_{\mu\nu} \tag{404}$$

Einstein's tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - (R/2) \times g_{\mu\nu}$. We simplify the equation before as

$$R_{\mu\nu} = G_{\mu\nu} + \left(\frac{R}{2}\right) \times g_{\mu\nu} = G_{\mu\nu} + \underline{G}_{\mu\nu} \tag{405}$$

Due to our definition, it is $R_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu}$ and $G_{\mu\nu} = A_{\mu\nu} + C_{\mu\nu}$. We rearrange the equation before as

$$A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu} = A_{\mu\nu} + C_{\mu\nu} + \left(\frac{R}{2}\right) \times g_{\mu\nu} \tag{406}$$

Simplifying equation, it follows that

$$B_{\mu\nu} + D_{\mu\nu} = \left(\frac{R}{2}\right) \times g_{\mu\nu} \tag{407}$$

or

$$D_{\mu\nu} = \left(\frac{R}{2}\right) \times g_{\mu\nu} - B_{\mu\nu} \tag{408}$$

Due to the decomposition of the stress-energy tensor as $\left(4 \times 2 \times \pi \times \gamma / \left(c^4\right)\right) \times T_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu}$, the stress-energy tensor of the electromagnetic field is $B_{\mu\nu} = \left(1/(4 \times \pi)\right) \times \left(\left(F_{\mu c} \times F_{\nu}^{c}\right) - \left((1/4) \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)$. Under conditions of general relativity, the tensor $D_{\mu\nu} = {}_0\underline{t}_{\mu\nu}$ as associated with the stress energy tensor of the electromagnetic field $B_{\mu\nu} = {}_0\underline{E}_{\mu\nu}$ follows as

$$D_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \equiv \left(\frac{R}{2}\right) \times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) \tag{409}$$

Quod erat demonstrandum.

3.27. Theorem. The Probability Tensor as Associated with the Tensor ${}_0\underline{t}_{\mu\nu}$

Claim.

In general, it is

$$p(D) \equiv p\left({}_0\underline{t}_{\mu\nu}\right) \equiv D_{\mu\nu} \cap {}_R^{*}\Psi_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \cap {}_R^{*}\Psi_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_0\underline{t}_{\mu\nu}}{R_{\mu\nu}} \tag{410}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{411}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{412}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{413}$$

Multiplying this equation by $D_{\mu\nu}$, it is

$$D_{\mu\nu} \equiv D_{\mu\nu} \tag{414}$$

or in general to

$$D_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \equiv \left(\left(\frac{R}{2}\right) \times g_{\mu\nu}\right) - \left(\left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}^{\ c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)\right) \tag{415}$$

where $B_{\mu\nu} = (1/(4 \times \pi)) \times \left(\left(F_{\mu c} \times F_{\nu}^{\ c}\right) - \left((1/4) \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)$ denotes the stress energy tensor of the electromagnetic field and $\Lambda \times g_{\mu\nu}$ denotes the cosmological "constant" Λ times the metric $g_{\mu\nu}$ term. Multiplying by the tensor ${}_R Y_{\mu\nu}$ it is

$$D_{\mu\nu} \cap {}_R Y_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \cap {}_R Y_{\mu\nu} \tag{416}$$

The *commutative multiplication* with the tensor ${}_R Y_{\mu\nu}$ yields the probability tensor as associated with the tensor $D_{\mu\nu}$.

$$p\left(D_{\mu\nu}\right) \equiv p\left({}_0\underline{t}_{\mu\nu}\right) \equiv D_{\mu\nu} \cap {}_R Y_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \cap {}_R Y_{\mu\nu} \tag{417}$$

Due to our theorem before, it is ${}_R Y_{\mu\nu} = {}_R^{*}\Psi_{\mu\nu} = \left(1_{\mu\nu}/R_{\mu\nu}\right)$. The equation before simplifies as

$$p(D) \equiv p\left({}_0\underline{t}_{\mu\nu}\right) \equiv D_{\mu\nu} \cap {}_R^{*}\Psi_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \cap {}_R^{*}\Psi_{\mu\nu} \equiv {}_0\underline{t}_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{{}_0\underline{t}_{\mu\nu}}{R_{\mu\nu}} \tag{418}$$

Quod erat demonstrandum.

Scholium.

Under conditions of general theory of relativity, the associated probability tensor follows as

$$p\left(D_{\mu\nu}\right) \equiv p\left({}_0\underline{t}_{\mu\nu}\right) \equiv \frac{\left(\frac{R}{2}\right) \times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4 \times \pi}\right) \times \left(\left(F_{\mu c} \times F_{\nu}^{\ c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)}{R_{\mu\nu}} \tag{419}$$

3.28. Theorem. The Tensor ${}_W g_{\mu\nu}$

Still, one of the major unsolved problems in physics is the unification of gravity with all the other interactions of nature. Such a unification would have to provide a theoretical framework of a theory of everything which fully would explain and link together all physical aspects of objective reality. Einstein's theoretical framework of the theory of general relativity focuses mostly on gravity as being curvature of space time. The curvature of space time is expressed mathematically using the metric tensor—denoted $g_{\mu\nu}$. Curvature itself is caused by the presence of energy/matter and accelerating energy/matter generate changes in this curvature. Changes in the curvature of space time propagate in a wave-like manner and are known as gravitational waves.

Claim.

In general, under conditions of general relativity, gravitational waves are determined by the equation

$$ {}_{W}g_{\mu\nu} = \frac{{}_{W}t_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu}} = \frac{R}{2_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}} \times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4_{\mu\nu} \times \pi_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}} \right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c} \right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu} \right) \right) \tag{420} $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{421} $$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{422} $$

or

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{423} $$

Multiplying this equation by the stress-energy tensor of general relativity $\left(4 \times 2 \times \pi \times \gamma / \left(c^4\right)\right) \times T_{\mu\nu}$, it is

$$ +1_{\mu\nu} \cap \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) = +1_{\mu\nu} \cap \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{424} $$

Due to Einstein's general relativity, the equation before is equivalent with

$$ R_{\mu\nu} - \left(\frac{R}{2} \times g_{\mu\nu} \right) + \left(\Lambda \times g_{\mu\nu} \right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{425} $$

By defining the Einstein tensor as $G_{\mu\nu} = R_{\mu\nu} - (R/2) \times g_{\mu\nu}$, it is possible to write the Einstein field equations in a more compact as

$$ G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu} \right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{426} $$

The equation can be rearranged as

$$ R_{\mu\nu} - \frac{R}{2} \times g_{\mu\nu} + \left(\Lambda \times g_{\mu\nu} \right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) \tag{427} $$

or as

$$ R_{\mu\nu} - \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu} \right) = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu} \right) \tag{428} $$

or as

$$ {}_{R}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu} \right) \tag{429} $$

This equation can be changed as

$$ {}_{R}t_{\mu\nu} + 0 = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu} \right) \tag{430} $$

or as

$$ {}_{R}t_{\mu\nu} - {}_{0}t_{\mu\nu} + {}_{0}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu} \right) \tag{431} $$

Due to our definition it is ${}_{W}t_{\mu\nu} \equiv {}_{R}t_{\mu\nu} - {}_{0}t_{\mu\nu}$. The equation changes to

$$ {}_{W}t_{\mu\nu} + {}_{0}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu} \right) \tag{432} $$

and at the end to

$$ {}_{W}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu}\right) - {}_{0}t_{\mu\nu} \tag{433} $$

Due to the theorem before it is ${}_{0}t_{\mu\nu} \equiv \left(\frac{1_{\mu\nu}}{4_{\mu\nu} \cap \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) - \Lambda \times g_{\mu\nu}$.

The equation above changes to

$$ {}_{W}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu}\right) - \left(\left(\frac{1_{\mu\nu}}{4_{\mu\nu} \cap \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) - \Lambda \times g_{\mu\nu}\right) \tag{434} $$

or to

$$ {}_{W}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\Lambda \times g_{\mu\nu}\right) - \left(\frac{1_{\mu\nu}}{4_{\mu\nu} \cap \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) + \Lambda \times g_{\mu\nu} \tag{435} $$

and at the end to

$$ {}_{W}t_{\mu\nu} = \frac{R}{2} \times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4_{\mu\nu} \cap \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) \tag{436} $$

Dividing the equation before by the c^2, we obtain

$$ {}_{W}g_{\mu\nu} = \frac{{}_{W}t_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu}} = \frac{R}{2 \cap c_{\mu\nu} \cap c_{\mu\nu}} \times g_{\mu\nu} - \left(\frac{1_{\mu\nu}}{4_{\mu\nu} \times \pi_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right) \tag{437} $$

Quod erat demonstrandum.

Scholium.

There are circumstances, where the tensor ${}_{W}g_{\mu\nu}$ is identical with *the tensor of the gravitational waves*. Whether this is the case in general is a point of further research. It is convenient to consider the existence of gravitational waves in analogous manner to electromagnetic waves. Before going on to discuss this aspect in more detail one could expect gravitational waves to carry energy away from a radiating source. However, there are some shortcomings of such an approach. Assigning an energy density to a gravitational field is notoriously difficult, both in principle and technically. In general relativity, the energy momentum of a gravitational field at one point in space-time has no real meaning. One way of circumventing such a problem is to take seriously the fact that all energy and momentum is contained within the stress-energy tensor. This has the important consequence that there is no energy and momentum left, which could be put within an own energy momentum tensor of the gravitational field.

3.29. Theorem. The Probability Tensor as Associated with Einstein's Tensor $G_{\mu\nu}$

Claim.

In general, it is

$$ p(G) \equiv p\left({}_{0}C_{\mu\nu}\right) \equiv G_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}C_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}C_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{G_{\mu\nu}}{R_{\mu\nu}} = 1_{\mu\nu} - \frac{\frac{R}{2} \times g_{\mu\nu}}{R_{\mu\nu}} \tag{438} $$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 \tag{439} $$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$ 1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{440} $$

or

$$ 1_{\mu\nu} = 1_{\mu\nu} \tag{441} $$

Multiplying this equation by Einstein's tensor $G_{\mu\nu}$, it is

$$G_{\mu\nu} \equiv G_{\mu\nu} \tag{442}$$

or in general to

$${}_{0}C_{\mu\nu} \equiv G_{\mu\nu} \equiv \left(R_{\mu\nu} - \left(\frac{R}{2}\right) \times g_{\mu\nu}\right) \tag{443}$$

Multiplying by the tensor ${}_{R}Y_{\mu\nu}$ it is

$$G_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{0}C_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \tag{444}$$

The *commutative multiplication* with the tensor ${}_{R}Y_{\mu\nu}$ yields the probability tensor as associated with the tensor $G_{\mu\nu}$.

$$p(G_{\mu\nu}) \equiv p({}_{0}C_{\mu\nu}) \equiv G_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{0}C_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \tag{445}$$

Due to our theorem before, it is ${}_{R}Y_{\mu\nu} = {}_{R}^{*}\Psi_{\mu\nu} = (1_{\mu\nu}/R_{\mu\nu})$. The equation before simplifies as

$$p(G) \equiv p({}_{0}C_{\mu\nu}) \equiv G_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}C_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}C_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{G_{\mu\nu}}{R_{\mu\nu}} = 1_{\mu\nu} - \frac{\frac{R}{2} \times g_{\mu\nu}}{R_{\mu\nu}} \tag{446}$$

Quod erat demonstrandum.

3.30. Theorem. The Probability Tensor as Associated with Anti Einstein's Tensor $\underline{G}_{\mu\nu}$

Claim.

In general, it is

$$p(\underline{G}) \equiv p({}_{0}\underline{C}_{\mu\nu}) \equiv \underline{G}_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}\underline{C}_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}\underline{C}_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{\underline{G}_{\mu\nu}}{R_{\mu\nu}} = \frac{\frac{R}{2} \times g_{\mu\nu}}{R_{\mu\nu}} \tag{447}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1 \tag{448}$$

Multiplying by the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu} \cap 1 = 1_{\mu\nu} \cap 1 \tag{449}$$

or

$$1_{\mu\nu} = 1_{\mu\nu} \tag{450}$$

Multiplying this equation by anti Einstein's tensor $\underline{G}_{\mu\nu}$, it is

$$\underline{G}_{\mu\nu} \equiv \underline{G}_{\mu\nu} \tag{451}$$

or in general to

$${}_{0}\underline{C}_{\mu\nu} \equiv \underline{G}_{\mu\nu} \equiv R_{\mu\nu} - \left(R_{\mu\nu} - \left(\frac{R}{2}\right) \times g_{\mu\nu}\right) \equiv \left(\frac{R}{2}\right) \times g_{\mu\nu} \tag{452}$$

Multiplying by the tensor ${}_{R}Y_{\mu\nu}$ it is

$$\underline{G}_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{0}\underline{C}_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \tag{453}$$

The *commutative multiplication* with the tensor ${}_{R}Y_{\mu\nu}$ yields the probability tensor as associated with the tensor $G_{\mu\nu}$.

$$p(\underline{G}_{\mu\nu}) \equiv p({}_{0}\underline{C}_{\mu\nu}) \equiv \underline{G}_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \equiv {}_{0}\underline{C}_{\mu\nu} \cap {}_{R}Y_{\mu\nu} \tag{454}$$

Due to our theorem before, it is ${}_{R}Y_{\mu\nu} = {}_{R}^{*}\Psi_{\mu\nu} = (1_{\mu\nu}/R_{\mu\nu})$. The equation before simplifies as

$$p(\underline{G}) \equiv p\left({}_{0}\underline{C}_{\mu\nu}\right) \equiv \underline{G}_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}\underline{C}_{\mu\nu} \cap {}_{R}^{*}\Psi_{\mu\nu} \equiv {}_{0}\underline{C}_{\mu\nu} \cap \frac{1_{\mu\nu}}{R_{\mu\nu}} \equiv \frac{\underline{G}_{\mu\nu}}{R_{\mu\nu}} = \frac{\frac{R}{2} \times g_{\mu\nu}}{R_{\mu\nu}} \tag{455}$$

Quod erat demonstrandum.

Scholium.

The following 2 × 2 table may illustrate the basic relationships between the tensors (**Table 5**).

Under conditions of general theory of relativity, in terms of probability tensors, we obtain the following table (**Table 6**).

3.31. Theorem. Einstein's Weltformel

As long as humans have been trying to understand the laws of objective reality, they have been proposing theories. In contrast to the well-known quantum theory, the most fundamental theory of matter currently available, Laplace's demon and Einstein's Weltformel are related more widely at least by standing out against the indeterminacy as stipulated by today's quantum theory. Randomness as such does not exclude a deterministic relationship between cause and effect, since every random event has its own cause. The purpose of this publication is to provide a satisfactory description of the microstructure of space-time by mathematising the deterministic relationship between cause and effect at quantum level in the form of a *mathematical formula of the causal relationship k*. Despite of our best and different approaches of theorists worldwide spanning more than thousands of years taken to describe the workings of the universe in general, to understand the nature at the most fundamental quantum level and to develop a theory of everything progress has been very slow. There are a lot of proposals and interpretations, some of them grounded on a picturesque interplay of observation and experiment with ideas. In short, the battle for the correct theory is not completely free of metaphysics. Yet, besides of the many efforts and attempts to reconcile quantum (field) theory with general relativity an ultimate triumph of human reason on this matter is not in sight. There is still no single theory which provides a genuine insight and understanding of gravity and quantum mechanics, one of the most cherished dreams of physics and of science as such. Einstein's Weltformel or a "final" or "ultimate" *theory of everything* (ToE) as a hypothetical theoretical framework of philosophy, mathematics and physics capable of describing all phenomena of objective reality should rest at least on general relativity (GR) and quantum (field) theory (Q(F)T). Still, physicists have experimentally confirmed that (GR) and (Q(F)T) as they are currently formulated are to some extent mutually incompatible and cannot both be right in the same respect. Thus far, some of today's front runners are the *string theory*, the *loop quantum gravity* et cetera and the *quantum field theory*. Among the numerous alternative proposals for reconciling quantum physics and general relativity theory, the mathematical and conceptual framework of quantum field theory

Table 5. The unified field ${}_{R}W_{\mu\nu}$.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	$A_{\mu\nu}$	$B_{\mu\nu}$	${}_{R}U_{\mu\nu}$
	no	$C_{\mu\nu}$	$D_{\mu\nu}$	${}_{R}\underline{U}_{\mu\nu}$
		${}_{0}W_{\mu\nu}$	${}_{0}\underline{W}_{\mu\nu}$	${}_{R}W_{\mu\nu}$

Table 6. The unified field in terms of probabitliy tensors.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	$p(A_{\mu\nu})$	$p(B_{\mu\nu})$	$p({}_{R}E_{\mu\nu})$
	no	$p(C_{\mu\nu})$	$p(D_{\mu\nu})$	$p({}_{R}t_{\mu\nu})$
		$p(G_{\mu\nu})$	$p(\underline{G}_{\mu\nu})$	$1_{\mu\nu}$

(Q(F)T) covers the electromagnetic, the weak and the strong interaction. *In quantum field theory*, *there is a field associated to each type of a fundamental particle that appears in nature*. However, quantization of a classical field proposed by quantum field theory is (philosophically) unsatisfactory since the very important and fundamental force in nature, gravitation, has defied quantization so far. The problems are related to the quantum mechanical framework as such. The usual axioms of quantum mechanics say that observables are represented by Hermitian operators which are not entirely true. At least one observable in quantum mechanics is not represented by a Hermitian operator: the time itself. Today, the time itself enters into the mathematical formalism of quantum mechanics but not as an eigenvalue of any operator. Our subsequent discussion will be restricted almost completely to both, the principles of general relativity and quantum theory.

Claim.

In general, the mathematical formula of the causal relationship k (Einstein's Weltformel) covariant under a class of general coordinates transformations *i.e.* the same in all the reference frames, namely in all coordinates systems, follows as

$$k\left({}_RU_{\mu\nu},{}_0W_{\mu\nu}\right)=\frac{\sigma\left({}_RU_{\mu\nu},{}_0W_{\mu\nu}\right)}{\sigma\left({}_RU_{\mu\nu}\right)\cap\sigma\left({}_0W_{\mu\nu}\right)}=\frac{\left(p\left({}_RU_{\mu\nu},{}_0W_{\mu\nu}\right)-p\left({}_RU_{\mu\nu}\right)\cap p\left({}_0W_{\mu\nu}\right)\right)}{\sqrt[2]{\left(p\left({}_RU_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_RU_{\mu\nu}\right)\right)\right)\cap\left(p\left({}_0W_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_0W_{\mu\nu}\right)\right)\right)}} \tag{456}$$

Direct proof.

As a rule, the point of departure is axiom I. In general, axiom I is determined as

$$+1=+1 \tag{457}$$

Multiplying be the tensor of the unified field $1_{\mu\nu}$, we obtain

$$1_{\mu\nu}\cap 1=1_{\mu\nu}\cap 1 \tag{458}$$

or

$$1_{\mu\nu}=1_{\mu\nu} \tag{459}$$

Multiplying this equation by the tensor of the cause ${}_RU_{\mu\nu}$, we obtain

$$1_{\mu\nu}\cap {}_RU_{\mu\nu}=1_{\mu\nu}\cap {}_RU_{\mu\nu} \tag{460}$$

or

$${}_RU_{\mu\nu}={}_RU_{\mu\nu} \tag{461}$$

Multiplying by the tensor of the effect ${}_0W_{\mu\nu}$, it is

$${}_RU_{\mu\nu}\cap {}_0W_{\mu\nu}={}_RU_{\mu\nu}\cap {}_0W_{\mu\nu} \tag{462}$$

Due to our definition of standard deviation of the cause, it is ${}_RU_{\mu\nu}\equiv\dfrac{\sigma\left({}_RU_{\mu\nu}\right)}{\sqrt[2]{\left(p\left({}_RU_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_RU_{\mu\nu}\right)\right)\right)}}$

Substituting this relation into the equation above, we obtain

$${}_RU_{\mu\nu}\cap {}_0W_{\mu\nu}=\frac{\sigma\left({}_RU_{\mu\nu}\right)}{\sqrt[2]{\left(p\left({}_RU_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_RU_{\mu\nu}\right)\right)\right)}}\cap {}_0W_{\mu\nu} \tag{463}$$

Due to our definition of standard deviation of effect, it is ${}_0W_{\mu\nu}\equiv\dfrac{\sigma\left({}_0W_{\mu\nu}\right)}{\sqrt[2]{\left(p\left({}_0W_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_0W_{\mu\nu}\right)\right)\right)}}$

Substituting this relation into the equation before, we obtain

$${}_RU_{\mu\nu}\cap {}_0W_{\mu\nu}=\frac{\sigma\left({}_RU_{\mu\nu}\right)\cap\sigma\left({}_0W_{\mu\nu}\right)}{\sqrt[2]{\left(p\left({}_RU_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_RU_{\mu\nu}\right)\right)\right)}\cap\sqrt[2]{\left(p\left({}_0W_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_0W_{\mu\nu}\right)\right)\right)}} \tag{464}$$

According to the definition of the co-variance of cause and effect, it is

$$ {}_{R}U_{\mu\nu} \cap {}_{0}W_{\mu\nu} \equiv \frac{\sigma\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right)}{\left(p\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right) - p\left({}_{R}U_{\mu\nu}\right) \cap p\left({}_{0}W_{\mu\nu}\right)\right)} $$

Substituting this relationship into the equation before, we obtain

$$ \frac{\sigma\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right)}{\left(p\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right) - p\left({}_{R}U_{\mu\nu}\right) \cap p\left({}_{0}W_{\mu\nu}\right)\right)} = \frac{\sigma\left({}_{R}U_{\mu\nu}\right) \cap \sigma\left({}_{0}W_{?}\right)}{\sqrt[2]{\left(p\left({}_{R}U_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left({}_{R}U_{\mu\nu}\right)\right)\right)} \cap \sqrt[2]{\left(p\left({}_{0}W_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left({}_{0}W_{\mu\nu}\right)\right)\right)}} \tag{465} $$

Rearranging equation, it is

$$ \frac{\sigma\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right)}{\sigma\left({}_{R}U_{\mu\nu}\right) \cap \sigma\left({}_{0}W_{\mu\nu}\right)} = \frac{\left(p\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right) - p\left({}_{R}U_{\mu\nu}\right) \cap p\left({}_{0}W_{\mu\nu}\right)\right)}{\sqrt[2]{\left(p\left({}_{R}U_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left({}_{R}U_{\mu\nu}\right)\right)\right)} \cap \sqrt[2]{\left(p\left({}_{0}W_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left({}_{0}W_{\mu\nu}\right)\right)\right)}} \tag{466} $$

Einstein's Weltformel, the mathematical formula of the causal relationship k, follows as

$$ k\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right) = \frac{\sigma\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right)}{\sigma\left({}_{R}U_{\mu\nu}\right) \cap \sigma\left({}_{0}W_{\mu\nu}\right)} = \frac{\left(p\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right) - p\left({}_{R}U_{\mu\nu}\right) \cap p\left({}_{0}W_{\mu\nu}\right)\right)}{\sqrt[2]{\left(p\left({}_{R}U_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left({}_{R}U_{\mu\nu}\right)\right)\right) \cap \left(p\left({}_{0}W_{\mu\nu}\right) \cap \left(1_{\mu\nu} - p\left({}_{0}W_{\mu\nu}\right)\right)\right)}} \tag{467} $$

Quod erat demonstrandum.

Scholium.

The range of the causal relationship is $-1_{\mu\nu} \leq k\left({}_{R}U_{\mu\nu}, {}_{0}W_{\mu\nu}\right) \leq +1_{\mu\nu}$. In last consequence, negative particles can be derived from Einstein's field equation.

Causality and determinism (and prediction) are often equated even if both are not really the same. For a variety of reasons such an approach to determinism and causality is fraught with many problems. A further problem is posed by the fact that, as today widely recognized, the fundamental, exceptionless laws of nature are governed by the laws of quantum mechanics which itself is widely thought to be a strongly non-deterministic [31]-[34] theory. Roughly speaking, Einstein's dream of a complete [35] theory of quantum mechanics (*i.e.* hidden variable theory) with the goal "to restore to the theory causality and locality" [36], determinism and definiteness to micro-reality became [37] partly mistaken and/or misleading but not impossible [38] [39] in principle. The causal relationship k, deeply connected with our understanding of objective reality, became a subject to clarification and mathematical analysis and has been investigated in a specific, well-defined theoretical context of the general theory of relativity as developed by the German-born theoretical physicist Albert Einstein. In order for us to gain a clear understanding of the concept of causality or unified field theory under conditions of the general theory of relativity further explanation and investigation is required. Causality has been given various, usually imprecise definitions. Many scholars contributed to the notion of causality and determinism, among them Nicolas de Condorcet, Baron D'Holbach and Laplace [40]. One of these definitions is the known Laplace demon (sometimes referred to as Laplace's Superman, after Hans Reichenbach). The mechanical determinism generally referred to as Laplace demon is of course incompatible with the mainstream interpretations of today quantum mechanics which stipulates indeterminacy, and was formulated by Laplace as follows: "Une intelligence qui, pour un instant donné, connaîtrait toutes les forces dont la nature est animée, et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule les mouvements des plus grand corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle, l'avenir comme le passé seraient presents à ses yeux" [41].

Laplace demon translated into English:

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes".

Thus far, to avoid certain major errors of definition, the geometrical tensorial representation of the mathematical formula of the causal relationship k (Einstein's Weltformel) as

$$k\left({}_{R}U_{\mu\nu},{}_{0}W_{\mu\nu}\right)=\frac{\sigma\left({}_{R}U_{\mu\nu},{}_{0}W_{\mu\nu}\right)}{\sigma\left({}_{R}U_{\mu\nu}\right)\cap\sigma\left({}_{0}W_{\mu\nu}\right)}=\frac{\left(p\left({}_{R}U_{\mu\nu},{}_{0}W_{\mu\nu}\right)-p\left({}_{R}U_{\mu\nu}\right)\cap p\left({}_{0}W_{\mu\nu}\right)\right)}{\sqrt[2]{\left(p\left({}_{R}U_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_{R}U_{\mu\nu}\right)\right)\right)\cap\left(p\left({}_{0}W_{\mu\nu}\right)\cap\left(1_{\mu\nu}-p\left({}_{0}W_{\mu\nu}\right)\right)\right)}} \tag{468}$$

is valid for a chaotic and random system too and cannot be reduced to Laplace demon and his articulation of causal or scientific determinism.

4. Discussion

Einstein had started unifying the electromagnetic and gravitational fields via pure geometry into a unified field [2] theory. In spite of failing success, Einstein tried to relate the macroscopic world of universal space-time to those in the physical phenomena in the submicroscopic world of the atom. Einstein's modest hope and the key to a more perfect quantum theory was his epistemological and methodological position that a "real state" of a physical system exists objectively and independent of any observation or measurement, independent of human mind and consciousness. Still only a rather small number of theoretical physicists devoted their work to the search for a unified theory and the unification of electromagnetism and gravitation has apparently faded into the background at least since the death of Einstein.

For the convenience of the reader, some of the mathematical formalism given by general relativity theory is repeated in a slightly extended form only as much as needed for an understanding of this paper. In general, for the geometrization and the quantization of the fields, various geometric frameworks can be chosen. The geometrical structures of the underlying probability field enable the transformation to different geometric frameworks.

Under conditions of general theory of relativity, we obtain the following relationships (**Table 7**).

4.1. Curvature Excludes Momentum and Vice Versa

Under conditions where curvature excludes momentum, the stress-energy tensor of ordinary matter ${}_{0}E_{\mu\nu}$ is equivalent to zero we obtain (**Table 8**) or the equation.

$${}_{0}E_{\mu\nu}=\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)=0 \tag{496}$$

From this assumption we obtain

$$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}=\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right) \tag{497}$$

Table 7. Unified field theory under conditions of the theory of general realtivity.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$ $-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$	$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$
	no	$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$ $-\Lambda\cap g_{\mu\nu}$	$-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$ $\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}-\Lambda\cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$R_{\mu\nu}$

Table 8. Curvature excludes momentum and vice versa.

		Curvature: yes	Curvature: no	
Energy/momentum	**yes**	0	$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$
	no	$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$ $-\Lambda\cap g_{\mu\nu}$	$-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$ $\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}-\Lambda\cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$R_{\mu\nu}$

Such a manifold is determined by the fact that all energy and momentum is contained within the stress-energy tensor of the electromagnetic field.

4.2. Momentum Implies Curvature

Under conditions of general relativity, there are circumstances where momentum implies curvature. Such manifolds are determined by the stress-energy tensor of the electro-magnetic field which is equal ${}_0\underline{E}_{\mu\nu} = 0$. Under these conditions we obtain

$$\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)=0 \tag{471}$$

The following 2 × 2 table may illustrate these circumstances (**Table 9**).

4.3. Without Momentum No Curvature

Under conditions of general relativity, manifolds can be determined by the fact that without momentum no curvature. Under these conditions it is

$$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)-\Lambda\cap g_{\mu\nu}=0 \tag{472}$$

and the stress energy tensor of the electromagnetic field is determined by the equation

$$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)=\Lambda\cap g_{\mu\nu} \tag{473}$$

The question of course is, are there circumstances at all, where the stress energy tensor of the electromagnetic field is determined by the equation before.The following 2 × 2 table may illustrate these circumstances (**Table 10**).

4.4. Momentum or Curvature

One feature of manifolds determined by momentum or curvature is the validity of the equation

$$\left(\frac{R}{2}\right)\cap g_{\mu\nu}-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}{}^{c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)=0 \tag{474}$$

Table 9. Momentum implies curvature.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	$\left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}}\right) \cap T_{\mu\nu}$	0	$\left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}}\right) \cap T_{\mu\nu}$
	no	$-\Lambda \cap g_{\mu\nu}$	$\left(\frac{R}{2}\right) \cap g_{\mu\nu}$	$\left(\frac{R}{2}\right) \cap g_{\mu\nu} - \Lambda \cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right) \cap g_{\mu\nu}$	$R_{\mu\nu}$

Table 10. Without momentum no curvature.

		Curvature		
		yes	**no**	
Energy/momentum	**yes**	$\left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}}\right) \cap T_{\mu\nu}$ $-\Lambda \cap g_{\mu\nu}$	$+\Lambda \cap g_{\mu\nu}$	$\left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}}\right) \cap T_{\mu\nu}$
	no	0	$-\Lambda \cap g_{\mu\nu}$ $\left(\frac{R}{2}\right) \cap g_{\mu\nu}$	$\left(\frac{R}{2}\right) \cap g_{\mu\nu} - \Lambda \cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right) \cap g_{\mu\nu}$	$R_{\mu\nu}$

Consequently, under these circumstances the stress energy tensor of the electromagnetic field is determined by the equation

$$\left(\left(\frac{1_{\mu\nu}}{4 \times \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)\right) = \left(\frac{R}{2}\right) \cap g_{\mu\nu} \tag{475}$$

The following 2 × 2 table may illustrate this manifold in more detail (**Table 11**).

4.5. Either Momentum or Curvature

Manifolds determined by *either* momentum *or* curvature are illustrated by the following 2 × 2 table (**Table 12**).

The *either* momentum *or* curvature manifold is determined by the equation

$$\left(\frac{R}{2}\right) \cap g_{\mu\nu} - \left(\left(\frac{1_{\mu\nu}}{4 \times \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)\right) = 0 \tag{476}$$

and by the equation

$$\left(\frac{4_{\mu\nu} \cap 2_{\mu\nu} \cap \pi_{\mu\nu} \cap \gamma_{\mu\nu}}{c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu} \cap c_{\mu\nu}}\right) \cap T_{\mu\nu} - \left(\left(\frac{1_{\mu\nu}}{4 \times \pi_{\mu\nu}}\right) \times \left(\left(F_{\mu c} \times F_{\nu}{}^{c}\right) - \left(\frac{1}{4} \times g_{\mu\nu} \times F_{d\nu} \times F^{d\nu}\right)\right)\right) = 0 \tag{477}$$

The following 2 × 2 table may illustrate this manifold in more detail (**Table 13**).

In last consequence, this manifold is determined by the equation

$$\left(\frac{R}{2}\right) \cap g_{\mu\nu} - \Lambda \cap g_{\mu\nu} + \left(\frac{R}{2}\right) \cap g_{\mu\nu} = R \cap g_{\mu\nu} - \Lambda \cap g_{\mu\nu} = \underline{\Lambda} \cap g_{\mu\nu} = R_{\mu\nu} \tag{478}$$

Table 11. Momentum or curvature.

		Curvature		
		yes	no	
Energy/momentum	yes	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$ $-\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$+\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$
	no	$+\left(\frac{R}{2}\right)\cap g_{\mu\nu}$ $-\Lambda\cap g_{\mu\nu}$	0	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}-\Lambda\cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$R_{\mu\nu}$

Table 12. *Either* momentum *or* curvature.

		Curvature		
		yes	no	
Energy/momentum	yes	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$ $-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$	$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$
	no	$+\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$ $-\Lambda\cap g_{\mu\nu}$	$-\left(\left(\frac{1_{\mu\nu}}{4\times\pi_{\mu\nu}}\right)\times\left(\left(F_{\mu c}\times F_{\nu}^{\ c}\right)-\left(\frac{1}{4}\times g_{\mu\nu}\times F_{d\nu}\times F^{d\nu}\right)\right)\right)$ $\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}-\Lambda\cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$R_{\mu\nu}$

Table 13. *Either* momentum *or* curvature.

		Curvature		
		yes	no	
Energy/momentum	yes	0	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$\left(\frac{4_{\mu\nu}\cap 2_{\mu\nu}\cap\pi_{\mu\nu}\cap\gamma_{\mu\nu}}{c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}\cap c_{\mu\nu}}\right)\cap T_{\mu\nu}$
	no	$+\left(\frac{R}{2}\right)\cap g_{\mu\nu}$ $-\Lambda\cap g_{\mu\nu}$	0	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}-\Lambda\cap g_{\mu\nu}$
		$G_{\mu\nu}$	$\left(\frac{R}{2}\right)\cap g_{\mu\nu}$	$R_{\mu\nu}$

At the end, *either* momentum *or* curvature manifolds are described by the equation

$$+\Lambda\cap g_{\mu\nu}+\underline{\Lambda}\cap g_{\mu\nu}=R\cap g_{\mu\nu} \tag{479}$$

where $\underline{\Lambda}$ denotes anti lamda, the anti cosmological constant. Under *these* conditions, anti lambda describes the geometrical structures underlying the unified hyper-field of electromagnetism and gravitation, the unifying of the electromagnetic and gravitational fields into a hyper-field via pure geometry.

5. Conclusion

For the geometrization of fields, various geometric frameworks can be chosen. This probability theory compatible approach to the unified field theory enables the use of different geometric frameworks depending upon circumstances. The relationship between cause and effect is expressed completely in the language of tensors while demonstrating the close relationship to Einstein's general theory of relativity and Einstein's field equation.

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Rigorous Coupled-Wave Approach for Sandwich Gratings

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Abstract

Rigorous Coupled-Wave Approach (RCWA) has been used successfully and accurately to study simple grating structures, such as one-layer gratings, one-whole gratings. In this paper, RCWA is expanded to solve Sandwich gratings (SG), which is composed of two identical planar dielectric gratings adjoined by thin metallic or dielectric film. The electromagnetic analytic expressions for each layer of SG structure are given and rigorous coupled-wave equations are deduced. The numerical investigations for the diffraction spectra of SG by our theoretical and computer programs are in good agreement with the results of classical RCWA in the condition when a Sandwish grating is simplified to a classical one-layer grating. The calculations by our programs of another condition when a Sandwish grating is degenerated to a classical single planar structure also conform to the results of classical electromagnetic theory. The research results above indicate that the extended theoretical formula has backwards compatibility and is self-consistent with the classical theory.

Keywords

Rigorous Coupled-Wave Approach, Sandwich Gratings, Self-Consistent

1. Introduction

Over the past 30 years RCWA formulated by Moharam and Gaylord [1]-[5] has been used successfully and accurately to analyze periodic structures including holographic gratings [1] [2] and arbitrary profiled dielectric or metallic surface-relief gratings [3] [6]-[8]. RCWA is almost used to study relatively simple structure, such as one-layer gratings [1]-[5] and one-whole gratings [6]-[13] which have arbitrary profiled surface-relief on both of top and bottom of monolithic materials. Owing to its complexity and difficulty, RCWA is seldom used to study multi-layers grating.

In this paper, RCWA is adopted to solve sandwich gratings (SG) structure, which is composed of two identical planar dielectric gratings adjoined by thin metallic or dielectric film. The electromagnetic analytic expressions for each layer of SG structure are given and rigorous coupled-wave equations are deduced. To verify the theory presented in the paper, the proposed RCWA for SG and classic electromagnetic theory are respectively used to research two degenerative SG structures, namely classical single grating and classical single planar structure. The results indicate that RCWA for SG has backwards compatibility and is self-consistent with the classical theory.

2. The Sandwich Grating Structure and Theoretical Formulas

A schematic diagram of the proposed Sandwich grating structure is shown in **Figure 1**. The configuration consists of two identical planar sinusoidal dielectric gratings of thickness *d* adjoined by continuous thin silver or dielectric film of thickness *h*. The lossless planar dielectric grating [1] [2] is characterized by a periodical medium. The relative permittivity can be depicted

$$\varepsilon_2(x,z) = \varepsilon_4(x,z) = \varepsilon_{avg} + \Delta\varepsilon\cos\left[K\left(x\sin\phi + z\cos\phi\right)\right] \quad (1)$$

where ε_{avg} is the average permittivity and $\Delta\varepsilon$ is the amplitude of the sinusoidal permittivity. ϕ is the grating slant angle and $K = 2\pi/\Lambda$, here Λ is the grating period. The permittivity in the region I ($z<0$) is ε_1 and the ones in the region V ($z > 2d+h$) is ε_5. While the permittivity of Ag film in the region III is ε_3. The complex permittivity of metallic films is described by the Drude model

$$\varepsilon_3(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + j\gamma\omega} \quad (2)$$

where $\omega_p = 1.37\times10^{16}$ rad/s is the plasma frequency for Ag and $\gamma = 7.29\times10^{13}$ rad/s is the collision frequency for Ag [14], $j=\sqrt{-1}$.

For E-mode polarization (the electric field is in the plane of incidence), the magnetic field is solely in the y direction. According to rigorous coupled-wave analysis theory (RCWA) [1]-[5], the normalized magnetic fields in each region may be expressed as:

$$\text{Region I} \quad H_{1y} = \mathrm{e}^{-j(k_{x0}x + k_{1z0}z)} + \sum_i R_{1i}\mathrm{e}^{-j(k_{xi}x - k_{1zi}z)} \quad (3)$$

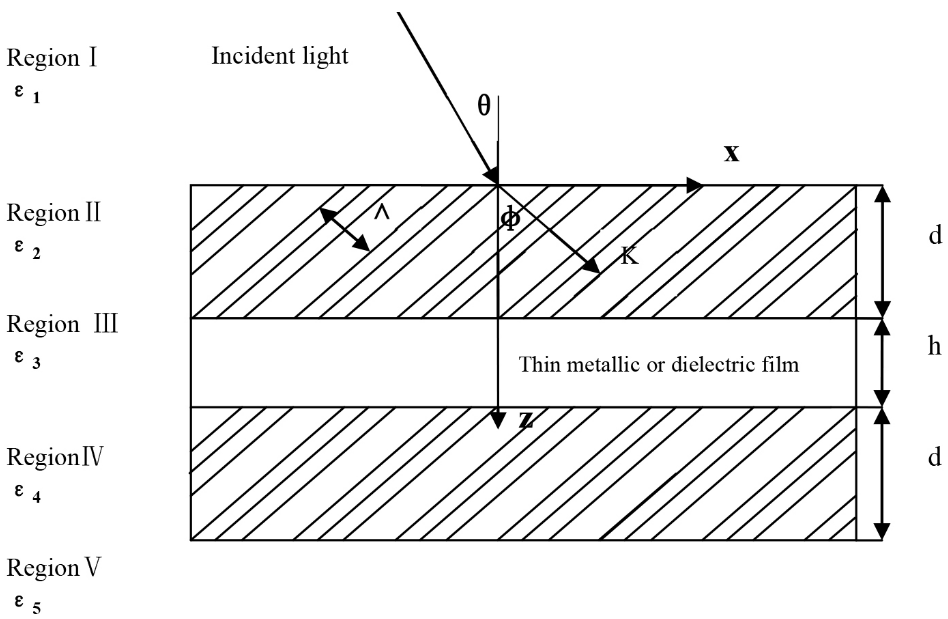

Figure 1. Schematic diagram of Sandwish grating structure.

$$\text{Region II} \quad H_{2y} = \sum_i U_{2i}(z) e^{-j(k_{xi}x + k_{2zi}z)} \tag{4}$$

$$\text{Region III} \quad H_{3y} = \sum_i T_{3i} e^{-j(k_{xi}x + k_{3zi}(z-d-h))} + \sum_i R_{3i} e^{-j(k_{xi}x - k_{3zi}(z-d-h))} \tag{5}$$

$$\text{Region IV} \quad H_{4y} = \sum_i U_{4i}(z-d-h) e^{-j(k_{xi}x + k_{4zi}(z-d-h))} \tag{6}$$

$$\text{Region V} \quad H_{5y} = \sum_i T_{5i} e^{-j(k_{xi}x + k_{5zi}(z-2d-h))} \tag{7}$$

And $U_{2i}(z)$, $U_{4i}(z)$ are the space harmonic magnetic-field amplitudes and satisfy coupled-wave equations in grating regions. The solutions for $U_{2i}(z)$, $U_{4i}(z)$ are referenced from the Ref. [2] and may be expressed as: $U_{2i}(z) = \sum_{n=1}^{2m} C_n \omega_{im} e^{\lambda_m z}$, $U_{4i}(z) = \sum_{n=2m+1}^{4m} C_n \omega_{im} e^{\lambda_m z}$ where λ_m and ω_{im} are the eigenvalues and eigenvectors. The second grating in Region IV is the same modulated as the first grating, so their eigenvalues and eigenvectors are also the same. But owing to the different boundary conditions of tangential electric and magnetic fields, the coefficients C_n are different. On the other hand, the normalized wave amplitudes of the thin connected region (Region III) are determined by the interactions between the forward-diffraction of the first grating and backward-diffraction of the second grating.

The symbols used in Equations (3)-(7) are as follows:

$$k_{xi} = 2\pi\sqrt{\varepsilon_1}/\lambda \sin\theta - iK \sin\phi \tag{8}$$

$$k_{lzi} = \sqrt{\left(k_l^2 - k_{xi}^2\right)}, k_l = 2\pi\sqrt{\varepsilon_l}/\lambda, l = 1,3,5 \tag{9}$$

$$k_{\xi zi} = \sqrt{\left(k_\xi^2 - k_{x0}^2\right)} - iK\cos\phi, k_\xi = 2\pi\sqrt{\varepsilon_\xi}/\lambda, \xi = 2,4 \tag{10}$$

where i is the space-harmonic index in grating Regions II and IV (analogous to the diffractive order index in Regions I, III and V), θ is the angle of incidence, λ is the free-space wavelength. R_{li}, T_{li} are the normalized amplitude of the ith reflected and transmitted wave of Region I, Region III or Region V.

The electromagnetic boundary conditions require that the tangential components of the electric field and the magnetic field must be continuous across planes $z = 0$, $z = d$, $z = d + h$ and $z = 2d + h$. The boundary conditions for tangential magnetic field (H_y) are respectively

$$U_{2i}(0) = \delta_{i0} + R_{1i} \tag{11}$$

$$U_{2i}(d) e^{-jk_{2zi}d} = T_{3i} e^{jk_{3zi}h} + R_{3i} e^{-jk_{3zi}h} \tag{12}$$

$$U_{4i}(0) = T_{3i} + R_{3i} \tag{13}$$

$$U_{4i}(d) e^{-jk_{4zi}d} = T_{5i} \tag{14}$$

The tangential electric field E_x may be obtained from the Maxwell curl equation $\nabla \times \boldsymbol{H} = j\omega\varepsilon_0\varepsilon_r \boldsymbol{E}$. The result is $E_x = j/\left(\omega\varepsilon_0\varepsilon_r(x,z)\right)\partial H_y/\partial z$ and boundary conditions for tangential electric field (E_x) are respectively

$$\frac{1}{\varepsilon_1}\left(R_{1i} - \delta_{i0}\right)\left(jk_{1zi}\right) = \sum_l \varepsilon_2^{-1}(l)\left(U'_{2(i-l)}(0) + U_{2(i-l)}(0)\left(-jk_{2z(i-l)}\right)\right) \tag{15}$$

$$\begin{aligned} &\frac{1}{\varepsilon_3}\left(T_{3i}\left(-jk_{3zi}\right) e^{jk_{3zi}h} + R_{3i}\left(jk_{3zi}\right) e^{-jk_{3zi}h}\right) \\ &= \sum_l \varepsilon_2^{-1}(l) e^{-jk_{2zi}d}\left(U'_{2(i-l)}(d) + U_{2(i-l)}(d)\left(-jk_{2z(i-l)}\right)\right) \end{aligned} \tag{16}$$

$$\frac{1}{\varepsilon_3}\left(T_{3i}\left(-jk_{3zi}\right)+R_{3i}\left(jk_{3zi}\right)\right)=\sum_l \varepsilon_4^{-1}(l)\left(U'_{4(i-l)}(0)+U_{4(i-l)}(0)\left(-jk_{4z(i-l)}\right)\right) \tag{17}$$

$$\frac{1}{\varepsilon_5}\left(T_{5i}\left(-jk_{5zi}\right)\right)=\sum_l \varepsilon_4^{-1}(l)\,\mathrm{e}^{-jk_{4zi}d}\left(U'_{4(i-l)}(d)+U_{4(i-l)}(d)\left(-jk_{4z(i-l)}\right)\right) \tag{18}$$

where $U'(z)=\mathrm{d}U(z)/\mathrm{d}z$.

If N values of *i* are retained in the analysis, there will be 4N unknown values of C_n and they will be determined from the boundary conditions. All the R_{li}, T_{li} may then be calculated.

The backward-wave diffraction efficiencies (Region I) are

$$DE_{1i}=\left|R_{1i}\right|^2 Re\left(k_{1zi}/k_{1z0}\right) \tag{19}$$

The forward-wave diffraction efficiencies (Region V) are

$$DE_{5i}=\left|T_{5i}\right|^2 Re\left(k_{5zi}\times\varepsilon_1/\left(\varepsilon_5\times k_{1z0}\right)\right) \tag{20}$$

3. Numerical Calculations and Discussions

In order to verify the deduced formulas above, the reflection and transmission characteristics of Sandwich gratings connected by thin silver film are studied in the condition of grating thickness $d=0$ at normal incidence. The other parameters are as follows, $\varepsilon_1=\varepsilon_5=1.33^2, h=40$ nm. The five calculated regions shown in **Figure 2(a)** are degenerated into three regions shown in **Figure 2(b)**. The proposed RCWA for SG and classic electromagnetic theory are respectively used to solve for the reflection and transmission characteristics. The efficiencies of results given by RCWA in the paper and classical theory are both shown in **Figure 3**. The discrepancy magnitude between two methods is only 10^{-5} shown in **Figure 4**, which proves our formulas and computer program codes to be true. Our work backwards contains the results of the classical Fresnel formulas of three regions.

Furthermore, when the connection layer of thin metallic film is absence, namely $h=0$ nm, the proposed Sandwich grating is simplified into an ordinary one-layer grating. The unslant grating has 400 nm grating period and 100 nm thickness, and its average permittivity is 2.25 with the modulation 0.33. The thickness of the connection layer of thin Ag film is zero. On this condition, the two same gratings are combined to be one whole thick grating of 200 nm, shown in **Figure 5(b)**. Suppose the grating in water, thus $\varepsilon_1=\varepsilon_5=1.33^2$.

The efficiencies of reflection and transmission of single layer gratings calculated by our theory and by classical Rigorous Coupled-wave Approach are both shown in **Figure 6**. The results are nearly the same.

From the above discussed, the correctness and efficiencies of RCWA for SG are verified. The theory given in the paper has backwards compatibility and is self-consistent with the classical theory.

4. Conclusion

Rigorous coupled-wave approach for SG is proposed in the paper. The proposed RCWA for SG and classic electromagnetic theory are respectively used to study two degenerative SG structures, and the reflection and

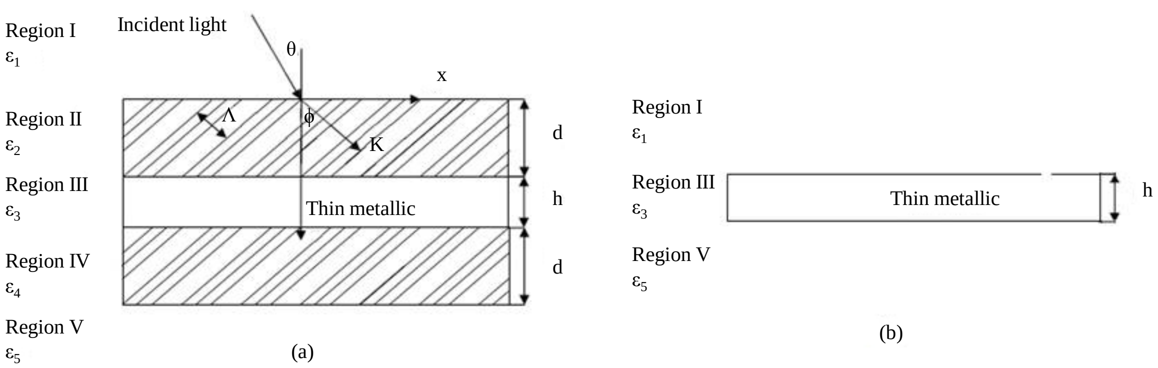

Figure 2. (a) Sandwish grating structure; (b) classical planar structure.

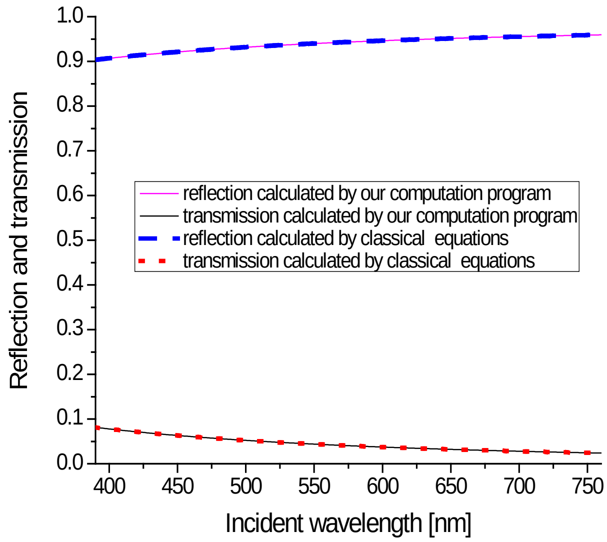

Figure 3. Reflection and transmission at normal incidence with $d = 0$, $\varepsilon_1 = \varepsilon_5 = 1.33^2$, $h = 40$ nm.

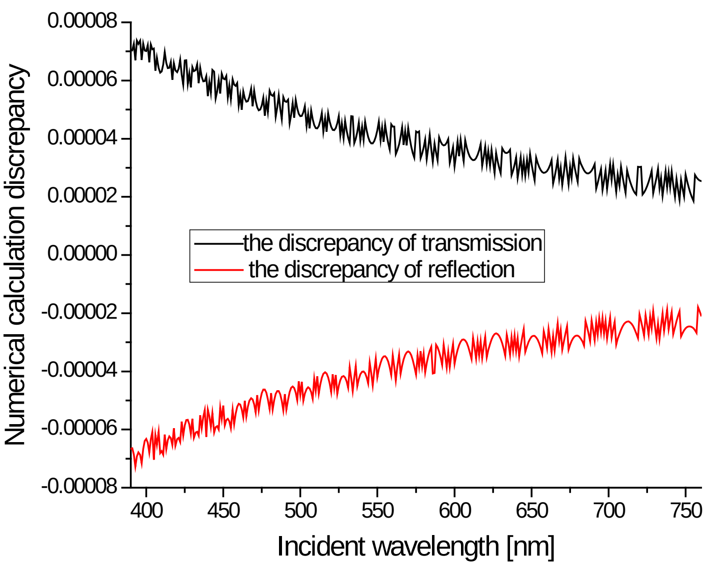

Figure 4. Calculation discrepancy between our program and Fresnel equations.

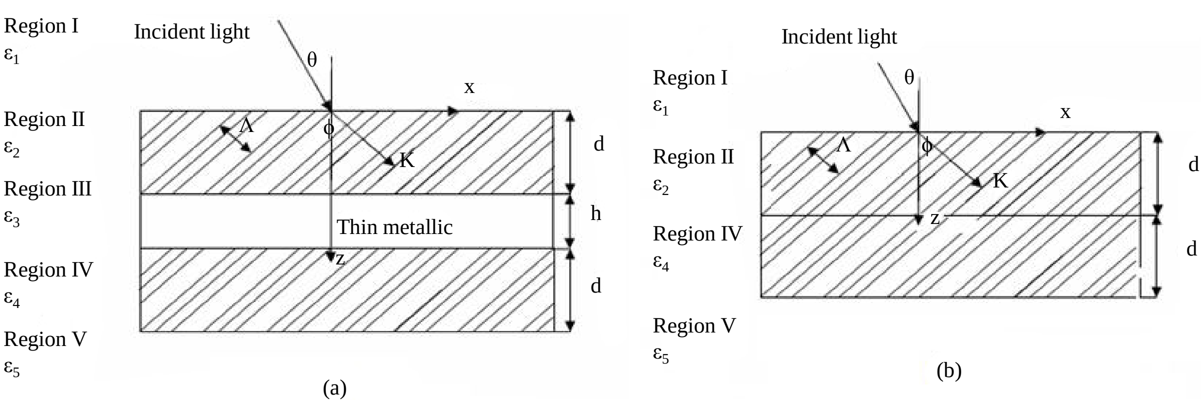

Figure 5. (a) Sandwish grating; (b) Combined into single layer grating.

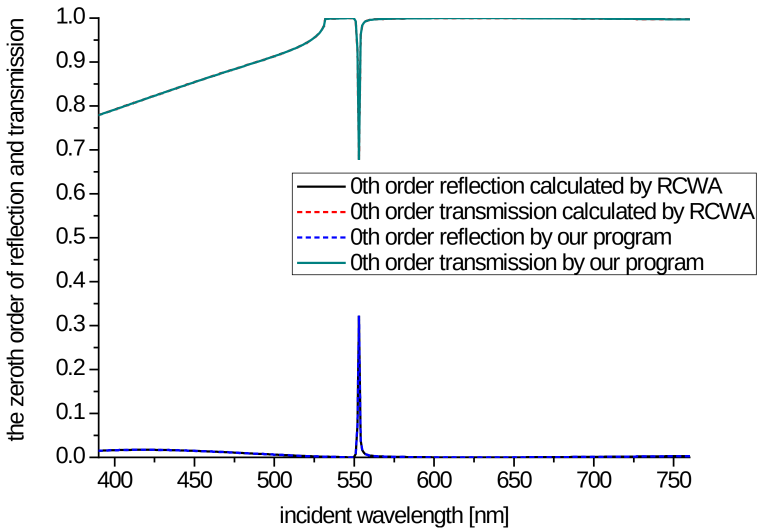

Figure 6. Reflection and transmission at normal incidence with $d = 100$ nm, $\varepsilon_1 = \varepsilon_5 = 1.33^2$, $h = 0$.

transmission spectra are almost the same. The results indicate that RCWA for SG has backwards compatibility and is self-consistent with the classical theory. The theoretical formula and computer codes lay the foundations for investigation of properties of the novel Sandwich grating and exploitation of nano-photonics devices.

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Septic B-Spline Solution of Fifth-Order Boundary Value Problems

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Abstract

A numerical method based on septic B-spline function is presented for the solution of linear and nonlinear fifth-order boundary value problems. The method is fourth order convergent. We use the quesilinearization technique to reduce the nonlinear problems to linear problems and use B-spline collocation method, which leads to a seven nonzero bands linear system. Illustrative example is included to demonstrate the validity and applicability of the proposed techniques.

Keywords

Septic B-Spline Function, Fifth-Order Boundary Value Problems, B-Spline Collocation Method, Nonlinear Problems

1. Introduction

Consider the following fifth-order boundary value problem.

$$Ly(x) = y^{(5)}(x) + p(x)y(x) = f(x), \quad c < x < d \tag{1}$$

With boundary conditions

$$\begin{aligned} &y(c) = \alpha_0, y^{(1)}(c) = \alpha_1, y^{(2)}(c) = \alpha_2, \\ &y(d) = \alpha_3, y^{(1)}(d) = \alpha_4, y^{(2)}(d) = \alpha_5 \end{aligned} \tag{2}$$

where $\alpha_i \ (i = 0,1,2,3,4,5)$ are known real constants, $p(x)$ and $f(x)$ are continuous on $[c,d]$. This problem arising in the mathematical modeling of viscoelastic flows [1] [2] has been studied by several authors [3]-[5]. A. Lamnii, H. Mraoui, D. Sbibih and A. Tijini studied the fifth-order boundary value problem based on splines quasi-interpolants and proved to be second order convergent.

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B-spline functions based on piece polynomials are useful wavelet basis functions, the resulting matrices are sparse, but always, banded. And that possess attractive properties: piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination, B-splines were introduced by Schoenberg in 1946 [6]. Up to now, B-spline approximation method for numerical solutions has been researched by various researchers [7]-[14].

In this paper, the septic B-spline function is used as a basis function and the B-spline collocation method is studied to solve the linear and nonlinear fifth-order boundary value problems. The method is fourth order convergent. We use the quesilinearization technique to reduce the nonlinear problems to linear problems. The present method is tested for its efficiency by considering two examples.

2. Septic B-Spline Interpolation

An arbitrary Nth order spline function with compact support of N. It is a concatenation of N sections of (N-1)th order polynomials, continuous at the junctions or "knots", and gives continuous (N-1)th derivatives at the junctions.

Let $\{x_i\}_{i=0}^{N}$ be a uniform partition of $[c,d]$ such that $x_i = c + ih$, $i = 0,1,2,\cdots,N$, where $h = (d-c)/N$. Let the septic B-spline function $\phi_i(x)$ with knots at the points x_i be given by

$$\phi_i(x) = \frac{1}{h^7}\begin{cases} (x-x_{i-4})^7 & x\in[x_{i-4},x_{i-3}] \\ (x-x_{i-4})^7 - 8(x-x_{i-3})^7 & x\in[x_{i-3},x_{i-2}] \\ (x-x_{i-4})^7 - 8(x-x_{i-3})^7 + 28(x-x_{i-2})^7 & x\in[x_{i-2},x_{i-1}] \\ (x-x_{i-4})^7 - 8(x-x_{i-3})^7 + 28(x-x_{i-2})^7 - 56(x-x_{i-1})^7 & x\in[x_{i-1},x_i] \\ (x_{i+4}-x)^7 - 8(x_{i+3}-x)^7 + 28(x_{i+2}-x)^7 - 56(x_{i+1}-x)^7 & x\in[x_i,x_{i+1}] \\ (x_{i+4}-x)^7 - 8(x_{i+3}-x)^7 + 28(x_{i+2}-x)^7 & x\in[x_{i+1},x_{i+2}] \\ (x_{i+4}-x)^7 - 8(x_{i+3}-x)^7 & x\in[x_{i+2},x_{i+3}] \\ (x_{i+4}-x)^7 & x\in[x_{i+3},x_{i+4}] \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The set of splines $\{\phi_{-3},\phi_{-2},\phi_{-1},\phi_0,\phi_1,\cdots,\phi_N,\phi_{N+1},\phi_{N+2},\phi_{N+3}\}$ forms a basis for the functions defined over $[c,d]$. The values of $\phi_i(x)$ and its derivatives are as shown in **Table 1**.

We seek the approximation $S(x)$ to the exact solution $y(x)$, which uses these septic B-splines:

$$S(x) = \sum_{i=-3}^{N+3} a_i \phi_i(x) \quad (4)$$

which satisfies the following interpolation conditions:

$$\begin{cases} s(x_i) = y(x_i), \quad (i = 0,1,2,\cdots,n) \\ s^{(1)}(c) = y^{(1)}(c), s^{(2)}(c) = y^{(2)}(c), \\ s^{(1)}(d) = y^{(1)}(d),\ s^{(2)}(d) = y^{(2)}(d), \end{cases} \quad (5)$$

where a_i are unknown real coefficients.

Using the septic B-spline function Equation (3) and the approximate solution Equation (4), the nodal values $S(x_j)$ and $S^{(5)}(x_j)$ at the node x_j are given in terms of element parameters by

$$S(x_j) = a_{j-3} + 120a_{j-2} + 1191a_{j-1} + 2416a_j + 1191a_{j+1} + 120a_{j+2} + a_{j+3} \quad (6)$$

$$S^{(5)}(x_j) = \frac{2520}{h^5}\left(-a_{j-3} + 4a_{j-2} - 5a_{j-1} + 5a_{j+1} - 4a_{j+2} + a_{j+3}\right) \quad (7)$$

Table 1. The values of $\phi_i(x)$ and its derivatives with knots.

x	x_{i-4}	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
$\phi_i(x)$	0	1	120	1191	2416	1191	120	1	0
$\phi_i^{(1)}(x)$	0	$-\frac{7}{h}$	$-\frac{392}{h}$	$-\frac{1715}{h}$	0	$\frac{1715}{h}$	$\frac{392}{h}$	$\frac{7}{h}$	0
$\phi_i^{(2)}(x)$	0	$\frac{42}{h^2}$	$\frac{1008}{h^2}$	$\frac{630}{h^2}$	$\frac{-3360}{h^2}$	$\frac{630}{h^2}$	$\frac{1008}{h^2}$	$\frac{42}{h^2}$	0
$\phi_i^{(3)}(x)$	0	$-\frac{210}{h^3}$	$-\frac{1680}{h^3}$	$\frac{3990}{h^3}$	0	$\frac{-3990}{h^3}$	$\frac{1680}{h^3}$	$\frac{210}{h^3}$	0
$\phi_i^{(4)}(x)$	0	$\frac{840}{h^4}$	0	$\frac{-7560}{h^4}$	$\frac{13440}{h^4}$	$\frac{-7560}{h^4}$	0	$\frac{840}{h^4}$	0
$\phi^{(5)}(x)$	0	$-\frac{2520}{h^5}$	$\frac{10080}{h^5}$	$-\frac{12600}{h^5}$	0	$\frac{12600}{h^5}$	$-\frac{10080}{h^5}$	$\frac{2520}{h^5}$	0

From Equations (4)-(7), we have

$$\begin{aligned}&s^{(5)}(x_{j-3})+120s^{(5)}(x_{j-2})+1191s^{(5)}(x_{j-1})+2416s^{(5)}(x_j)+1191s^{(5)}(x_{j+1})+120s^{(5)}(x_{j+2})+s^{(5)}(x_{j+3})\\&=\frac{2520}{h^5}\left(-y_{j-3}+4y_{j-2}-5y_{j-1}+5y_{j+1}-4y_{j+2}+y_{j+3}\right),\end{aligned}\tag{8}$$

Using operator notations $Ey(x)=y(x+h), Dy(x)=y'(x), Iy(x)=y(x), E=\mathrm{e}^{hD}$, we obtain

$$s^{(5)}(x_j)=\frac{2520}{h^5}\left(\frac{-E^{-3}+4E^{-2}-5E^{-1}+5E^{+1}-4E^{+2}+E^{+3}}{E^{-3}+120E^{-2}+1191E^{-1}+2416I+1191E^{+1}+120E^{+2}+E^{+3}}\right)y_j \tag{9}$$

Expanding them in powers of hD, we obtain

$$s^{(5)}(x_j)=y^{(5)}(x_j)-\frac{h^4}{240}y^{(9)}(x_j)-\frac{59h^6}{15120}y^{(11)}(x_j)+\frac{259h^8}{226800}y^{(13)}(x_j)+O(h^{10}) \tag{10}$$

Hence we get

$$y(x_j)=a_{j-3}+120a_{j-2}+1191a_{j-1}+2416a_j+1191a_{j+1}+120a_{j+2}+a_{j+3} \tag{11}$$

$$y^{(5)}(x_j)=\frac{2520}{h^5}\left(-a_{j-3}+4a_{j-2}-5a_{j-1}+5a_{j+1}-4a_{j+2}+a_{j+3}\right)+O(h^4) \tag{12}$$

3. Spline Collocation Method

3.1. Linear Problems

From Equation (1) and Equation (12), we can get

$$\begin{aligned}&\frac{2520}{h^5}\left(-a_{j-3}+4a_{j-2}-5a_{j-1}+5a_{j+1}-4a_{j+2}+a_{j+3}\right)+O(h^4)\\&+p_j\left(a_{j-3}+120a_{j-2}+1191a_{j-1}+2416a_j+1191a_{j+1}+120a_{j+2}+a_{j+3}\right)=f_j\end{aligned}\tag{13}$$

Using the boundary conditions and by neglecting the error of Equation (13), we can obtain following linear equations

$$\begin{aligned}&\left[-\frac{2520}{h^5}+p_j\right]a_{j-3}+\left[\frac{10080}{h^5}+120p_j\right]a_{j-2}+\left[-\frac{12600}{h^5}+1191p_j\right]a_{j-1}\\&+2416p_ja_j+\left[\frac{12600}{h^5}+1191p_j\right]a_{j+1}+\left[-\frac{10080}{h^5}+120p_j\right]a_{j+2}+\left[\frac{2520}{h^5}+p_j\right]a_{j+3}=f_j\end{aligned}\tag{14}$$

Or

$$Ba = r \tag{15}$$

where

$$B = \begin{bmatrix} 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & \cdots & 0 \\ -\frac{7}{h} & -\frac{392}{h} & -\frac{1715}{h} & 0 & \frac{1715}{h} & \frac{392}{h} & \frac{7}{h} & 0 & \cdots & 0 \\ \frac{42}{h^2} & \frac{1008}{h^2} & \frac{630}{h^2} & -\frac{3360}{h^2} & \frac{630}{h^2} & \frac{1008}{h^2} & \frac{42}{h^2} & 0 & \cdots & 0 \\ LB_{4,1} & LB_{4,2} & LB_{4,3} & LB_{4,4} & LB_{4,5} & LB_{4,6} & LB_{4,7} & 0 & \cdots & 0 \\ 0 & LB_{5,2} & LB_{5,3} & LB_{5,4} & LB_{5,5} & LB_{5,6} & LB_{5,7} & LB_{5,8} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & LB_{4+N,1+N} & LB_{4+N,2+N} & LB_{4+N,3+N} & LB_{4+N,4+N} & LB_{4+N,5+N} & LB_{4+N,6+N} & LB_{4+N,7+N} \\ 0 & \cdots & 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ 0 & \cdots & 0 & -\frac{7}{h} & -\frac{392}{h} & -\frac{1715}{h} & 0 & \frac{1715}{h} & \frac{392}{h} & \frac{7}{h} \\ 0 & \cdots & 0 & \frac{42}{h^2} & \frac{1008}{h^2} & \frac{630}{h^2} & -\frac{3360}{h^2} & \frac{630}{h^2} & \frac{1008}{h^2} & \frac{42}{h^2} \end{bmatrix}$$

$$a = \left(a_{-3}, a_{-2}, a_{-1}, a_0, a_{1,} \cdots, a_N, a_{N+1}, a_{N+2}, a_{N+3}\right)^{\mathrm{T}}$$

$$r = \left(y(c), y^{(1)}(c), y^{(2)}(c), f(x_0), f(x_1), \cdots, f(x_N), y(d), y^{(1)}(d), y^{(2)}(d)\right)^{\mathrm{T}}$$

where

$$LB_{4+k,1+k} = -\frac{2520}{h^5} + p_k,\ LB_{4+k,2+k} = \frac{10080}{h^5} + 120p_k,\ LB_{4+k,3+k} = -\frac{12600}{h^5} + 1191p_k$$

$$LB_{4+k,4+k} = 2416p_k,\ LB_{4+k,5+k} = \frac{12600}{h^5} + 1191p_k,\ LB_{4+k,6+k} = -\frac{10080}{h^5} + 120p_k$$

$$LB_{4+k,7+k} = \frac{2520}{h^5} + p_k,\ p_k = p(a+kh),\ f_k = f(x_k), k = 0,1,2,\cdots,N$$

T denoting transpose.

In which B is a square matrix of order $N + 7$ with seven nonzero bands. Since B is nonsingular, after solving the linear system Equation (15) for $a_{-3}, a_{-2}, \cdots, a_{N+2}, a_{N+3}$, we can obtain the septic spline approximate solution $S(x) = \sum_{i=-3}^{N+3} a_i \phi_i(x)$ with the accuracy being $O(h^4)$.

3.2. Nonlinear Problems

Consider the nonlinear fifth order boundary value problem

$$y^{(5)}(x) = F(x, y, y') \tag{16}$$

with boundary conditions

$$\begin{aligned} &y(c) = \alpha_0, y^{(1)}(c) = \alpha_1, y^{(2)}(c) = \alpha_2, \\ &y(d) = \alpha_3, y^{(1)}(d) = \alpha_4, y^{(2)}(d) = \alpha_5 \end{aligned} \tag{17}$$

We use the quesilinearization technique to reduce the above nonlinear problem to a sequence of linear problems. Expanding the right hand side of Equation (16), we have

$$\left(y^{(5)}\right)_{k+1} = F\left(x, y, y'\right) \approx F\left(y_k, \left(y'\right)_k\right) + \left(\frac{\partial F}{\partial y}\right)_{(x, y_k)} \left(y_{k+1} - y_k\right) + \left(\frac{\partial F}{\partial y'}\right)_{(x, y_k)} \left(\left(y'\right)_{k+1} - \left(y'\right)_k\right) \tag{18}$$

Equation (18) can be rewritten as

$$y_{k+1}^{(5)} + q_k\left(x\right) y_{k+1}^{(1)} + p_k\left(x\right) y_{k+1} = f_k\left(x\right),\ c < x < d \tag{19}$$

where

$$q_k\left(x\right) = -\left(\frac{\partial F}{\partial y'}\right)_{(x, y_k)},\ p_k\left(x\right) = -\left(\frac{\partial F}{\partial y}\right)_{(x, y_k)},$$
$$f_k\left(x\right) = F\left(y_k, \left(y'\right)_k\right) - \left(\frac{\partial F}{\partial y}\right)_{(x, y_k)} y_k - \left(\frac{\partial F}{\partial y'}\right)_{(x, y_k)} \left(y'\right)_k$$

Equation (19) once the initial values ($k = 0$, $q_k(x)$, $p_k(x)$, $f_k(x)$) has been computed from the initial conditions, Equation (19) becomes into a linear equations with constant coefficients. Equation (19) can be solved by using iterative method.

Subject to the boundary conditions

$$\begin{aligned} &y_{k+1}(c) = \alpha_0, y_{k+1}^{(1)}(c) = \alpha_1, y_{k+1}^{(2)}(c) = \alpha_2, \\ &y_{k+1}(d) = \alpha_3, y_{k+1}^{(1)}(d) = \alpha_4, y_{k+1}^{(2)}(d) = \alpha_5 \end{aligned} \tag{20}$$

Instead of solving nonlinear problem (16) with boundary conditions (17), we solve a sequence of linear problems (19) with boundary conditions (20), we consider $y_{k+1}(x)$ as the numerical solution to nonlinear problem (16) with boundary conditions (17).

4. Computation of Error

The relative error of numerical solution is given by

$$E^r = \frac{\sqrt{\sum_{i=1}^{N}\left(S\left(x_i\right) - y\left(x_i\right)\right)^2}}{\sqrt{\sum_{i=1}^{N}\left(y\left(x_i\right)\right)^2}} \tag{21}$$

The pointwise errors are given by

$$E\left(x_i\right) = \left|S\left(x_i\right) - y\left(x_i\right)\right| \tag{22}$$

The maximum pointwise errors are given by

$$E^N = \max_{0 \le i \le N}\left|y\left(x_i\right) - S\left(x_i\right)\right| \tag{23}$$

5. Numerical Tests

In the section, we illustrate the numerical techniques discussed in the previous section by the following problems.

Example 1. Consider the following equation [15]-[17]:

$$\begin{aligned} &y^{(5)}(x) - y(x) = f(x),\ 0 \le x \le 1 \\ &f(x) = -(15 + 10x)\mathrm{e}^x, \end{aligned}$$

With boundary conditions

$$\begin{aligned} &y(0) = 0, y^{(1)}(0) = 1, y^{(2)}(0) = 0, \\ &y(1) = 0, y^{(1)}(1) = -\mathrm{e}, y^{(2)}(1) = -4\mathrm{e} \end{aligned}$$

The exact solution is given by

$$y(x) = x(1-x)e^{x}$$

The numerical results are shown in **Table 2**, the comparison of maximum absolute errors are given by **Table 3**. The relative errors for different values of h are seen in **Figure 1**. The pointwise errors of example are given in **Figure 2**. The maximum pointwise errors for different values of h are given in **Figure 3**.

Example 2. Consider the following nonlinear equation [15] [18] [19].

Table 2. Maximum absolute errors, relative error for example 1.

h	E^N	E^r	CPU time (seconds)
1/8	1.718286042191597e−004	3.294552154146086e−004	3.218
1/10	6.447839923018339e−005	1.351847113984665e−004	8.172
1/16	1.028885985859818e−005	2.068672045465237e−005	2.953
1/20	4.192815893866442e−006	8.479871033239017e−006	5.391
1/32	6.368030709968942e−007	1.296221338426021e−006	6.922
1/40	2.475231232201836e−007	5.045908672756208e−007	8.688
1/50	1.671824484961171e−007	3.420027458407504e−007	8.172
1/64	3.108106372273767e−008	6.343766826003454e−008	6.921

Table 3. Comparison of maximum absolute errors for example 1.

h	E^N	E^N	E^N	E^N
	Our method	Caglar *et al.* [15]	Shahid.*et al.* [16]	Khan *et al.* [17]
1/10	6.447839923018339E−5	0.1570	2.259E−4	4.025E−3
1/20	4.192815893866442E−6	0.0747	1.33E−5	3.911E−3
1/40	2.475231232201836E−7	0.0208	5.2812E−7	1.145E−2

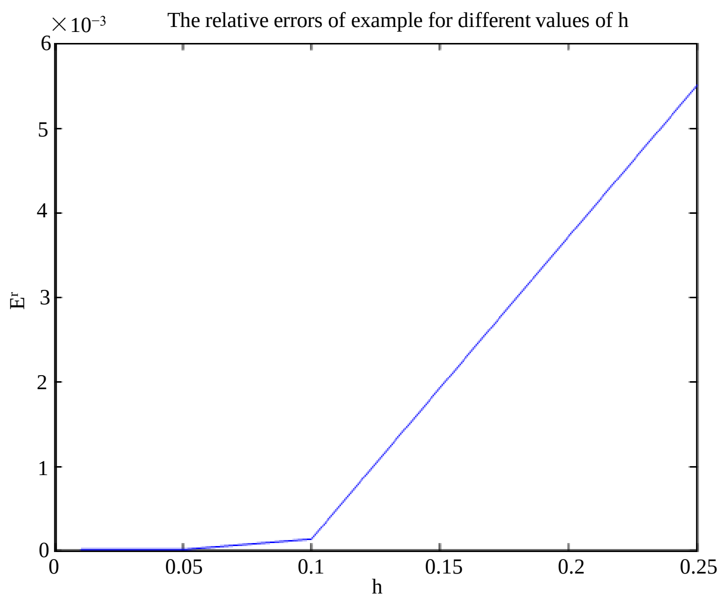

Figure 1. The relative errors of example 1 for different values of h.

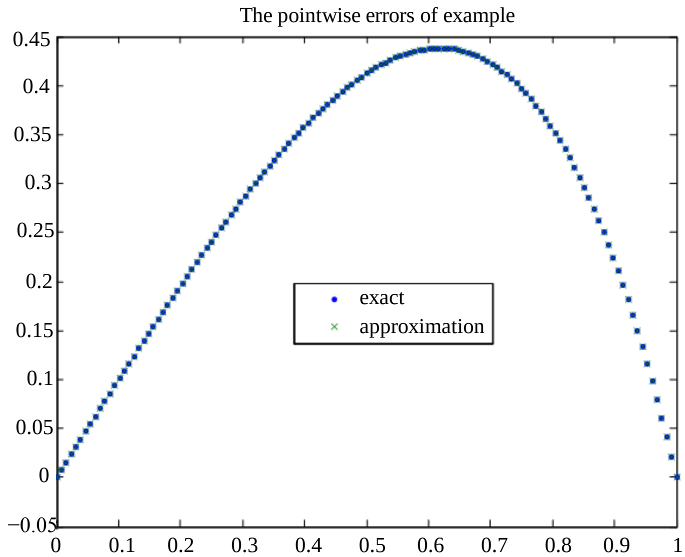

Figure 2. The pointwise errors of example 1.

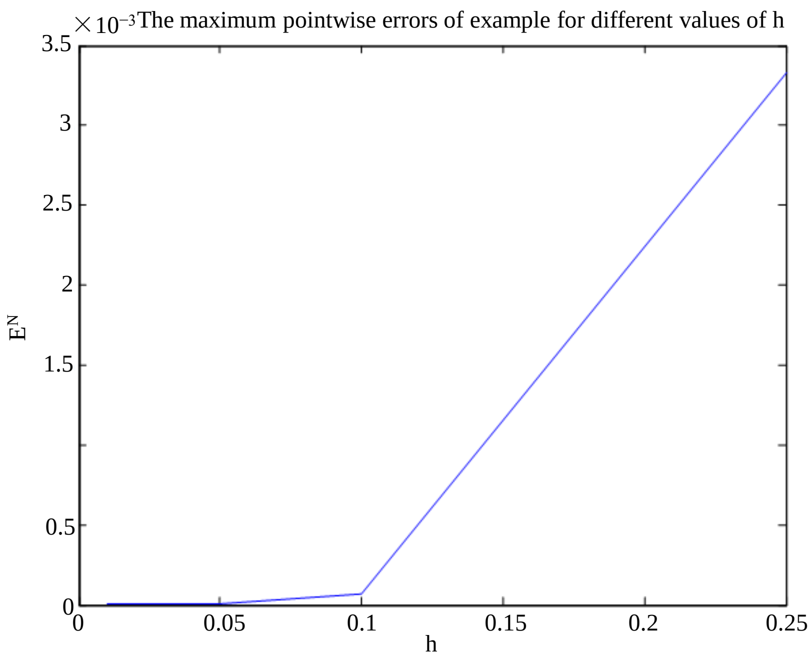

Figure 3. The maximum pointwise errors of example 1 for different values of h.

$$y^{(5)}(x) = e^{-x} y^2(x), \ 0 \le x \le 1$$

With boundary conditions

$$y(0) = y^{(1)}(0) = y^{(2)}(0) = 1$$
$$y(1) = y^{(1)}(1) = y^{(2)}(1) = e$$

The exact solution is given by $y(x) = e^x$.

Comparison of numerical results and pointwise errors are given in Table 4. The numerical result is found in good agreement with exact solution.

Table 4. Example 2. Comparison of results and pointwise errors.

X	Numerical	Exact	Our errors	Errors of [15]	Errors of [17]	Errors of [18]	Errors of [19]
0.1	1.10517089134327	1.10517091807565	2.673237986527965e−008	7.0e−4	1.3e−7	2.3e−7	0
0.2	1.22140266137546	1.22140275816017	9.678471002416700e−008	7.2e−4	4.2e−7	1.6e−6	1.0e−5
0.3	1.34985864259601	1.34985880757600	1.649799901137783e−007	4.1e−4	7.2e−7	4.6e−6	1.0e−5
0.4	1.49182448262282	1.49182469764127	2.150184499338792e−007	4.6e−4	9.4e−7	8.9e−6	1.0e−4
0.5	1.64872127070013	1.64872103467892	2.360212099095094e−007	4.7e−4	1.0e−6	1.3e−5	3.2e−4
0.6	1.82211858273888	1.82211880039051	2.176516300522735e−007	4.8e−4	9.3e−7	1.6e−5	3.6e−4
0.7	2.01375254082381	2.01375270747048	1.666466702410219e−007	3.9e−4	7.1e−7	1.6e−6	1.4e−4
0.8	2.22554083163489	2.22554092849247	9.685758017852209e−008	3.1e−4	4.1e−7	1.2e−5	3.1e−4
0.9	2.45960308048908	2.45960311115695	3.066787002126148e−008	1.6e−4	1.3e−7	5.1e−6	5.8e−4

6. Conclusion

In the paper, the fifth-order boundary value problems are solved by means of septic B-splines collocation method. We use the quesilinearization technique to reduce the nonlinear problems to linear problems and reduce a boundary value problem to the solution of algebraic equations with seven nonzero bands. The numerical results show that the present method is relatively simple to collocate the solution at the mesh points and easily carried out by a computer and approximates the exact solution very well.

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Three-Level Λ-Type Atomic System Localized by the Parameters of the Two Orthogonal Standing-Wave Fields

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Abstract

Localization of the three-level Λ-type atomic system interacting with two orthogonal standing-wave fields is proposed. Two equal and tunable peaks in the 2D plane are obtained by the detunings corresponding to the two orthogonal standing-wave fields when the decreasing intensities of spontaneously generated coherence (SGC) arise in the three-level Λ-type atomic system, while one circular ring with shrinking radii in the 2D plane is obtained by the adjusted phases and wave vectors of the standing-wave fields when the increasing intensities of SGC occur in the three-level Λ-type atomic system. 2D atom localization with the single ring with shrinking radii realized by the multiple parametric manipulations demonstrated the flexibility for our scheme.

Keywords

Three-Level Λ-Type Atomic System, Atom Localization, Spontaneously Generated Coherence (SGC)

1. Introduction

Because precise position measurement of a single atom has some potential applications, such as laser cooling and trapping of neutral atoms [1] [2], atom nanolithography [3], Bose-Einstein condensation [4], and measurement of center-of-mass wave function of moving atoms [5] [6], etc., atomic localization has attracted considerable investigation. Utilizing the interacting between an atom and the standing-wave field, one dimensional (1D) atom localization within the optical wavelength domain is realized by the measurement of the phase shift [7] [8], homodyne detection [9], quantum trajectories [10], the dual quadrature field [11], the upper level population [12]

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[13], the probe field absorption or gain [14]-[16], two-photon spontaneous emission [17], the coherent population trapping [18] and the reservoir modes [19]-[21], etc.

More recently, the researchers proposed two-dimensional (2D) atom localization schemes [22]-[26], in which the two orthogonal standing-wave fields are employed. The 2D atom localization was obtained via measurement of the population in the upper or any ground state in a four-level tripod system [22], in which some interesting spatial periodic structures, such as spikes, craters, and waves are observed. Wan *et al*. suggested the 2D atom localization scheme via incorporating the quantum interference phenomenon in a coherently driven inverted-Y system [23]. Recently, some other schemes, such as via spontaneous emission in a coherently driven five-level M-type atomic system [24], and via the probe absorption in microwave-driven atomic system [25] [26] have been proposed for 2D atom localization.

On the other hand, the phrase of SGC is a well-known concept in quantum optics, which refers to the interference of spontaneous emission channels [27] firstly suggested by Agarwal [28] who showed that the spontaneous emission from a degenerate Λ-type three-level atom is sensitive to the mutual orientation of the atomic dipole moments. And SGC is responsible for many important physical phenomena involving potential application [29]-[36] in lasing without population inversion, coherent population trapping (CPT), group velocity reduction, ultra fast all-optical switching and transparent high-index materials, high-precision spectroscopy and magnetometer and modified quantum beats, etc. Inspired by these studies, we here utilize this quantum interference to explore an efficient scheme of 2D atom localization in a three-level Λ-type system. When the parameters corresponding to the standing-wave fields are changed, finding an atom is sensitive to the SGC intensities at a particular position within a wavelength domain. And a better resolution for 2D atom localization can be theoretically achieved in our scheme.

2. Model and Equations

We consider a Λ-type system as shown in **Figure 1**. The excited state $|1\rangle$ is coupled to the lower levels $|2\rangle$ and $|3\rangle$ via a standing-wave field $E_{x,y}$ and a week probe field E_p. $2\gamma_1$ and $2\gamma_2$ are the spontaneous decay rates of the excited state $|1\rangle$ to the ground states $|2\rangle$ and $|3\rangle$. When the two lower levels $|2\rangle$ and $|3\rangle$ are closely spaced such that the two transitions to the excited state interact with the same vacuum mode, SGC can be present. The standing-wave field $E_{x,y}$ is the superposition of two orthogonal standing-wave fields, *i.e.*, one is in the *x* direction and the second is along *y* direction [26] [37]. The Rabi frequency corresponding to the probe field E_p is $\Omega_p = E_p\mu_{13}/2\hbar$, and the position dependent Rabi frequency corresponding to the standing-wave field $E_{x,y}$ is $\Omega_c(x,y) = E_{x,y}\mu_{12}/2\hbar$ where μ_{13} and μ_{12} are the corresponding dipole matrix elements.

The position-dependent Rabi frequency $\Omega_c(x,y)$ corresponding to the field $E_c(x,y)$ which is the superposition of two standing-wave fields $E_c(x)$ and $E_c(y)$ is defined as [26]

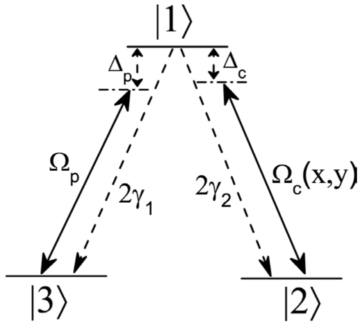

Figure 1. The position-dependent Rabi frequency $\Omega_c(x,y)$ corresponding to the atomic transition from $|1\rangle$ to $|2\rangle$ is due to the superposition of two standing wave fields, *i.e.*, one is along the x direction and the second is along the y direction. The transition from $|1\rangle$ to $|3\rangle$ is coupled via a weak probe field Ω_p, $2\gamma_1$ and $2\gamma_2$ are the atomic decay rates.

$$\Omega_c(x,y)=\Omega_0\left[\sin(\kappa_1 x+\delta)+\sin(\kappa_2 y+\eta)\right] \tag{1}$$

where $\kappa_i=2\pi/\lambda_i,(i=1,2)$ is the wave vector with wavelengths $\lambda_i,(i=1,2)$ of the corresponding standing wave fields. The parameters δ and η are the phase shifts associating with the standing-wave fields having wave vectors κ_1 and κ_2, respectively. We assume that the center-of-mass position of the atom along the direction of the standing-wave field is nearly constant. Therefore we neglect the kinetic-energy part of the Hamiltonian under the Raman-Nath approximation. In the interaction picture, Hamiltonian of this system in a rotating-wave frame is then given by

$$H_I=\Delta_c|1\rangle\langle 1|+(\Delta_c-\Delta_p)|3\rangle\langle 3|+\left[\Omega_p|1\rangle\langle 3|+\Omega_c|1\rangle\langle 2|+H.c.\right]. \tag{2}$$

Here, $\Delta_c=\omega_{12}-\nu_c$ and $\Delta_p=\omega_{13}-\nu_p$ are the field detunings corresponding to the atomic transitions $|1\rangle$-$|2\rangle$ and $|1\rangle$-$|3\rangle$, respectively. ν_c and ν_p are the frequencies for the coupling standing-wave field and week probe field.

Under the rotating-wave approximation [38], the systematic density matrix in the interaction picture involving the SGC can be written as

$$\dot\rho_{11}=-2(\gamma_1+\gamma_2)\rho_{11}+i\Omega_p\rho_{31}+i\Omega_c\rho_{21}-i\Omega_c^*\rho_{12}-i\Omega_p^*\rho_{13}, \tag{3}$$

$$\dot\rho_{22}=2\gamma_2\rho_{11}+i\Omega_c^*\rho_{12}-i\Omega_c\rho_{21}, \tag{4}$$

$$\dot\rho_{33}=2\gamma_1\rho_{11}+i\Omega_p^*\rho_{13}-i\Omega_p\rho_{31}, \tag{5}$$

$$\dot\rho_{12}=-(\gamma_1+\gamma_2+i\Delta_c)\rho_{12}+i\Omega_p\rho_{32}-i\Omega_c(\rho_{11}-\rho_{22}), \tag{6}$$

$$\dot\rho_{13}=-(\gamma_1+\gamma_2+i\Delta_p)\rho_{13}+i\Omega_c\rho_{23}-i\Omega_p(\rho_{11}-\rho_{33}), \tag{7}$$

$$\dot\rho_{23}=-i(\Delta_p-\Delta_c)\rho_{23}+2p\sqrt{\gamma_1\gamma_2}\rho_{11}+i\Omega_c^*\rho_{13}-i\Omega_p\rho_{21}, \tag{8}$$

The above equations are constrained by $\rho_{11}+\rho_{22}+\rho_{33}=1$ and $\rho_{ij}^*=\rho_{ji}$. The effect of SGC is very sensitive to the orientations of the atomic dipole moments μ_{13} and μ_{12}. Here, the parameter *p* denotes the alignment of the two dipole moments and is defined as $p=\boldsymbol{\mu}_{13}\cdot\boldsymbol{\mu}_{12}/|\boldsymbol{\mu}_{13}\cdot\boldsymbol{\mu}_{12}|=\cos\theta$ with θ being the angle between the two dipole moments. So the parameter *p* depicts the intensity of SGC in the atomic system. The terms with $p\sqrt{\gamma_1\gamma_2}$ represent the quantum interference resulting from the cross coupling between spontaneous emission paths $|1\rangle$-$|2\rangle$ and $|1\rangle$-$|3\rangle$. With the restriction of each field acting only on one transition, the Rabi frequencies Ω_c and Ω_p are connected to the angle θ and represented by $\Omega_0=\Omega_{c0}\sin\theta$, $\Omega_p=\Omega_{p0}\sin\theta$. It should be noted that only for small energy spacing between the two lower levels are the interference terms in the systematic density matrix significant; otherwise the oscillatory terms will average out to zero and thereby the SGC effect vanishes.

Our goal here is to obtain the information about the atomic position from the susceptibility of the system [14]-[16] at the probe field frequency. The nonlinear Raman susceptibility χ is then given by

$$\chi=\frac{2N|\mu_{13}|^2}{\varepsilon_0\Omega_p\hbar}\rho_{13}, \tag{9}$$

where *N* is the atom number density in the medium and μ_{13} is the magnitude of the dipole-matrix element between $|1\rangle$ and $|3\rangle$. ε_0 is the permittivity in free space. For simplicity we assume Ω_p and Ω_0 to be real. The general steady-state analytical solution for ρ_{13} can be written as

$$\rho_{13}=\frac{\Omega_p\Omega_c^2\left[B-A+(\Delta_p+2i)C\right]C}{A^2(A-B)+(2A-B)C(A-2\Omega_c^2)\Delta_c+\left[(4+\Delta_c^2)A+(4\Omega_p^2+B)\Omega_c^2-2\Omega_c^4\right]C^2+(2\Delta_c+C)\Omega_c^2C^3} \tag{10}$$

with $A=\Omega_p^2+\Omega_c^2$, $B=2\Omega_c p\cos\theta\cdot\Omega_p$, $C=\Delta_p-\Delta_c$, and we have set $\gamma_1=\gamma_2=\gamma$. All the parameters are reduced to dimensionless units by scaling with γ. Thus the linear susceptibility χ at the probe frequency can therefore be calculated using Equation (9), which consists of both real and imaginary parts, *i.e.*, $\chi=\chi'+i\chi''$.

The imaginary part of the susceptibility gives the absorption profile of the probe field which can be written as

$$\chi'' = \frac{2N|\mu_{13}|^2}{\epsilon_0 \hbar} Im\left[\frac{\rho_{13}}{\Omega_P}\right] = \alpha Im\left[\frac{\rho_{13}}{\Omega_P}\right], \tag{11}$$

where $\alpha = \frac{2N|\mu_{13}|^2}{\epsilon_0 \hbar}$. Here we are interested in the precise position measurement of the atom using the absorption process of the probe field. Equation (11) is the main result and reflects the position probability distribution of the atom [14] [15]. It can be seen that the probe absorption depends on the position dependent SGC intensities, therefore, we can obtain the position information of the atom by measuring the probe absorption.

3. Results and Discussion

The schematic our considered in **Figure 1** can be understood more clearly when the combination of standing-wave fields with the corresponding position-dependent Rabi frequency $\Omega_c(x,y)$ is replaced by a simple driving field with the corresponding Rabi frequency Ω_c [39]. As mentioned earlier [14]-[16] [26], it is clear that the expression (11), which exhibits the probe field absorption, depends on the controllable parameters like the intensities and phase shifts of the standing-wave fields, the detunings of the probe field. Our aim is to investigate the precise location of the 2D atom localization via χ''. Here, we consider different controllable parameters for the atomic position localization, *i.e.*, the interference between spontaneous emission channels, *i.e.*, SGC. The expression (11) reflecting the atomic position probability distribution associating with the intensities p of SGC is rather cumbersome. Hence, we follow the numerical approach and analyze the position probability distribution via χ''.

Initially, we set the detuning $\Delta_c = 0$, $\Delta_p = 15\gamma$, the phase shifts associating with standing-wave fields $\delta = \eta = 0$, and the Rabi frequency $\Omega_{c0} = 10\gamma$, $\Omega_{p0} = 0.3\gamma$. The wavelengths' parameters were set $\kappa_1 = \kappa_2 = \pi/4$, which means wavelengths for the standing-wave fields are 8 wavelength units. For these choices of parameters, we consider the atomic position probability distribution dependent different intensities of SGC (depicted by p).

The position-dependent the intensities of SGC is shown in **Figure 2**, in which χ'' is plotted versus position x and y within the optical wavelength. The intensities of SGC are **Figure 2** (a) p = 0.87, (b) p = 0.81, (c) p = 0.71, (d) p = 0.50. Two spike-like localization peaks sit in the second and fourth quadrants of the x-y plane, and their same amplitudes are about 0.02 in **Figure 2(a)** when p = 0.92. The amplitude of the position probability distribution increases to 0.04 when the intensities of SGC was tuned to 0.81 in **Figure 2(b)**. The double spike-like peaks become more sharp and their peak values reach 0.3 in **Figure 2(c)** for p = 0.71. However, two crater-like structures show in the second and fourth quadrants of the x-y plane when p = 0.50 in **Figure 2(d)**, in spite of the amplitude of the localization peak increasing considerably. Under these setting parameters, the best resolution for the 2D atomic localization is obtained when the intensity of SGC with p = 0.71 in **Figure 2(c)**, and the resolution becomes ambiguous when p = 0.5 in **Figure 2(d)**.

We next study how the detuning associating with standing-wave fields brings changes in the 2D atomic localization, *i.e*, $\Delta_c = -1.5\gamma$. In **Figure 3**, χ'' was plotted versus position x and y with other parameters being the same as those in **Figure 2**. The decreasing intensities of SGC are equal to **Figure 3** (a) p = 0.92, (b) p = 0.87, (c) p = 0.81, and (d) p = 0.79. The results presented in **Figure 3** from **Figure 3(a)** to **Figure 3(d)** are gratifying. The dual spike-like peaks with increasing amplitudes are shown in **Figure 3** from **Figure 3(a)** to **Figure 3(d)**, and their values are 0.015, 0.03, 0.08, 0.10, respectively. We noted that the increasing resolution for 2D atomic localization can be obtained when the standing-wave fields couples the transition $|1\rangle \leftrightarrow |2\rangle$ off-resonantly.

After studying the 2D atomic localization dependent the detuning associating with standing-wave fields, we further study how the phase shifts δ, η associating with standing-wave fields bring changes in the 2D atomic localization. Due to the periodicity associated with the position-dependent Rabi frequency $\Omega_c(x,y)$), there will be more than one position probability distribution for the 2D atom localization in the x-y plane. The multiple peaks are much more ambiguous than the unique peak for 2D atom localization. We look for the roles of phases δ and η associated with the standing-wave fields in the 2D atomic localization. In **Figure 4**, the density plots of the 2D position probability distribution are shown for different intensities of SGC, *i.e.*, (a) p = 0.5, (b) p = 0.71, (c) p = 0.81, and we set $\delta = \eta = \pi/2$, $\kappa_1 = \kappa_2 = \pi/8$, $\Delta_p = 12\gamma$, other parameters are the same as those

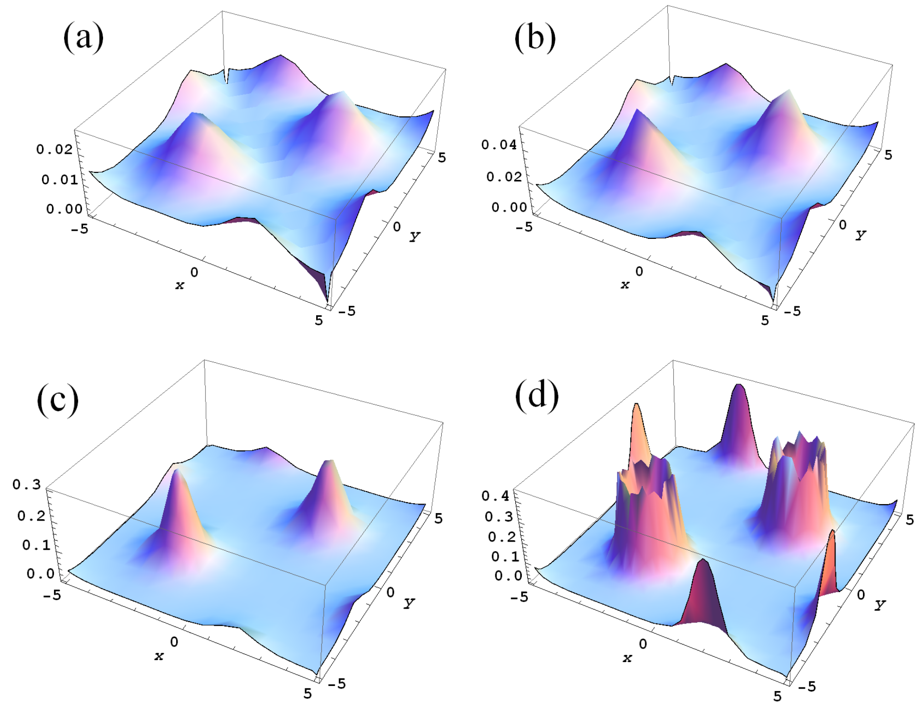

Figure 2. (Color online) Plots for 2D atom localization: χ'' versus the positions x and y for different intensities p of SGC. (a) p = 0.87, (b) p = 0.81, (c) p = 0.71, (d) p = 0.50. Other parameters are $\Delta_p = 15\gamma$, $\Delta_c = 0$, $\alpha = \gamma$, and $\gamma_1 = \gamma_2 = \gamma$ where γ is the scaling parameter.

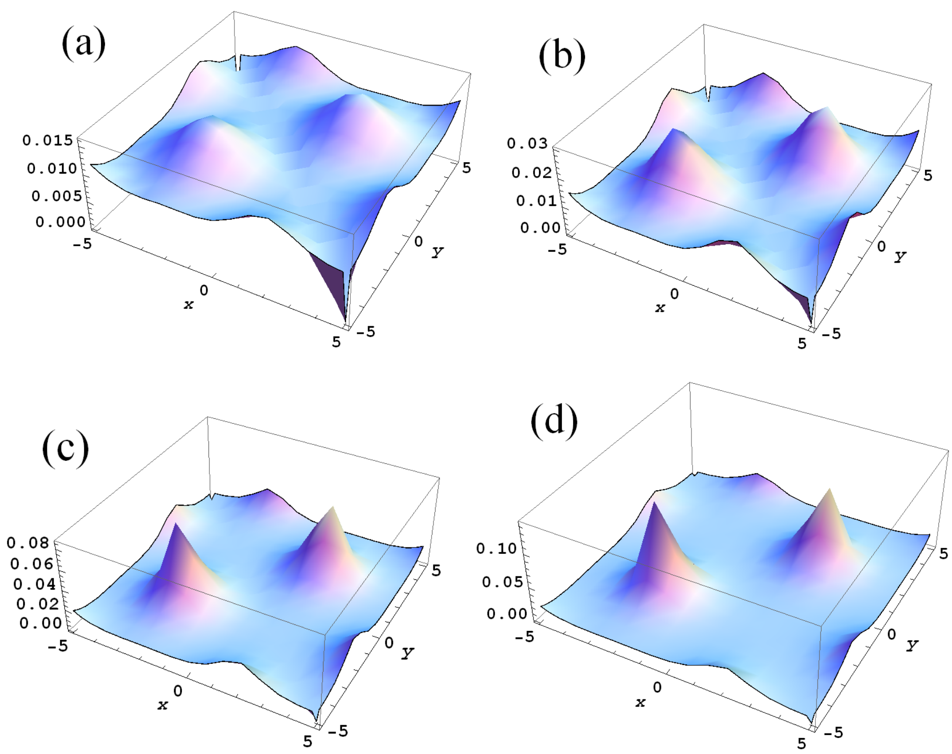

Figure 3. (Color online) Plots for 2D atom localization: χ'' versus the positions x and y for different intensities p of SGC. (a) p = 0.92, (b) p = 0.87, (c) p = 0.81, (d) p = 0.79, $\Delta_c = -1.5\gamma$. The other parameters are the same as in Figure 2.

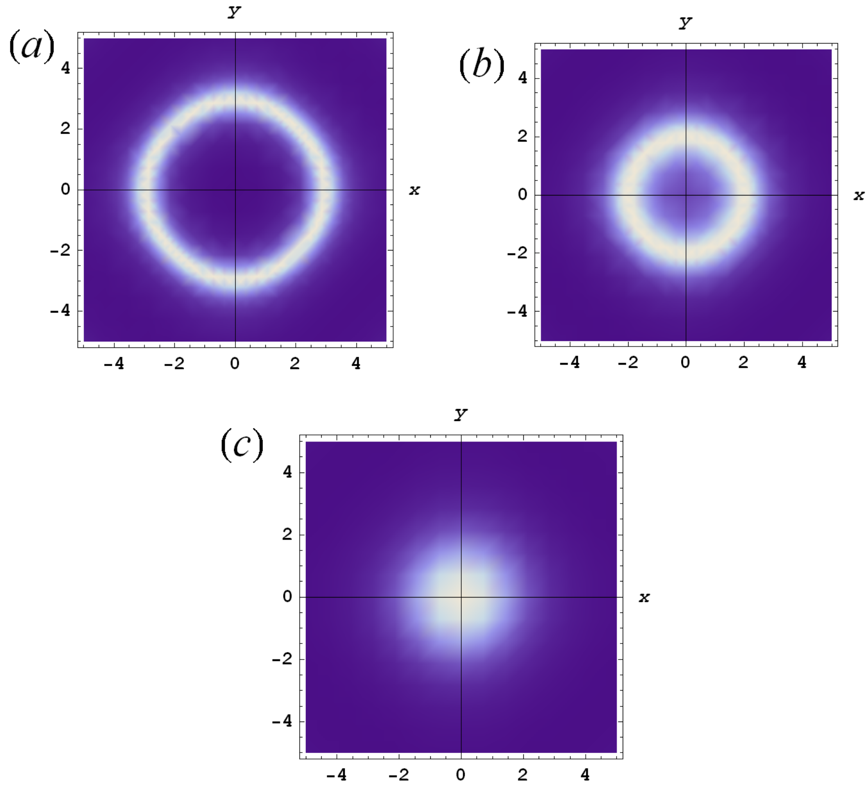

Figure 4. (Color online) Density plots of 2D atom localization: Plots of χ'' for different intensities p of SGC. (a) p = 0.5, (b) p = 0.71, (c) p = 0.81. κ_1 and $\kappa_2 = \pi/8$, $\delta = \eta = \pi/2$, $\Delta_p = 12\gamma$. All the other parameters are the same as in **Figure 2**.

4. Conclusion

On the basis of a three-level Λ-type atom model, we investigated its 2D atom localization via different parameters of the two orthogonal standing-wave fields with the decreasing or increasing intensities of SGC. Two spike-like peaks with flexible amplitudes for the atom localization are shown in the x-y plane by the decreasing intensities of SGC when the detuning corresponding to the two orthogonal standing-wave fields is varied, while the increasing intensities of SGC reduce the resolution of 2D atom localization strongly. When the phases and wave vectors corresponding to the standing-wave fields are changed, one circular ring with shrinking radii for 2D atom localization is obtained by the increasing intensities of SGC. Comparing the phases and wave vectors with the detuning corresponding to the two orthogonal standing-wave fields, a better resolution can be obtained by the phases and wave vectors corresponding to the two orthogonal standing-wave fields when SGC was manipulated. Considering the proposed three-level atom-field system to be a simple system which would be realized experimentally, such as the bichromatic EIT in cold rubidium atoms (^{87}Rb), our scheme and results may be of great interest for the researchers.

Fund

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A New Approach for Dispersion Parameters

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Abstract

This paper presents a new approach to identify and estimate the dispersion parameters for bivariate, trivariate and multivariate correlated binary data, not only with scalar value but also with matrix values. For this direction, we present some recent studies indicating the impact of over-dispersion on the univariate data analysis and comparing a new approach with these studies. Following the property of McCullagh and Nelder [1] for identifying dispersion parameter in univariate case, we extended this property to analyze the correlated binary data in higher cases. Finally, we used these estimates to modify the correlated binary data, to decrease its over-dispersion, using the Hunua Ranges data as an ecology problem.

Keywords

Measures of Association, Correlated Binary Data, Dispersion Parameters, Scaled Deviance, Scalar Value, Scalar Matrix

1. Introduction

The dispersion parameter should be the unity in case of the univariate Bernoulli data, but there may be deviation if there is a sequence of the Bernoulli outcomes included in a study that may lead to a binomial variable. The over-dispersion is happened if the variance of actual response is more than the nominal variance, $Var(Y) > V(\mu)$, as a function of the mean, μ. The estimation of dispersion parameter in the univariate case can be obtained easily using the Pearson's Chi-square or the deviance function. Many studies have devoted the over-dispersion criteria in the univariate case, namely, when the binomial data are used. It is difficult to extend these methods to estimate the dispersion parameters in the bivariate case, because in the bivariate case, the association between correlated response variables may be happened. So, we must take this association into account when estimate the dispersion parameter. But in the independence case, the estimate of dispersion parameter is performed as in

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the univariate case. The estimate of dispersion parameters for the bivariate correlated binary data can be obtained using different methods. The first one when the dispersion parameter is scalar. The second one when we have a matrix values of dispersion parameters. These estimates can be extended to the trivariate and multivariate correlated binary data. So, we present a new approach to identify and estimate the dispersion parameters, in scalar and matrix values, for the bivariate, trivariate and multivariate correlated binary data. Also, after obtaining these estimates we can modify the correlated binary data, this happens to obtain a dispersion parameter equal or near to the unity.

This paper can be organized as follows: Some of the previous studies are presented in the Section 2.

A proposed approach for identifying and estimating the dispersion parameters in a scalar and matrix values, and the impact of over-dispersion in the case of bivariate, trivariate and multivariate binary outcomes associated with covariates, are demonstrated in the Sections 3, 4 and 5, respectively.

Finally, the numerical examples for the vectorized generalized additive model, VGAM, or vectorized generalized linear model, VGLM, Yee and Wild [2], and the alternative quadratic exponential form, AQEF, measure, El-Sayed *et al*. [3], are demonstrated in Section 6.

2. Previous Studies

In this section, we present some studies on the over-dispersion problem as shown below:

(1) Smith and Heitjan [4] provided an appropriate statistical tool to detect extra binomial variation (over-dispersion). To test the nominal dispersion in the *i*-th ($i=1,2,\cdots,d$) margin, it is important to give the relation, for m_i trials,

$$Var(Y_i)=\phi_i m_i \pi_i (1-\pi_i). \tag{1}$$

The hypothesis testing problem is formulated as

$$H_0:\phi_i=1 \quad \text{vs} \quad H_1:\phi_i>1.$$

An appropriate procedure to test H_0 is the score statistic suggested by Smith and Heitjan

$$\chi^2=J_i' A_i^{-1} J_i, \tag{2}$$

where $J_i=(J_{1i},J_{2i},\cdots,J_{pi})$ is a random vector that registers the difference between actual information and nominal information, in the *i*-th margin with respect to every *j*-th ($j=1,2,\cdots,p$) parameter, for $k(k=1,2,\cdots,n)$ observations, namely

$$J_{ji.}=\frac{1}{2}\sum_{k=1}^{n}\left[\left(y_{jik}-m_i\pi_i\right)^2-m_i\pi_i\left(1-\pi_i\right)\right]x_{jik}^2, \tag{3}$$

And A_i is the covariance matrix of J_i corrected for estimation of linear predictors, θ_i, where $\theta_i=\log\frac{\pi_i}{1-\pi_i}$. Under the null hypothesis, H_0, the asymptotic distribution of statistic (2) is the χ^2 distribution with *p* degrees of freedom. The eventual rejection of H_0 will be a clear evidence that $Var(Y_i)>m_i\pi_i(1-\pi_i)$.

(2) Cook and Ng [5] described a bivariate logistic-normal mixture model for over-dispersed two state Markov processes. The use of these mixed models cause increase in the standard error of marginal probability estimates. They did not specify the explicit form for the over-dispersion estimate, but display the log-likelihood function for the full sample of *m* subjects, as

$$\begin{aligned}\ell(y,\theta)=\sum_{i=1}^{m}\log\left[E_{\alpha_i}\left\{\prod_{k=1}^{2}p_k^{n_{kli}}\left(x_{ki},\beta_k\mid\alpha_{ki}\right)\times\left(1-p_k\right)^{n_{k1i}+n_{k2i}-n_{kli}}\left(x_{ki},\beta_k\mid\alpha_{ki}\right)\right\}\right],\\ i=1,2,\cdots,m,\quad k\neq l=1,2,\end{aligned} \tag{4}$$

where, the expectation, E_{α_i}, is taken with respect to the bivariate normal distribution, hence $\alpha_i=(\alpha_{1i},\alpha_{2i})'\sim BVN(\mu,\Sigma)$, $\beta_k\,(k=1,2)$, are regression parameters.

(3) Saefuddin *et al.* [6] showed the effect of over-dispersion on the hypothesis test of logistic regression.

A simple method proposed by William, [7], was used to correct the effect of over-dispersion by taking inflation factor into consideration. This method takes account of adjusting the estimate of the standard error of the parameter resulting from the over-dispersion. Modeling of the over-dispersion is often expressed in the equation of the variance of response variable, Y_i, for binomial case for n_i trials, as follows

$$Var(Y_i) = n_i\pi_i(1-\pi_i)\left[1+(n_i-1)\phi\right], \tag{5}$$

where $\left[1+(n_i-1)\phi\right]$ is the over-dispersion scale and ϕ denote inflation factor. When the over-dispersion does not occur or very small over-dispersion occurs, ϕ will be approximately equal to zero, so Y_i exactly follows binomial distribution, $Bin(n_i,\pi_i)$, and $Var(Y_i) = n_i\pi_i(1-\pi_i)$, Collett [8]. However, when over-dispersion exists, ϕ exceeds zero and leads $Var(Y_i)$ to be greater than $n_i\pi_i(1-\pi_i)$. The parameter estimate of ϕ, is obtained by equating X^2 statistic of the model to its approximate expected value, written as

$$X^2 = \sum_{i=1}^{n}\frac{w_i(y_i - n_i\hat{\pi}_i)^2}{n_i\hat{\pi}_i(1-\hat{\pi}_i)}, \quad \text{and} \quad E(X^2) \simeq \sum_{i=1}^{n} w_i(1-w_i\upsilon_i d_i)\left[1+(n_i-1)\phi\right], \tag{6}$$

where $\upsilon_i = n_i\pi_i(1-\pi_i)$, w_i is the weight and d_i is the diagonal element of the variance-covariance matrix of the linear predictor, say $\hat{\eta}_i = \sum_{j=1}^{p}\hat{\beta}_j x_{ji}$. The value of X^2 statistic depends on $\hat{\phi}$, so iteration process is needed to find the optimum value. This procedure was the first introduced by William, [7], and is known as William method.

The algorithm of the William method is described as follows:

1. Assume $\phi = 0$, calculate parameter estimate of logistic regression parameter, $\hat{\beta}$, using maximum likelihood method. Calculate the X^2 statistics of fitted model.

2. Compare X^2 statistics to χ^2_{n-p} distribution. If X^2 statistic is too large, conclude that $\phi > 0$ and calculate the initial estimates of ϕ using following formula

$$\hat{\phi}_{0i} = \frac{X^2 - (n-p)}{\sum_{i=1}^{n}(n_i-1)(1-\upsilon_i d_i)}. \tag{7}$$

3. Using the initial weights

$$w_{0i} = \left[1+(n_i-1)\hat{\phi}_0\right]^{-1}, \tag{8}$$

we can recalculate the value of $\hat{\beta}$ and X^2 statistic.

4. If X^2 statistic close to its degrees of freedom, $n-p$, then the estimated value of ϕ is sufficient. If not, re-estimate ϕ using following expression:

$$\hat{\phi} = \frac{X^2 - \sum_{i=1}^{n} w_i(1-w_i\upsilon_i d_i)}{\sum_{i=1}^{n} w_i(n_i-1)(1-w_i\upsilon_i d_i)}. \tag{9}$$

If X^2 statistic remains large, return to step (3) until optimum value of estimated ϕ is obtained. Once ϕ has been estimated by $\hat{\phi}$, $w_i = \left[1+(n_i-1)\phi\right]^{-1}$ could be used as weights in fitting the new model, Collett [8], and William [7]. We conclude that the over-dispersion problem causes lower standard errors of the estimates of parameters.

(4) Davila *et al.* [9] introduced a new approach for modeling the multivariate marginals over-dispersed binomial data. They illustrate this approach by analyzing the data using the Gaussian copula with Beta-binomial margins. In order to model the over-dispersion, they used the Beta-binomial model, a generalization of binomial distribution, Casella and Berger [10]. In this model, it is supposed that $Y_i | P_i \sim Bin(m_i, P_i)$, whereas $P_i \sim Beta(\alpha_i, \beta_i)$. Then, they make the assumption that each margin, Y_i, follows a Beta-binomial distribution. Therefore, unconditionally the compound density, with respect to the counting measure of Y_i, is given by

$$f\left(y_i,\alpha_i,\beta_i\right)=\binom{m_i}{y_i}\frac{Beta\left(y_i+\alpha_i,m_i-y_i+\beta_i\right)}{Beta\left(\alpha_i,\beta_i\right)},\quad y_i\in\{0,1,\cdots,m_i\},\tag{10}$$

where, $\alpha_i>0,\beta_i>0$. Conditional to P_i, the expectation is given by

$$E\left(Y_i\mid P_i\right)=m_i\pi_i=m_i\left(\frac{\alpha_i}{\alpha_i+\beta_i}\right),\quad i=1,2,\cdots,d,\tag{11}$$

The conditional variance is

$$Var\left(Y_i\mid P_i\right)=m_i\pi_i\left(1-\pi_i\right)\frac{\alpha_i+\beta_i+m_i}{\alpha_i+\beta_i+1}=m_i\pi_i\left(1-\pi_i\right)\left[1+\phi_i\left(m_i-1\right)\right],\quad i=1,2,\cdots,d.\tag{12}$$

From the relation (12), we see that the marginal dispersion parameter is

$$\phi_i=\frac{1}{\alpha_i+\beta_i+1}.\tag{13}$$

Comparing the relation (1) with the relation (12), it is noted that the later has a greater variance. In their study, as compared with the multivariate normal (MVN), the marginal GLM, and the marginal over-dispersion model (ODM), they have shown that the model based on the Beta-binomial model (BBM) displayed the higher standard errors associated to estimated parameters.

(5)-The vectorized generalized additive model (VGAM) introduced by Yee and Wild [2] and implemented by Yee [11] [12]. The conditional distribution of VGAM function for bivariate correlated binary responses, $\left(Y_1,Y_2\right)$ given that some covariates, *x*, is:

$$\log f\left(y_1,y_2\mid x\right)=u_0\left(x\right)+u_1\left(x\right)y_1+u_2\left(x\right)y_2+u_{12}\left(x\right)y_1y_2,\tag{14}$$

where, $u_0\left(x\right)$ is the normalizing constant,

$$\begin{bmatrix}u_1\left(x\right)\\u_2\left(x\right)\\u_{12}\left(x\right)\end{bmatrix}=\eta\left(x\right)=\begin{bmatrix}\eta_1\left(x\right)\\\eta_2\left(x\right)\\\eta_3\left(x\right)\end{bmatrix},$$

And the η_j, $j=1,2,3$, are additive predictors. If all the functions are constrained to be linear, then the resulting model is a vector generalized linear model (VGLM).

The conditional distribution of VGAM family function for trivariate binary responses, $\left(Y_1,Y_2,Y_3\right)$ given that some covariates, *x*, is

$$\log f\left(y_1,y_2,y_3\mid x\right)=u_0\left(x\right)+u_1\left(x\right)y_1+u_2\left(x\right)y_2+u_3\left(x\right)y_3+u_{12}\left(x\right)y_1y_2+u_{13}\left(x\right)y_1y_3+u_{23}\left(x\right)y_2y_3.\tag{15}$$

Note that a third order association parameter, u_{123}, for the product, $\left(y_1y_2y_3\right)$, is assumed to be zero for this family, Yee and Wild [2].

The conditional distribution of VGAM (VGLM) function for multivariate correlated binary responses, $\left(Y_1,Y_2,Y_3,\cdots,Y_k\right)$, given that some covariates, *x*, is

$$\log f\left(y_1,y_2,\cdots,y_k\mid x\right)=u_0\left(x\right)+\sum_{j=1}^{k}u_j\left(x\right)y_j+\sum_{j<l}^{k}u_{jl}\left(x\right)y_jy_l,\tag{16}$$

where $u_0\left(x\right)$ is the normalizing constant.

In the next section, we suggest a new approach to estimate the dispersion parameter, ϕ, using a scalar and a matrix values of the dispersion parameters and indicate how the dispersion parameter may influence on the analysis of correlated binary data, specially on the standard errors, the Wald statistics and the LRTs for the bivariate, trivariate and multivariate binary outcomes variables associated with covariates. For fitting the correlated binary data, we use the log-likelihood function for the alternative quadratic exponetial form (AQEF) measure, [3], in the bivariate, trivariate and multivariate case, respectively.

Using the following notations which imply to the link functions which enable us to use the regression model:

$$\left.\begin{aligned} &\beta_1' = \left(\beta_{10} \quad \beta_{11}\right), \quad \mu_1(x) = p_1(x) = \frac{e^{\beta_1' x}}{1+e^{\beta_1' x}}, \quad x' = \left(1 \quad x_i\right), \quad i = 1, 2, \cdots, n, \\ &\beta_2' = \left(\beta_{20} \quad \beta_{21}\right), \quad \mu_2(x) = p_2(x) = \frac{e^{\beta_2' x}}{1+e^{\beta_2' x}}, \\ &\beta_3' = \left(\beta_{30} \quad \beta_{31}\right), \quad \mu_3(x) = p_3(x) = \frac{e^{\beta_3' x}}{1+e^{\beta_3' x}}, \\ &\alpha_1' = \left(\alpha_{10} \quad \alpha_{11}\right), \\ &\alpha_2' = \left(\alpha_{20} \quad \alpha_{21}\right), \\ &\alpha_3' = \left(\alpha_{30} \quad \alpha_{31}\right), \\ &\alpha_4' = \left(\alpha_{40} \quad \alpha_{41}\right), \end{aligned}\right\} \tag{17}$$

we have the log-likelihood function for the bivariate AQEF measure as

$$\ell\left(y, \beta_1, \beta_2, \alpha\right) = \sum_{i=1}^{n}\left[\beta_1' x y_{1i} + \beta_2' x y_{2i} + \alpha' x y_{1i} y_{2i} - \log\left(1 + e^{\beta_1' x} + e^{\beta_2' x} + e^{\beta_1' x + \beta_2' x + \alpha' x}\right)\right]. \tag{18}$$

The log-likelihood function for the trivariate AQEF measure is

$$\ell\left(y, \beta, \alpha\right) = \beta_1' x y_1 + \beta_2' x y_2 + \beta_3' x y_3 + \alpha_1' x y_1 y_2 + \alpha_2' x y_1 y_3 + \alpha_3' x y_2 y_3 + \alpha_4' x y_1 y_2 y_3 - \log A\left(\beta, \alpha\right), \tag{19}$$

where,

$$A\left(\beta, \alpha\right) = 1 + e^{\beta_1' x} + e^{\beta_2' x} + e^{\beta_3' x} + e^{\beta_1' x + \beta_2' x + \alpha_1' x} + e^{\beta_1' x + \beta_3' x + \alpha_2' x} + e^{\beta_2' x + \beta_3' x + \alpha_3' x} + e^{\beta_1' x + \beta_2' x + \beta_3' x + \alpha_1' x + \alpha_2' x + \alpha_3' x + \alpha_4' x}$$

Finally, the log-likelihood function for the multivariate AQEF measure is

$$\ell\left(y, \beta, \alpha\right) = \sum_{i=1}^{n}\left[\sum_{j=1}^{k} \beta_j' x y_{ji} + \sum_{1 \le j < l}^{k} \alpha_{jl}' x y_{ji} y_{li} + \cdots + \alpha_{12\cdots k}' x y_{1i} y_{2i} \cdots y_{ki} - \log A\left(\beta, \alpha\right)\right], \tag{20}$$

where,

$$A\left(\beta, \alpha\right) = 1 + \sum_{j=1}^{k} \exp\left[\beta_j' x\right] + \sum_{1 \le j < l}^{k} \exp\left[\beta_j' x + \beta_l' x + \alpha_{jl}' x\right] + \cdots + \exp\left[\sum_{j=1}^{k}\left(\beta_j' x\right) + \cdots + \alpha_{123\cdots k}' x\right]. \tag{21}$$

3. Dispersion Parameters in Bivariate Case

In this section, we determine the identification and estimation of a fixed value for dispersion parameter, ϕ, and also a matrix of dispersion parameters to extend the effect of over-dispersion on the analysis of bivariate correlated binary data.

3.1. Scalar Dispersion Parameter

We can use the variance-covariance matrix of Y_1 and Y_2 to estimate a scalar dispersion parameter, ϕ, in the bivariate binary outcomes. So, we can define the response vector

$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ and its mean vector $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$.

Following the GLM property, the variance-covariance matrix of Y is

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \phi \begin{bmatrix} V\left(\mu_1\right) & V\left(\mu_1, \mu_2\right) \\ V\left(\mu_2, \mu_1\right) & V\left(\mu_2\right) \end{bmatrix},$$

where,

$$\sigma_{11} = E\left(Y_1 - \mu_1\right)^2, \quad \sigma_{22} = E\left(Y_2 - \mu_2\right)^2, \quad \sigma_{12} = \sigma_{21} = E\left(Y_1 - \mu_1\right)\left(Y_2 - \mu_2\right),$$

And,

$$V(\mu_1)=\mu_1(1-\mu_1),\quad V(\mu_2)=\mu_2(1-\mu_2),\quad Cov(\mu_1,\mu_2)=Cov(\mu_2,\mu_1)=\rho_{12}\sqrt{V(\mu_1)V(\mu_2)},$$

Then, the estimator of ϕ, for n observations, is

$$\hat{\phi}=\frac{1}{n-p}\sum_{i=1}^{n}\begin{bmatrix} y_{1i}-\hat{\mu}_{1i} & y_{2i}-\hat{\mu}_{2i}\end{bmatrix}\begin{bmatrix} V(\hat{\mu}_{1i}) & Cov(\hat{\mu}_{1i},\hat{\mu}_{2i}) \\ Cov(\hat{\mu}_{2i},\hat{\mu}_{1i}) & V(\hat{\mu}_{2i})\end{bmatrix}^{-1}\begin{bmatrix} y_{1i}-\hat{\mu}_{1i} \\ y_{2i}-\hat{\mu}_{2i}\end{bmatrix}. \tag{22}$$

Hence, we can show that

$$\hat{\phi}=\frac{1}{n-p}\sum_{i=1}^{n}\frac{(y_{1i}-\hat{\mu}_{1i})^2 V(\hat{\mu}_{2i})+(y_{2i}-\hat{\mu}_{2i})^2 V(\hat{\mu}_{1i})-2Cov(\hat{\mu}_{1i},\hat{\mu}_{2i})(y_{1i}-\hat{\mu}_{1i})(y_{2i}-\hat{\mu}_{2i})}{V(\hat{\mu}_{1i})V(\hat{\mu}_{2i})-\left[Cov(\hat{\mu}_{1i},\hat{\mu}_{2i})\right]^2}. \tag{23}$$

Then,

$$\sum_{i=1}^{n}\begin{bmatrix} y_{1i}-\mu_{1i} & y_{2i}-\mu_{2i}\end{bmatrix}\begin{bmatrix} V(\mu_{1i}) & Cov(\mu_{1i},\mu_{2i}) \\ Cov(\mu_{2i},\mu_{1i}) & V(\mu_{2i})\end{bmatrix}^{-1}\begin{bmatrix} y_{1i}-\mu_{1i} \\ y_{2i}-\mu_{2i}\end{bmatrix}$$

Follows the non-central χ^2_{n-p}. Under independence, this quantity follows, approximately, χ^2_{n-p}. An estimator of ϕ in this case is

$$\hat{\phi}=\frac{1}{n-p}\sum_{i=1}^{n}\left[\frac{(y_{1i}-\hat{\mu}_{1i})^2}{\hat{\mu}_{1i}(1-\hat{\mu}_{1i})}+\frac{(y_{2i}-\hat{\mu}_{2i})^2}{\hat{\mu}_{2i}(1-\hat{\mu}_{2i})}\right]. \tag{24}$$

3.2. Matrix of Dispersion Parameters

Now, we use different values for dispersion parameter, such that $\phi_{11},\phi_{22},\phi_{12}$ and ϕ_{21}, here, $\phi_{12}=\phi_{21}$. The variance-covariance matrix of Y is

$$\Sigma=\begin{bmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{bmatrix}=\begin{bmatrix}\phi_{11}V(\mu_1) & \phi_{12}V(\mu_1,\mu_2) \\ \phi_{21}V(\mu_2,\mu_1) & \phi_{22}V(\mu_2)\end{bmatrix}. \tag{25}$$

The estimator of dispersion parameters matrix is

$$\begin{bmatrix}\hat{\phi}_{11} & \hat{\phi}_{12} \\ \hat{\phi}_{21} & \hat{\phi}_{22}\end{bmatrix}=\begin{bmatrix}\dfrac{\hat{\sigma}_{11}}{V(\hat{\mu}_1)} & \dfrac{\hat{\sigma}_{12}}{Cov(\hat{\mu}_1,\hat{\mu}_2)} \\ \dfrac{\hat{\sigma}_{21}}{Cov(\hat{\mu}_2,\hat{\mu}_1)} & \dfrac{\hat{\sigma}_{22}}{V(\hat{\mu}_2)}\end{bmatrix},$$

Then,

$$\begin{bmatrix}\hat{\phi}_{11} & \hat{\phi}_{12} \\ \hat{\phi}_{21} & \hat{\phi}_{22}\end{bmatrix}=\frac{1}{n-p}\begin{bmatrix}\displaystyle\sum_{j=1}^{n}\frac{(y_{1i}-\hat{\mu}_{1i})^2}{V(\hat{\mu}_{1i})} & \displaystyle\sum_{j=1}^{n}\frac{(y_{1i}-\hat{\mu}_{1i})(y_{2i}-\hat{\mu}_{2i})}{Cov(\hat{\mu}_{1i},\hat{\mu}_{2i})} \\ \displaystyle\sum_{j=1}^{n}\frac{(y_{2i}-\hat{\mu}_{2i})(y_{1i}-\hat{\mu}_{1i})}{Cov(\hat{\mu}_{2i},\hat{\mu}_{1i})} & \displaystyle\sum_{j=1}^{n}\frac{(y_{2i}-\hat{\mu}_{2i})^2}{V(\hat{\mu}_{2i})}\end{bmatrix}. \tag{26}$$

From the equation (26), we have

$$\sum_{i=1}^{n}\begin{bmatrix} y_{1i}-\mu_{1i} & y_{2i}-\mu_{2i}\end{bmatrix}\begin{bmatrix} \phi_{11i}V(\mu_{1i}) & \phi_{12i}Cov(\mu_{1i},\mu_{2i}) \\ \phi_{21i}Cov(\mu_{2i},\mu_{1i}) & \phi_{22i}V(\mu_{2i})\end{bmatrix}^{-1}\begin{bmatrix} y_{1i}-\mu_{1i} \\ y_{2i}-\mu_{2i}\end{bmatrix}$$

Follows the non-central χ^2_{n-p}. Under independence, this quantity follows, approximately, χ^2_{n-p}. If $\hat{\phi}_{12}=\hat{\phi}_{21}=0$, and $\hat{\phi}_{11}=\hat{\phi}_{22}=\hat{\phi}$, then the estimator of ϕ is same as (24).

We can correct the data using the estimates of dispersion parameters, $\hat{\phi}_{11},\hat{\phi}_{22}$, and Equation (25), for the *i*-th observation, in the bivariate case as

$$\left.\begin{aligned}
&Var(Y_{1i}) = \phi_{11i}V(\mu_{1i}) \Leftrightarrow Var\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}}\right) = V(\mu_{1i}),\\
&Var(Y_{2i}) = \phi_{22i}V(\mu_{2i}) \Leftrightarrow Var(\frac{Y_{2i}}{\sqrt{\varphi_{22i}}}) = V(\mu_{2i}),\\
&Cov(Y_{1i}, Y_{2i}) = \phi_{12i}Cov(\mu_{1i}, \mu_{2i}) \Leftrightarrow Cov\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}}, \frac{Y_{2i}}{\sqrt{\phi_{22i}}}\right) = Cov(\mu_{1i}, \mu_{2i}).
\end{aligned}\right\} \tag{27}$$

4. Dispersion Parameters in Trivariate Case

We can define the response vector

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \text{ and its mean vector } \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}.$$

4.1. Scalar Dispersion Parameter

The variance-covariance matrix of Y can be written as

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} V(\mu_1) & Cov(\mu_1, \mu_2) & Cov(\mu_1, \mu_3) \\ Cov(\mu_2, \mu_1) & V(\mu_2) & Cov(\mu_2, \mu_3) \\ Cov(\mu_3, \mu_1) & Cov(\mu_3, \mu_2) & V(\mu_3) \end{bmatrix}, \tag{28}$$

where,

$$V(\mu_1) = \mu_1(1-\mu_1), \quad V(\mu_2) = \mu_2(1-\mu_2), \quad V(\mu_3) = \mu_3(1-\mu_3),$$

$$Cov(\mu_1, \mu_2) = \rho_{12}\sqrt{V(\mu_1)V(\mu_2)}, \; Cov(\mu_1, \mu_3) = \rho_{13}\sqrt{V(\mu_1)V(\mu_3)}, \quad Cov(\mu_2, \mu_3) = \rho_{23}\sqrt{V(\mu_2)V(\mu_3)}.$$

The estimator of ϕ, for n observations, is

$$\hat{\phi} = \frac{1}{n-p}\sum_{i=1}^{n}\begin{bmatrix} y_{1i} - \hat{\mu}_{1i} & y_{2i} - \hat{\mu}_{2i} & y_{3i} - \hat{\mu}_{3i} \end{bmatrix} \begin{bmatrix} V(\hat{\mu}_{1i}) & V(\hat{\mu}_{1i}, \hat{\mu}_{2i}) & V(\hat{\mu}_{1i}, \hat{\mu}_{3i}) \\ V(\hat{\mu}_{2i}, \hat{\mu}_{1i}) & V(\hat{\mu}_{2i}) & V(\hat{\mu}_{2i}, \hat{\mu}_{3i}) \\ V(\hat{\mu}_{3i}, \hat{\mu}_{1i}) & V(\hat{\mu}_{3i}, \hat{\mu}_{2i}) & V(\hat{\mu}_{3i}) \end{bmatrix}^{-1} \begin{bmatrix} y_{1i} - \hat{\mu}_{1i} \\ y_{2i} - \hat{\mu}_{2i} \\ y_{3i} - \hat{\mu}_{3i} \end{bmatrix}. \tag{29}$$

Since,

$$\sum_{i=1}^{n}\begin{bmatrix} y_{1i} - \mu_{1i} & y_{2i} - \mu_{2i} & y_{3i} - \mu_{3i} \end{bmatrix} \begin{bmatrix} V(\mu_{1i}) & Cov(\mu_{1i}, \mu_{2i}) & Cov(\mu_{1i}, \mu_{3i}) \\ Cov(\mu_{2i}, \mu_{1i}) & V(\mu_{2i}) & Cov(\mu_{2i}, \mu_{3i}) \\ Cov(\mu_{3i}, \mu_{1i}) & Cov(\mu_{3i}, \mu_{2i}) & V(\mu_{3i}) \end{bmatrix}^{-1} \begin{bmatrix} y_{1i} - \mu_{1i} \\ y_{2i} - \mu_{2i} \\ y_{3i} - \mu_{3i} \end{bmatrix}$$

Follows the non-central χ^2_{n-p}. Then, under independence, this quantity follows, approximately, χ^2_{n-p}. Under independence, the estimator of ϕ is

$$\hat{\phi} = \frac{1}{n-p}\sum_{i=1}^{n}\left[\frac{(y_{1i} - \hat{\mu}_{1i})^2}{\hat{\mu}_{1i}(1-\hat{\mu}_{1i})} + \frac{(y_{2i} - \hat{\mu}_{2i})^2}{\hat{\mu}_{2i}(1-\hat{\mu}_{2i})} + \frac{(y_{3i} - \hat{\mu}_{3i})^2}{\hat{\mu}_{3i}(1-\hat{\mu}_{3i})}\right]. \tag{30}$$

4.2. Matrix of Dispersion Parameters

The variance-covariance matrix of Y can be displayed as

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \phi_{11}V(\mu_1) & \phi_{12}Cov(\mu_1, \mu_2) & \phi_{13}Cov(\mu_1, \mu_3) \\ \phi_{21}Cov(\mu_2, \mu_1) & \phi_{22}V(\mu_2) & \phi_{23}Cov(\mu_2, \mu_3) \\ \phi_{31}Cov(\mu_3, \mu_1) & \phi_{32}Cov(\mu_3, \mu_2) & \phi_{33}V(\mu_3) \end{bmatrix}. \tag{31}$$

The estimator of dispersion parameters, $\begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} & \hat{\phi}_{13} \\ \hat{\phi}_{21} & \hat{\phi}_{22} & \hat{\phi}_{23} \\ \hat{\phi}_{31} & \hat{\phi}_{32} & \hat{\phi}_{33} \end{bmatrix}$, are

$$\hat{\phi} = \frac{1}{n-p}\begin{bmatrix} \sum_{i=1}^{n}\frac{\left(y_{1i}-\hat{\mu}_{1i}\right)^2}{V\left(\hat{\mu}_{1i}\right)} & \sum_{i=1}^{n}\frac{\left(y_{1i}-\hat{\mu}_{1i}\right)\left(y_{2i}-\hat{\mu}_{2i}\right)}{Cov\left(\hat{\mu}_{1i},\hat{\mu}_{2i}\right)} & \sum_{i=1}^{n}\frac{\left(y_{1i}-\hat{\mu}_{1i}\right)\left(y_{3i}-\hat{\mu}_{3i}\right)}{Cov\left(\hat{\mu}_{1i},\hat{\mu}_{3i}\right)} \\ \sum_{i=1}^{n}\frac{\left(y_{2i}-\hat{\mu}_{2i}\right)\left(y_{1i}-\hat{\mu}_{1i}\right)}{Cov\left(\hat{\mu}_{2i},\hat{\mu}_{1i}\right)} & \sum_{i=1}^{n}\frac{\left(y_{2i}-\hat{\mu}_{2i}\right)^2}{V\left(\hat{\mu}_{2i}\right)} & \sum_{i=1}^{n}\frac{\left(y_{2i}-\hat{\mu}_{2i}\right)\left(y_{3i}-\hat{\mu}_{3i}\right)}{Cov\left(\hat{\mu}_{2i},\hat{\mu}_{3i}\right)} \\ \sum_{i=1}^{n}\frac{\left(y_{3i}-\hat{\mu}_{3i}\right)\left(y_{1i}-\hat{\mu}_{1i}\right)}{Cov\left(\hat{\mu}_{3i},\hat{\mu}_{1i}\right)} & \sum_{i=1}^{n}\frac{\left(y_{3i}-\hat{\mu}_{3i}\right)\left(y_{2i}-\hat{\mu}_{2i}\right)}{Cov\left(\hat{\mu}_{3i},\hat{\mu}_{2i}\right)} & \sum_{i=1}^{n}\frac{\left(y_{3i}-\hat{\mu}_{3i}\right)^2}{V\left(\hat{\mu}_{3i}\right)} \end{bmatrix}. \quad (32)$$

Since,

$$\sum_{i=1}^{n}\left[y_{1i}-\mu_{1i} \quad y_{2i}-\mu_{2i} \quad y_{3i}-\mu_{3i}\right]\begin{bmatrix} \phi_{11i}V\left(\mu_{1i}\right) & \phi_{12i}Cov\left(\mu_{1i},\mu_{2i}\right) & \phi_{13i}Cov\left(\mu_{1i},\mu_{3i}\right) \\ \phi_{21i}Cov\left(\mu_{2i},\mu_{1i}\right) & \phi_{22i}V\left(\mu_{2i}\right) & \phi_{23i}Cov\left(\mu_{2i},\mu_{3i}\right) \\ \phi_{31i}Cov\left(\mu_{3i},\mu_{1i}\right) & \phi_{32i}Cov\left(\mu_{3i},\mu_{2i}\right) & \phi_{33i}V\left(\mu_{3i}\right) \end{bmatrix}^{-1}\begin{bmatrix} y_{1i}-\mu_{1i} \\ y_{2i}-\mu_{2i} \\ y_{3i}-\mu_{3i} \end{bmatrix}$$

Follows the non-central χ^2_{n-p}. Under independence, this quantity follows, approximately, χ^2_{n-p}. If $\hat{\phi}_{12}=\hat{\phi}_{13}=\hat{\phi}_{23}=0$ and $\hat{\phi}_{11}=\hat{\phi}_{22}=\hat{\phi}_{33}=\hat{\phi}$, then the estimator of ϕ is same as (30).

Similarly, we can correct the data using the estimates of dispersion parameters, $\hat{\phi}_{11},\hat{\phi}_{22}$ and $\hat{\phi}_{33}$, and the equation (31), for the *i*-th observation, in the trivariate case as

$$\left.\begin{aligned} &Var\left(Y_{1i}\right)=\phi_{11i}V\left(\mu_{1i}\right)\Leftrightarrow Var\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}}\right)=V\left(\mu_{1i}\right), \\ &Var\left(Y_{2i}\right)=\phi_{22i}V\left(\mu_{2i}\right)\Leftrightarrow Var\left(\frac{Y_{2i}}{\sqrt{\phi_{22i}}}\right)=V\left(\mu_{2i}\right), \\ &Var\left(Y_{3i}\right)=\phi_{33i}V\left(\mu_{3i}\right)\Leftrightarrow Var\left(\frac{Y_{3i}}{\sqrt{\phi_{33i}}}\right)=V\left(\mu_{3i}\right), \\ &Cov\left(Y_{1i},Y_{2i}\right)=\phi_{12i}Cov(\mu_{1i},\mu_{2i})\Leftrightarrow Cov\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}},\frac{Y_{2i}}{\sqrt{\phi_{22i}}}\right)=Cov\left(\mu_{1i},\mu_{2i}\right), \\ &Cov\left(Y_{1i},Y_{3i}\right)=\phi_{13i}Cov(\mu_{1i},\mu_{3i})\Leftrightarrow Cov\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}},\frac{Y_{3i}}{\sqrt{\phi_{33i}}}\right)=Cov\left(\mu_{1i},\mu_{3i}\right), \\ &Cov\left(Y_{2i},Y_{3i}\right)=\phi_{23i}Cov(\mu_{2i},\mu_{3i})\Leftrightarrow Cov\left(\frac{Y_{2i}}{\sqrt{\phi_{22i}}},\frac{Y_{3i}}{\sqrt{\phi_{33i}}}\right)=Cov\left(\mu_{2i},\mu_{3i}\right). \end{aligned}\right\} \quad (33)$$

5. Dispersion Parameters in Multivariate Case

We can define the response vector

$Y=\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix}$ and its mean vector $\mu=\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}$.

5.1. Scalar Dispersion Parameter

The variance-covariance matrix of Y can be written as

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{bmatrix} = \begin{bmatrix} V(\mu_1) & Cov(\mu_1,\mu_2) & \cdots & Cov(\mu_1,\mu_k) \\ Cov(\mu_2,\mu_1) & V(\mu_2) & \cdots & Cov(\mu_2,\mu_k) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(\mu_k,\mu_1) & Cov(\mu_k,\mu_2) & \cdots & V(\mu_k) \end{bmatrix}, \tag{34}$$

where,

$$V(\mu_1) = \mu_1(1-\mu_1), V(\mu_2) = \mu_2(1-\mu_2), \cdots, V(\mu_k) = \mu_k(1-\mu_k),$$
$$Cov(\mu_1,\mu_2) = \rho_{12}\sqrt{V(\mu_1)V(\mu_2)}, Cov(\mu_1,\mu_3) = \rho_{13}\sqrt{V(\mu_1)V(\mu_3)}, \cdots, Cov(\mu_{k-1},\mu_k) = \rho_{k-1,k}\sqrt{V(\mu_{k-1})V(\mu_k)}.$$

The estimator of ϕ, for n observations, is

$$\hat{\phi} = \frac{1}{n-p}\sum_{i=1}^{n}\begin{bmatrix} y_{1i}-\hat{\mu}_{1i} & y_{2i}-\hat{\mu}_{2i} & \cdots & y_{ki}-\hat{\mu}_{ki} \end{bmatrix} \begin{bmatrix} V(\hat{\mu}_{1i}) & V(\hat{\mu}_{1i},\hat{\mu}_{2i}) & \cdots & V(\hat{\mu}_{1i},\hat{\mu}_{ki}) \\ V(\hat{\mu}_{2i},\hat{\mu}_{1i}) & V(\hat{\mu}_{2i}) & \cdots & V(\hat{\mu}_{2i},\hat{\mu}_{ki}) \\ \vdots & \vdots & \ddots & \vdots \\ V(\hat{\mu}_{ki},\hat{\mu}_{1i}) & V(\hat{\mu}_{ki},\hat{\mu}_{2i}) & \cdots & V(\hat{\mu}_{ki}) \end{bmatrix}^{-1} \begin{bmatrix} y_{1i}-\hat{\mu}_{1i} \\ y_{2i}-\hat{\mu}_{2i} \\ \vdots \\ y_{ki}-\hat{\mu}_{ki} \end{bmatrix}. \tag{35}$$

Since,

$$\sum_{i=1}^{n}\begin{bmatrix} y_{1i}-\mu_{1i} & y_{2i}-\mu_{2i} & .. & y_{ki}-\mu_{ki} \end{bmatrix} \begin{bmatrix} V(\mu_{1i}) & V(\mu_{1i},\mu_{2i}) & \cdots & V(\mu_{1i},\mu_{ki}) \\ V(\mu_{2i},\mu_{1i}) & V(\mu_{2i}) & \cdots & V(\mu_{2i},\mu_{ki}) \\ \vdots & \vdots & \ddots & \vdots \\ V(\mu_{ki},\mu_{1i}) & V(\mu_{ki},\mu_{2i}) & \cdots & V(\mu_{ki}) \end{bmatrix}^{-1} \begin{bmatrix} y_{1i}-\mu_{1i} \\ y_{2i}-\mu_{2i} \\ \vdots \\ y_{ki}-\mu_{ki} \end{bmatrix}.$$

Follows non-central χ^2_{n-p}. Then, under independence, this quantity follows, approximately, χ^2_{n-p}. Under independence, the estimator of ϕ is

$$\hat{\phi} = \frac{1}{n-p}\sum_{i=1}^{n}\left[\frac{(y_{1i}-\hat{\mu}_{1i})^2}{\hat{\mu}_{1i}(1-\hat{\mu}_{1i})} + \frac{(y_{2i}-\hat{\mu}_{2i})^2}{\hat{\mu}_{2i}(1-\hat{\mu}_{2i})} + \frac{(y_{3i}-\hat{\mu}_{3i})^2}{\hat{\mu}_{3i}(1-\hat{\mu}_{3i})} + \cdots + \frac{(y_{ki}-\hat{\mu}_{ki})^2}{\hat{\mu}_{ki}(1-\hat{\mu}_{ki})}\right]. \tag{36}$$

5.2. Matrix of Dispersion Parameters

The variance-covariance matrix of Y can be displayed as

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_{kk} \end{bmatrix} = \begin{bmatrix} \phi_{11}V(\mu_1) & \phi_{12}Cov(\mu_1,\mu_2) & \cdots & \phi_{1k}Cov(\mu_1,\mu_k) \\ \phi_{21}Cov(\mu_2,\mu_1) & \phi_{22}V(\mu_2) & \cdots & \phi_{2k}Cov(\mu_2,\mu_k) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1}Cov(\mu_k,\mu_1) & \phi_{k2}Cov(\mu_k,\mu_2) & \cdots & \phi_{kk}V(\mu_k) \end{bmatrix}. \tag{37}$$

The estimator of dispersion parameters, $\begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} & \cdots & \hat{\phi}_{1k} \\ \hat{\phi}_{21} & \hat{\phi}_{22} & \cdots & \hat{\phi}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\phi}_{k1} & \hat{\phi}_{k2} & \cdots & \hat{\phi}_{kk} \end{bmatrix}$, are

$$\hat{\phi} = \frac{1}{n-p}\begin{bmatrix} \sum_{i=1}^{n}\frac{(y_{1i}-\hat{\mu}_{1i})^2}{V(\hat{\mu}_{1i})} & \sum_{i=1}^{n}\frac{(y_{1i}-\hat{\mu}_{1i})(y_{2i}-\hat{\mu}_{2i})}{Cov(\hat{\mu}_{1i},\hat{\mu}_{2i})} & \cdots & \sum_{i=1}^{n}\frac{(y_{1i}-\hat{\mu}_{1i})(y_{ki}-\hat{\mu}_{ki})}{Cov(\hat{\mu}_{1i},\hat{\mu}_{ki})} \\ \sum_{i=1}^{n}\frac{(y_{2i}-\hat{\mu}_{2i})(y_{1i}-\hat{\mu}_{1i})}{Cov(\hat{\mu}_{2i},\hat{\mu}_{1i})} & \sum_{i=1}^{n}\frac{(y_{2i}-\hat{\mu}_{2i})^2}{V(\hat{\mu}_{2i})} & \cdots & \sum_{i=1}^{n}\frac{(y_{2i}-\hat{\mu}_{2i})(y_{ki}-\hat{\mu}_{ki})}{Cov(\hat{\mu}_{2i},\hat{\mu}_{ki})} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}\frac{(y_{ki}-\hat{\mu}_{ki})(y_{1i}-\hat{\mu}_{1i})}{Cov(\hat{\mu}_{ki},\hat{\mu}_{1i})} & \sum_{i=1}^{n}\frac{(y_{ki}-\hat{\mu}_{ki})(y_{2i}-\hat{\mu}_{2i})}{Cov(\hat{\mu}_{ki},\hat{\mu}_{2i})} & \cdots & \sum_{i=1}^{n}\frac{(y_{ki}-\hat{\mu}_{ki})^2}{V(\hat{\mu}_{ki})} \end{bmatrix}. \quad (38)$$

Since,

$$\sum_{i=1}^{n}\left[y_{1i}-\mu_{1i} \quad y_{2i}-\mu_{2i} \quad \cdots \quad y_{ki}-\mu_{ki}\right]\begin{bmatrix} \phi_{11i}V(\mu_{1i}) & \phi_{12i}Cov(\mu_{1i},\mu_{2i}) & \cdots & \phi_{1ki}Cov(\mu_{1i},\mu_{ki}) \\ \phi_{21i}Cov(\mu_{2i},\mu_{1i}) & \phi_{22i}V(\mu_{2i}) & \cdots & \phi_{2ki}Cov(\mu_{2i},\mu_{ki}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1i}Cov(\mu_{ki},\mu_{1i}) & \phi_{k2i}Cov(\mu_{ki},\mu_{2i}) & \cdots & \phi_{kki}V(\mu_{ki}) \end{bmatrix}^{-1}\begin{bmatrix} y_{1i}-\mu_{1i} \\ y_{2i}-\mu_{2i} \\ \vdots \\ y_{ki}-\mu_{ki} \end{bmatrix}$$

Follows non-central χ^2_{n-p}. Under independence, this quantity follows, approximately, χ^2_{n-p}. If $\hat{\phi}_{12} = \hat{\phi}_{13} = \cdots = \hat{\phi}_{k-1,k} = 0$ and $\hat{\phi}_{11} = \hat{\phi}_{22} = \cdots = \hat{\phi}_{kk} = \hat{\phi}$, then the estimator of ϕ is same as (36). Similarly, we can correct the data using the estimates of dispersion parameters, $\hat{\phi}_{11}, \hat{\phi}_{22}, \cdots, \hat{\phi}_{kk}$, and the equation (37), for the i-th observation, in the multivariate case as

$$\left.\begin{aligned} &Var(Y_{1i}) = \phi_{11i}V(\mu_{1i}) \Leftrightarrow Var\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}}\right) = V(\mu_{1i}), \\ &Var(Y_{2i}) = \phi_{22i}V(\mu_{2i}) \Leftrightarrow Var\left(\frac{Y_{2i}}{\sqrt{\phi_{22i}}}\right) = V(\mu_{2i}), \\ &\qquad\vdots \\ &Var(Y_{ki}) = \phi_{kki}V(\mu_{ki}) \Leftrightarrow Var\left(\frac{Y_{ki}}{\sqrt{\phi_{kki}}}\right) = V(\mu_{ki}), \\ &Cov(Y_{1i},Y_{2i}) = \phi_{12i}Cov(\mu_{1i},\mu_{2i}) \Leftrightarrow Cov\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}},\frac{Y_{2i}}{\sqrt{\phi_{22i}}}\right) = Cov(\mu_{1i},\mu_{2i}), \\ &Cov(Y_{1i},Y_{3i}) = \phi_{13i}Cov(\mu_{1i},\mu_{3i}) \Leftrightarrow Cov\left(\frac{Y_{1i}}{\sqrt{\phi_{11i}}},\frac{Y_{3i}}{\sqrt{\phi_{33i}}}\right) = Cov(\mu_{1i},\mu_{3i}), \\ &\qquad\vdots \\ &Cov(Y_{k-1,i},Y_{ki}) = \phi_{k-1,k,i}Cov(\mu_{k-1,i},\mu_{ki}) \Leftrightarrow Cov\left(\frac{Y_{k-1,i}}{\sqrt{\phi_{k-1,k-1,i}}},\frac{Y_{ki}}{\sqrt{\phi_{kki}}}\right) = Cov(\mu_{k-1,i},\mu_{ki}). \end{aligned}\right\} \quad (39)$$

6. Numerical Examples

In this section, we present two examples. The first one applies to the bivariate correlated binary data. This example presents the results obtained by using AQEF measure and the VGLM measure which are similar in the bivariate case. The second one applies on the trivariate binary data. However, the third association is absent in the VGAM (VGLM) measure. In both examples, we will use the Hunua Ranges data, Yee [11] [12]. These data were collected from the Hunua Ranges, a small forest in the Southern Auckland, New Zealand.

At 392 sites in the forest, the presence/absence of 17 plant species was recorded along with the altitude. Each site was of area size 200 m^2. The Hunua Ranges data frame has 392 rows and 18 columns. Altitude is a continuous variable, and there are binary responses (presence = 1, absence = 0) for 17 plant species. These data frame contains the following columns: agaaus, beitaw, corlae, cyadea, cyamed, daccup, dacdac, eladen, hedarb, hohpop, kniexc, kuneri, lepsco, metrob, neslan, rhosap, vitluc and altitude (meters above the sea level).

6.1. Application to Bivariate Case

Hence, we will use the first two columns, agaaus and beitaw, as correlated binary outcome variables, Y_1 and Y_2, respectively. A third column, corlae, is used as the explanatory binary variable, *X*.

We will use the estimates, $\hat{\phi}_{11}$ and $\hat{\phi}_{22}$, to modify the correlated data according to the relationship (27).

From **Table 1** and **Table 2**, we demonstrate the conclusions after modifying the correlated data by the estimates of dispersion parameters, as follows:

1. The estimates of the regression parameters are changed.
2. The standard errors are decreased for the estimates of association parameters. This leads to a significant association between the two outcomes binary variables, (Y_1, Y_2), associated with covariate, *x*.
3. The Wald statistic test shows lower values, this confirms a significant association between the two outcomes binary variables, (Y_1, Y_2), associated with covariate, *x*.
4. The LRT is increased, this also confirms the conclusion observed from the Wald statistic.
5. The estimate of a scalar dispersion parameter, ϕ, is increased.
6. The estimates of the matrix of dispersion parameters, ϕ_{11}, ϕ_{22} and ϕ_{12}, increased and close to the unity.
7. The scaled deviance value is increased.

6.2. Application to Trivariate Case

We will use the columns, cyadea, beitaw and kniexc, as the dependent correlated binary variables, Y_1, Y_2 and Y_3, respectively. On the other hand, we will use the column "altitude", meters above sea level, as the continuous explanatory variable, *X*. The estimates of the regression parameters and their tests for the association parameters can be determined for the AQEF and VGLM measures, before and after modifying the correlated data by the estimates of dispersion parameters, ϕ_{11}, ϕ_{22} and ϕ_{33}, as shown in **Table 3**.

Table 1. Results of AQEF and VGLM before modifying the data.

Parameters	Estimates	Standard Errors	Wald Statistic	Parameters/Tests	Estimates
β_{10}	−0.9320	0.1487	−6.2663	ϕ	1.4458
β_{11}	−1.0139	1.0793	−0.9393	S. Deviance	13.8206
β_{20}	−0.2389	0.1191	−2.0057	LRT: $\alpha = 0$	14.5761
β_{21}	1.0656	0.4686	2.2742	ϕ_{11}	0.8579
α_{10}	−0.9598	0.2876	−3.3377	ϕ_{22}	0.9906
α_{11}	−10.9504	154.7484	−0.0708	ϕ_{21}	0.7977

Hence, the LRT's will be compared with $\chi^2(0.05,1) = 3.8415$. Log-likelihood = −454.1039.

Table 2. Results of AQEF and VGLM after modifying data.

Parameters	Estimates	Standard Errors	Wald Statistic	Parameters/Tests	Estimates
β_{10}	−0.8212	0.1456	−5.6397	ϕ	1.4748
β_{11}	−1.0256	1.0457	−0.9808	S. Deviance	26.5973
β_{20}	−0.2106	0.1203	−1.7503	LRT: $\alpha = 0$	16.1546
β_{21}	1.0645	0.4720	2.2552	ϕ_{11}	0.9549
α_{10}	−0.9820	0.2788	−3.5225	ϕ_{22}	1.0853
α_{11}	−10.9610	149.3405	−0.0734	ϕ_{21}	0.8361

Hence, the LRTs will be compared with $\chi^2(0.05,1) = 3.8415$. Log-likelihood = −461.6315.

Table 3. Results before and after modifying data.

Estimates and Tests	Before modifying the data		After modifying the data	
Model	AQEF	VGLM	AQEF	VGLM
$\hat{\beta}_{10}$	−0.2910	−0.9517	−0.2917	−1.0348
$\hat{\beta}_{11}$	−0.0023	−0.0006	−0.0026	−0.0002
$\hat{\beta}_{20}$	−0.5336	−2.8037	−0.4942	−0.6708
$\hat{\beta}_{21}$	0.0009	0.0093	0.0009	0.0062
$\hat{\beta}_{30}$	−0.0139	−0.7867	−0.0724	−0.6435
$\hat{\beta}_{31}$	0.0015	0.0048	0.0010	0.0032
$\hat{\alpha}_{10}$	−0.1245	0.9098	−0.1340	0.6782
$\hat{\alpha}_{11}$	−0.0014	−0.0016	−0.0003	−0.0013
$\hat{\alpha}_{20}$	−0.1180	−0.1369	−0.1269	0.0400
$\hat{\alpha}_{21}$	−0.0007	0.0016	−0.0006	0.0008
$\hat{\alpha}_{30}$	0.0443	2.2313	0.0184	1.5890
$\hat{\alpha}_{30}$	0.0006	−0.0072	0.0007	−0.0048
$\hat{\alpha}_{40}$	0.0438	None	0.0243	None
$\hat{\alpha}_{41}$	0.0053	None	0.0044	None
Scaled Deviance	248.8728	119.2507	272.9934	134.7810
Log-likelihood	−762.1282	−738.9422	−767.4405	−717.1155
LRT: $\alpha_1 = 0$	34.6890	53.2214	31.8918	112.5485
LRT: $\alpha_2 = 0$	2.7690	47.3497	0.4542	143.5384
LRT: $\alpha_3 = 0$	6.4283	57.8120	76.6875	136.8635
LRT: $\alpha_4 = 0$	23.9190	None	3.8179	None
$\hat{\phi}_{11}$	1.0600	1.2202	1.0235	0.9937
$\hat{\phi}_{22}$	0.9802	2.4209	0.9336	1.8922
$\hat{\phi}_{33}$	1.1670	1.2720	1.0416	0.9494
$\hat{\phi}_{12}$	1.1933	1.8250	1.1978	1.0703
$\hat{\phi}_{13}$	1.1767	1.6050	1.1295	1.0700
$\hat{\phi}_{23}$	0.9434	1.7933	0.8912	0.9673
$\hat{\phi}$	1.8424	3.3679	1.6760	2.0798

Hence, the LRT's will be compared with $\chi^2(0.05,1) = 3.8415$.

From **Table 3**, we demonstrate the conclusions after modifying the data by the estimates of dispersion parameters, as follows:

1. The estimates of regression parameters in the two measures are changed.
2. The scaled deviance is increased for the two measures.
3. The estimate of a scalar dispersion parameter, ϕ, is decreased for the two measures.
4. The estimates of values of dispersion parameters, ϕ_{11}, ϕ_{22} and ϕ_{33}, are decreased for the two measures, but close to the unity for the AQEF measure. On the other hand, the estimates of dispersion parameters, ϕ_{12}, ϕ_{13} and ϕ_{23}, are decreased for the two measures, but close to the unity for the VGLM measure.
5. For the VGLM measure, the LRTs reflect significant association between the pairwise outcome variables, (Y_1, Y_2), (Y_1, Y_3) and (Y_2, Y_3), associated with covariates, *x*.

For the AQEF measure, the LRTs also reflect significant association between the pairwise outcome variables, (Y_1, Y_2) and (Y_2, Y_3), associated with covariates, *x*.

However, no significant association is observed between the correlated binary outcome variables, (Y_1, Y_3), associated with covariates, *x*.

6. The LRT for the third association, which is observed from the AQEF measure, reflects no significant association between the correlated binary outcome variables, (Y_1, Y_2, Y_3), associated with covariates, *x*.

So, when modifying the correlated data, the estimates of dispersion parameters, ϕ_{11}, ϕ_{22} and ϕ_{33}, tend to the unity. This leads to no significant association between the outcome variables, Y_1, Y_2 and Y_3, associated with covariates, *x*.

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For all my professors.

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Fundamental Limit for Universal Entanglement Detection

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Abstract

The entanglement, one of the central mysteries of the quantum mechanics, plays a significant role in a variety of applications of the quantum information theory. A natural question in theoretical and experimental importance is whether it is possible to detect a universal entanglement without full diagnostics. The diagnostics relies on a set of quantum trajectories and their records from measurements. This model reflects the probability that each of the measurements may be damaged from interference and decoherence, and may also be associated with recording of continuous signals for an end-time period. The goal is then to retrieve the quantum state such as it had been in the beginning of this measurement process. The proposed solution relies on explicit expression of the probability function through effective matrices contained in the quantum approximation and solutions of ad-joint quantum filters. In this article, we prove a no-go theorem, which outlines this possibility for non-adaptive schemes, which use only single-copy measurements. We also examine in detail a previously conducted experiment, for which it is claimed that detects the entanglement of two-qubit states through adaptive single-copy measurements without full diagnostics. With the conduct of the experiment and the analysis of the data, we demonstrate that the information collected is really sufficient to reconstruct the state. These results reveal a fundamental limit of the single-copy measurements upon the entanglement detection, and provide a common framework for learning the detection of other interesting properties of the quantum states, such as the positivity of partial transposition and the k-symmetric-extendibility.

Keywords

Quantum Operators, Universal Entanglement, Quantum Diagnostics

1. Introduction

Extraction of the state from a dynamic and non-ideal prepared system is of fundamental importance for the quantum computational physics. Contrary to the classical physics, the determination of the quantum state $\bar{\rho}$, its diagnostics, requires a large number of N independent measurements [1] [2]. The state of a quantum system in fact is determined by statistical quantities so the probability of the results for any upcoming measurement can be encoded. The entanglement is one of the central mysteries of the quantum mechanics-two or more parties may be committed in some way that is much stronger than the classical way.

In this article, we offer a method for measurement of the parameters of a partially unknown or partially coherent superposition of system quantum states. Such a circuit is created, when connecting the magnetic sublevels $m = -1$, $m = 0$, and $m = 1$ of $J = 1$ level with $J = 0$ level with laser pulses with linear (π), right circular ($\sigma+$), and left circular ($\sigma-$) polarizations. There are two ways to select the signal-through optical pumping of the population density of the upper state or through fast pulse excitation. The population density ρee (t) depends from the unknown elements of the superposition, as well as from the laser parameters (intensities and phases). By measuring the signal at different laser parameters, we receive a system of algebraic equations for the unknown elements of the superposition, the latter of which we define. The unknown factor F can be eliminated, taking relations of signals, because it does not depend from the laser parameters. The method of optical pumping has priority over the fast pulse excitation, because it is not compulsory at each new measurement of the signal, the Raby frequencies and the frequency differences to be one and the same.

Upon different laser phases and ellipticities are necessary 16% - 20% more independent measurements of the signal for the determination of the elements of the matrix density, compared to the cases when the unknown state is clear.

By default, the measurement process is assumed to be instantaneous and without defects from the decoherence. In specific experimental situations it is not difficult to be taken into account certain measuring imperfections. As a whole, the obtaining of a probability function in the presence of imperfections and decoherence during the measurement has not been examined sufficiently in depth. This is one of the goals of this document-to determine how the deficiencies and decoherence for quantum filtering affect the measurement

More specifically, the bipartite quantum state p_{AB} of the systems A and B, is separable, if it can be written as a combination of states of the products $p_{AB} = \sum_i p_i \rho_{iA} \otimes \rho_{iB}$ as $pi \geq 0$ and $\sum_i p_i = 1$, for certain states ρ_{iA} of the system A and ρ_{iB} of the system B; otherwise p_{AB} is entangled [3]. But not every entangled state p_{AB} violates the Bell's inequalities-some entangled states do not allow hidden variable descriptions [3].

In practice, an entanglement can be detected by measurement of the "entanglement witnesses", physical observables with known values, which prove the existence of quantum entanglement in a given state p_{AB} [4]. But none one of these entanglement witnesses can prove with certainty whether an arbitrary state is entangled or not. On the other hand, the "measurement" of the entanglement plays a similar universal role. According to the generally accepted axioms, the quantum state p_{AB} is entangled only if there is a non-zero value of any entanglement measurement [5]. Unfortunately, the entanglement measurement is not physically observable.

These widely known limitations on the Bell's inequalities, the entanglement witnesses and the entanglement measurement raise a fundamental question: How do we universally detect the entanglement by physical observables? The traditional method for solving this problem is to fully characterize the quantum state by diagnostic of the state [6]-[8], a method that provides full information about the state including, of course, the measurements of the entanglement of the state. But the application of diagnostics of the quantum state requires multiple measurements, a very courageous task for the growing sizes of the system.

A natural solution is to find a way to obtain the value of the entanglement measurement without full diagnostics of the state (FDS). In fact, many efforts were made in this aspect in the last decade [9]-[12]. But the usual techniques to achieve this goal rely mainly on collective measurements of many identical copies of the state p_{AB}. There is a need of joint measurement of more than one copy of the state ($\rho \otimes r$ for some integer $r > 1$). This is a bad news for the experimentalists, since the collective measurements are difficult for application rather than the measurement of single-copy observables. Therefore, it is very desirable to be found a method, which detects the entanglement without full diagnostics of the state by measuring only the single-copy observables. The searching of such a method is very active in the recent years with both theoretical simulations and the realization of experiments, leading to positive results in the implementation of such an attractive task [13] [14].

In this article, we study the possibility for detecting entanglement without full diagnostics of the state by measuring only the single-copy observables. Surprisingly, despite the previous signs, we find that this attractive task, unfortunately is impossible, if only single-copy observables are measured. This means that there is no way to determine with certainty each entanglement measurement or to determine even whether the value is zero or not, without full diagnostics of the state. For greater accuracy, this means that for any set of informationally-incomplete measurements, there are always two different states, an entangled p_{AB}, and a separable σ_{AB}, which give the same results of the study under this measurement. This sounds a bit counter-intuitive at first glance, but as the entanglement is only a single value, while the diagnostics of the quantum state requires measurement of a set of observables, which are informationally-incomplete, there is a need to determine the scale through square of the Hilbert space of the system.

Our observation is that the universal detection of any characteristic without full diagnostics of the state imposes strong geometrical structural conditions for the set of states that have that characteristic. The set of separable states does not fulfill these conditions due to their non-linear nature, and hence the universal detection of entanglement without full diagnostics of the state through the use of single-copy measurements is impossible. There is a good geometric picture of this fact: unless the form of these separable states is not "cylindrical", it is not possible to find a projection of the space of the state on a lower measurable hyperplane with non-overlapping image for the set of separable states or entangled states.

If adaptive measurements are allowed (the observable that must be measured may depend on previous results of measurement), a protocol is applied in [13], with which it is claimed that was discovered an entanglement of two-qubit state p_{AB}, through measurements of one copy without full diagnostics of the state. The protocol includes local filters, which require repeated diagnostics on each single qubit, which leads to a bound of the concurrence [15] on the entanglement measurement of p_{AB}, in case that the single-qubit matrices with reduced density p_A and p_B, are not maximally combined.

We design an experiment with which to apply this adaptive protocol as proposed in [13], and to show that for some p_{AB}, with the collected experimental data, the state p_{AB}, is already fully defined. In other words, after the concurrence of p_{AB}, is defined, the protocol is already leading to full diagnostics of the state of p_{AB}, *i.e.* the protocol does not lead to universal entanglement detection without FDS. This complements our constant result with non-adaptive measurements.

In addition, it is worthwhile to underline that to the best of our knowledge that this is the first experimental realization of the quantum filters (or equivalently of the channel for reducing the amplitude) through an auxiliary method. In comparison with the optical platform that does not require additional ancilla qubits for realization of the channel for reduction of the amplitude [13] [16] [17], our method is more common and may include other systems directly.

We demonstrate also that if joint measurements of r-copies (*i.e.* $\rho_{AB}\otimes r$) are allowed even for $r = 2$ can be found protocols that detect the entanglement of p_{AB}, without FDS. Therefore, our constant result reveals a fundamental limit of the single-copy measurements, and provides a common framework for studying other interesting quantities for a bipartite quantum state, such as the positivity of the partial transposition and k-symmetric-extendibility [18].

2. Results

We discuss the constant result, claiming that it is impossible to determine universally whether a given state is entangled or not without FDS, with single-copy measurements. We first prove a no-go theorem for the non-adaptive measurements, and then examine protocol with adaptive measurements as proposed in detail in [19]. We develop an experiment in order to apply this adaptive protocol, and we demonstrate that the information collected is really sufficient for reconstruction of the state.

Non-adaptive measurement. For any given bipartite state p_{AB}, is allowed only measurement of the physical observables of one copy of this given state. *I.e.* we can only measure the Hilbert's operators S_K, which act on $A\otimes B$. For simplicity, we consider the case in which both A, B, are qubits. Our method naturally includes also the general case of any bipartite system (see "Additional information" for details).

Now we are looking at a two-qubit state p_{AB}. To obtain information on p, we measure a range of S physical observables $S = \{S_1, S_2, \cdots, S_k\}$. The informationally full range of observables contains $k = 15$ linearly regardless of S_1. A simple selection of S is the set of all two-qubit matrices of Pauli, except identical with $i, j = 0, 1, 2, 3$, where $\sigma_0 = I$, $\sigma_1 = X$, $\sigma_2 = Y$, $\sigma_3 = Z$ and $(i, j \neq (0, 0)$.

Let's suppose that we can solve universally whether a random p_{AB} is entangled or not, without a measurement of informationally-complete range of observables. *I.e.* there is a set S of almost $\kappa = 14$ physical observables such that, as we measure S, we can say whether p_{AB} is entangled or not. For this purpose, it is sufficient to assume that

$\kappa = 14$.

The set of all two-qubit states p_{AB}, designated with A, is characterized with 15 real parameters, which form a convex set in R^{15}. The separable two-qubit states S form a convex subset A. It is common knowledge that S has non-vanishing volume [20] [21]. We designate the set of entangled states with E. *i.e.* $E = A/S$.

The set of measurements S with $k = 14$ may be rendered as a definition of the projections of A (and hence of S) on a 14-dimensional hyperplane. If the measurement of the observables in S can demonstrate with certainty whether p_{AB} is entangled or not, the images on the hyperplane of the separable states S and the entangled state E should not overlap. We illustrate this geometric idea in **Figure 1**.

In fact, the only possibility to separate any set from the other states without FDS is the set of the intersection of the set of all states (*i.e*. set A as in **Figure 1**) with a generalized cylinder (*i.e*. a set of the form $\Omega\ (-\infty, +\infty)$, where Ω is a convex set of dimension 14). In this sense, we call these sets "cylindrical", where the relevant states can be separated from the other states from a projection with 14 (or less) dimensions.

So to show that the detection of entanglement without diagnostics is impossible, it is sufficient to prove that S is not "cylindrical" (at R^{15}). To do this, we show that for each projection on the hyperplane with 14 dimensions with normal direction R, there always is a two-qubit state p, which is on the border of the set S, as $R + tR$ is entangled for some t (see "Additional information" for details). *i.e.* p and $p + tR$ have the same image of the hyperplane with 14 dimensions.

To be useful, the quantum computers ultimately must include a large number of qubits. At the same time the scientists studying mainly the quantum physics aim at examining larger and larger objects. For example, an entanglement of up to fourteen ions or eight photons has been generated and checked. However, strictly analyzing the quantum properties of these entangled states in these expanded systems with instruments, which are available is cumbersome, mostly because of difficulties like time consuming and difficult data processing. To overcome these restrictions, we need innovative techniques for the analysis of the quantum state. A possible approach is the testing of an entanglement in a range of previously unexplored quantum systems.

Because of their non-local structure, the entanglement witnesses may not directly be provided from local measurements, but it is still easy to obtain the expectation value: Several types of correlations between the quantum systems can simply be measured and then by a linear combination are retrieved the mean values for calculating the value of non-local observables. These ratios may take the form of experimental measurements of polarization on spatially separated photons, where the ratios may be such that two photons will always have opposite polarization. Geometrically, the set of all separable states is convex, which means that for any two points in it, the line for connection is also in the set. This means that can be found such entanglement witnesses W for any entangled state, since there is always a possibility for separation of entangled from the separable states.

This geometric picture leads to a common framework for studying the detection of other interesting quantities for the bipartite quantum state with single-copy measurements. Indeed, our proof also showed that the states sets with positive partial transposition (PPT) are not "cylinder-like", and it follows herefrom that they cannot be

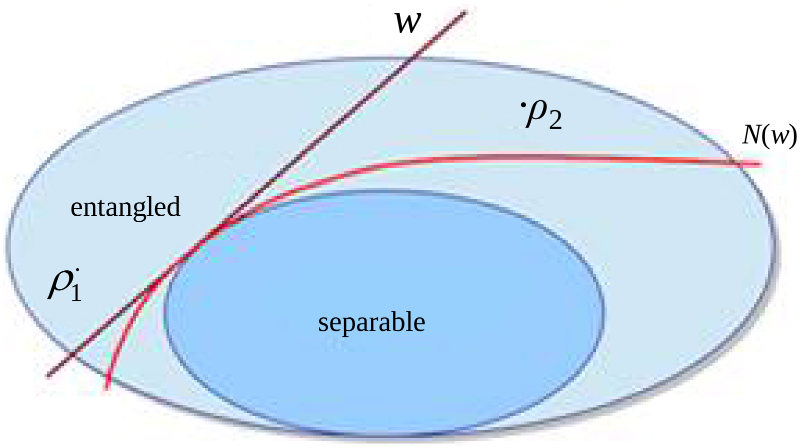

Figure 1. Geometry of the separable and entangled states. In the upper part of the space of all matrices is the multitude of all quantum states (light blue), the non-entangled states, forming a convex subset (dark blue).

detected by single-copy measurements without full diagnostics of the state. With a similar method, we can show that the states sets allowing k-symmetric-extendibility, are also not "cylinder-like", even for a two-qubit system. This shows the fundament limit of the single-copy measurements, *i.e*., a full diagnostics of the state is required in order to universally reveal the non-trivial properties of the quantum states (*i.e*. separability, PPT, k-symmetric-extendibility, see "Additional information" for details).

Adaptive measurement. In case of adaptive protocols, the observable, which must be measured at each step may depend on the previous study results. The type of protocol for measurement may be formulated as follows. First, the observable H_1 is selected, and $\mathrm{tr}(H_1 p)$ is measured. Let's suppose that the result of the study is α_1. On the basis of α_1, the observable $H_{2\alpha 1}$ is selected and $\mathrm{tr}(H_{2\alpha 1} p)$ is measured. Let's assume that the outcome of the survey is α_2. On the basis of α_1, α_2, the observable $H_{3\alpha 1\alpha 2,}$ is selected $\mathrm{tr}(H_{3\alpha 1\alpha 2} p)$, is measured and so on.

The protocol in [13], which should determine the concurrence [15] of a two-qubit state without FDS, falls within the category of adaptive measurements. We implement this protocol and show that with the experimental data collected for a given state $p_{AB,}$ this protocol does not actually lead to FDS of p_{AB}. *i.e*., this protocol does not lead to universal entanglement detection without FDS.

Let's first briefly introduce the idea for distillation of the entanglement by guiding filtering procedure (33). For unknown two-qubit state p_{AB}^0, we measure the local reduced matrices of the density $p_A^0 = \mathrm{tr}_B\left(p_{AB}^0\right)$ and $p_B^0 = \mathrm{tr}_A\left(p_{AB}^0\right)$ for both qubits. In case that p_A^0 and p_B^0 are not fully mixed, we produce the first filter $F_B^0 = 1/\sqrt{2}\, p_A^0$, on the basis of the information for p_A^0 and change p_{AB}^0 to p_{AB}^1. In a similar way, the same procedure is repeated for qubit B. The iterative applications of the filter continue and at step k, the reduced matrices of the density of the qubits will be p_A^κ and p_B^κ.

In case that p_A^0 and p_B^0 are not identity, the iterative procedure described above leads to a "distillation" of the matrices p_A^κ and p_B^κ and it is ensured that both merge into one identity. All of the reduced matrices of the density p_A^i and p_B^i. ($i = 0,1,\cdots,k$) are recorded at the time of the iterative procedure. At step k, when p_A^κ and p_B^κ almost reached the identity, they may be used to reconstruct the boundary of the entanglement value in p_{AB}^0 through the optimum witness W (p_{AB}^0), which depends only on p_A^i and p_B^i. ($i = 0,1,\cdots,k$) (up to local uniform transformation), whose value says whether p_{AB}^0 is entangled or not [13].

At first glance, the above procedure seems acceptable to determine the entanglement value without FDS because only the single-qubit matrices of the density p_A^i and p_B^i. ($i = 0,1,\cdots,k$) are measured for the second time and only local uniform transformations are used for the construction of the optimal witness. *i.e*. it appears that the two-qubit correlations of p_{AB}^0 are never measured, which does not lead to FDS. But more detailed examination shows that this is not the case. The key observation is that the "local" filters in fact are "weak" measurements, which record the correlations in p_{AB}^0. This is because the filters cannot be applied with a probability of one, so that the correlation in p_{AB}^0 is "encoded" in the information that all filters are applied successfully. In other words, what the local filters and the local diagnostics make on each qubit, is in fact a FDS of p_{AB}^0.

To demonstrate the relationship between the local filters and FDS, we simulate a procedure with a local filter as we choose another number of applied filters. It appears that in many cases $k = 4$ (five filter) is sufficient in order to uniquely determine p_{AB}^0 on the basis of p_A^i and p_B^i. ($i = 0,1,\cdots,k$). So the information of p_A^i and p_B^i. leads to FDS of p_{AB}^0.

As an example, we illustrate the simulation of the input state as equation (5) with $\lambda = 0.2$. Initially, we have 15 real parameters (*i.e*. degrees of freedom, DOF for short), which to determine p_{AB}^0 (by ignoring the part of the identity, due to the condition of the normalization). When applying more and more filters, DOF is reduced finally because we obtain more information for the original input state. For example, the initial local reduced matrices of the density p_A^0 and p_B^0 before the application of any filter can reduce DOF up to 9; p_B^1 after the first filter imposes more restrictions so that DOF reduces to 6, etc. It is proven that with 5 filters the input state p_{AB}^0 may be determined uniquely through the information gathered on the reduced matrices of density. And this procedure works in a similar way for many other two-qubit states p_{AB}^0, where 5 filters are sufficient to reconstruct p_{AB}^0, which we will show later in the results of the experiments.

Experimental protocol in the setup of nuclear magnetic resonance. In order to apply experimentally the protocol, we first must discuss how to realize the local filters at a NMR system. Without loss of generality, we can look at the local filter F_A applied on qubit A for example. For any F_A, it may always be decomposed to the form $U_A\Lambda_A V_A$ through decomposition of the only value where U_A and V_A are inseparable values and Λ_A is the diagonal operator of Kraus:

$$\Lambda_A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma_A} \end{pmatrix} \tag{1}$$

$\gamma_A \in [0,1]$ relies on F_A and indicates the probability that the excited state $|1\rangle$ to be decomposed to the ground state $|0\rangle$, when the system undergoes Λ_A. Although it is not inseparable, Λ_A may be extended up to a two-qubit unitary with the assistance of the ancilla qubit 1. Basically, if a two-qubit unitary can transform

$$|0\rangle_1 |0\rangle_A \to |0\rangle_1 |0\rangle_A, |0\rangle_1 |1\rangle_A \to \sqrt{1-\gamma_A}\,|0\rangle_1 |1\rangle_A + \sqrt{\gamma_A}\,|1\rangle_1 |1\rangle_A \tag{2}$$

the quantum channel of the system qubit A, will be Λ_A. By selecting later the subspace, where the ancilla qubit $|0\rangle$ is. One possible unitary transformation, which satisfies the equation (2) is

$$U_{1A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma_A} & 0 & \sqrt{\gamma_A} \\ 0 & 0 & 1 & 0 \\ 0 & -\sqrt{\gamma_A} & 0 & \sqrt{1-\gamma_A} \end{pmatrix} \tag{3}$$

The operation U_{1A} in this way is a controlled rotation: when the qubit of the system A is $|0\rangle$, the ancilla remains invariant; when A is $|1\rangle$, the ancilla undergoes a rotation $R_y(0_A) = e^{i0\tau y/2}$, where $0_A = 2\arccos\sqrt{1-Y_A}$. Therefore, in a system with ancillaries, when initiating the ancilla with $|0\rangle$, the local filter F_A may be implemented through a two-qubit gate $(I\otimes U_A)1_A(I\otimes V_A)$ followed by post-selection of a subspace, in which the ancilla is $|0\rangle$.

The schematic circuit for the application of a proposal for a filter-based distillation of the entanglement for an unknown two-qubit state p^0_{AB}. $F^I_{A,B} = 1/\sqrt{2}\, p^i_{A,B}$ $(i \geq 0)$ is i local filter, applied to A and B, where $p^i_A = \mathrm{tr}_B\left(p^i_{AB}\right)$ and $p^i_B = \mathrm{tr}_A\left(p^i_{AB}\right)$. A single-qubit diagnostic is applied. A variation is simulated of concurrence and fidelity by increasing the number $1 \leq m < 6$ of the applied filters. The simulated state is selected as equation $\Lambda = 0.2$ For any given m, we collected all available reduced matrices of the density at this stage and reconstructed 100 likely input states. When $m \leq 4$, the reconstructed state is not unique because of the absence of restrictions, so that both the concurrence and fidelity have known distributions. When $m = 5$, the input state may be uniquely defined and the concurrence and fidelity merge into one point.

Application of NMR. In order to implement the abovementioned protocol based filter of the distillation of the entanglement in NMR, we need a 4-qubit quantum processor, which shall consist of two system qubits A and B and two ancilla qubits 1 and 2. Our 4-qubit sample is ^{13}C with trans-crotonic acid, dissolved in d6-acetone. The Methyl-group M, H_1 and H_2 are distinguished through all experiments. The internal operator of Hamilton of this system may be described as

$$H_{int} = \sum\nolimits_{j=1}^{4} \pi \upsilon_j \sigma_z^j + \sum\nolimits_{j<k\leq 1}^{4} \frac{\pi}{2} J_{jk} \sigma_z^j \sigma_z^k \tag{4}$$

where υ_j is the chemical change of the j-th rotation and J_{jk} is the J-th strength of the connection between rotations j and k. We denoted C_3 and C_2 as the system qubits A and B, and C_4 and C_1 as ancilla qubits 1 and 2 which can help at in imitating the filters, respectively. All experiments were carried out on a Bruker DRX 700MHz spectrometer at room temperature.

Our target input state was selected as a combined state, including one proportion of the state of the Bell and two proportions of the state of the product, as the state of the Bell can be adjusted. This state is written as:

$$P^0_{AB} = \lambda|\phi_B\rangle\langle\phi_B| + (1-\lambda)\left(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\right)/2 \tag{5}$$

where

$$\begin{aligned} |\phi_B\rangle &= \left(|00\rangle + |11\rangle\right)/\sqrt{2} \\ |\phi_1\rangle &= \left(|0\rangle - i|1\rangle\right)\left(|0\rangle + |1\rangle\right)/\sqrt{2}, \\ |\phi_2\rangle &= \left(|0\rangle + |1\rangle\right)\left(|0\rangle - 2i|1\rangle\right)/\sqrt{2} \end{aligned} \tag{6}$$

have matches 1, 0 and 0 respectively. The parameter λ in (0.1) is thus proportional to the entanglement value p_{AB}^0. In the experiment, we varied with λ from 0.2 to 0.7 with a step with size 0.1 for each point and implement the relevant proposal. As we take into account the two ancilla, the overall input state of our 4-qubit system thus is $|0\rangle\langle 0| \otimes p_{AB}^0 \otimes |0\rangle\langle 0|$. We prepared a pseudo-pure state through average spatial technique, and then we created the three components $|\phi_B\rangle$, $|\phi_1\rangle$ and $|\phi_2\rangle$ on the system qubits, respectively. As a result, each component undergoes the whole filtering and the stage of reading of the single qubit, as the final result is obtained by a summary of all three experiments.

A sequence of NMR, which to realize the proposal based on filter, for the distillation of the entanglement. More precisely, this sequence shows how to realize the first two filters F_A^0 and F_B^0 regarding the terms of NMR pulse. All other sequences can be obtained in an analogous manner. The first 4-qubit system is prepared for positive partial state (PPS) by the average spatial technique that is applied before the step of starting. Then, the system qubits are started to $|\phi_B\rangle$, $|\phi_1\rangle$ and $|\phi_2\rangle$ by three independent experiments. The part after the step of the starting is a sequence for realization of the filters. V_A, V_B, U_A, U_B, θ_1 and θ_2 all depend on the measured results of the reduced matrices of density.

The two-qubit diagnostics of the state is applied on the system qubits after the creation of p_{AB}^0. p_0^e was reconstructed in the experiment and its fidelity compared to the expected p_{AB}^0 is above 98% for any λ. This two-qubit diagnostics of the state is not required in the initial proposal in which are necessary measurements with only single qubit. But as we claim that the proposal based on filter already has provided sufficient information to reconstruct the initial two-qubit state p_0^e, we must compare with p_f^e, which is reconstructed after the passing of the entire proposal. To prove our point of view, we must show that p_0^e and p_f^e are the same at small errors in the experiments. This comparison is the sole purpose of carrying out the diagnostics of the two-qubit state.

Now we should demonstrate how to realize the local operations for filtering in NMR. By measuring the local reduced matrix of the density p_A^0 of the input state p_{AB}^0, the first filter is calculated through $F_A^0 = 1/\sqrt{2}\, p_A^0$ and is decomposed to $U_A^0 V_A^0 \Lambda_A^0$. Since U_A^0 and V_A^0 are simply local unitaries of qubit A, they may be realized through pulsation of a local radio frequency, straightforwardly. V_A^0, which may be extended to a 2-qubit controlled rotation U_{1A}, (see the equation (3)) in a larger space of Hilbert, proceeded by a combination of pulsation of the local radio frequency and evolution of the J-coupling (45).

$$U_{1A} = R_{-x}^1\left(\frac{\pi}{2}\right) U\left(\frac{\theta_A}{2\pi J_{1A}}\right) R_x^1\left(\frac{\pi}{2}\right) R_{-y}^1\left(\frac{\theta_A}{2}\right) \tag{7}$$

where $U\left(\frac{\theta_A}{2\pi J_{1A}}\right)$ is the evolution of the J-coupling $e-i\theta A\sigma z\ \sigma z/4$ between the qubits 1 and A and

$0_A = 2\arccos\sqrt{1-Y_A}$, which depends from Λ_A^0. After this filter, the system is changed to p_{AB}^1, and a diagnostics with single qubit was applied, The same procedure is repeated for a qubit B, in order to realize the second filter $F_B^1 = 1/\sqrt{2}\, p_B^1$. In the experiment, these two filters F_A^0 and F_B^1 are applied at the same time by using the technique of partial decoupling, with additional Z rotations to the end in order to be compensated the unwanted phases, caused by the evolution of the chemical changes. The pulses are used for realizing $R1\ y(\theta A/2)$ and $R2\ y(\theta B/2)$, respectively, and the free time of evolution τ_1 and τ_2 are defined as

$$\begin{aligned} \tau_1 &= \theta_1/4\pi J_{1A} + \theta_2/4\pi J_{B2} \\ \tau_2 &= \theta_1/2\pi J_{1A} - \theta_2/2\pi J_{B2}. \end{aligned} \tag{8}$$

Here we have assumed that $\tau_2 > 0$ ($\theta_1/2\pi J_{1A} > \theta_2/2\pi J_{B2}$). When $\tau_2 < 0$, the circuit needs to be slightly modified by adjusting the positions of refocusing of the π pulses. All other filters have similar structures and they are always applied on qubit A and B simultaneously from their beginning.

Every time after executing one local filter, we implement diagnostics with single qubit on the other qubit, rather than the working qubit at which was applied the filter. The reason is that the working qubit has developed to identity due to the properties of this filter. The tomographic result was used for development of the next filter on the other qubit. In principle, before the application of any filters, it is necessary to be set the two ancilla qubits to $|00\rangle$. Since it is difficult to be loaded again the rotations in NMR, in our experiments was adopted an alternative way. For example, to realize F_A^2, we bind it with F_A^0, and generate a new operator. It can be seen as 2 in 1 filter and to be applied in the same way. Thus we avoided the resetting of the operations during the experiments and for any separate experiment, we simply start from the original two-qubit state p_{AB}^{0}. These operations based on the feedback continue to be executed until 5 filters applied and seven 1-qubit tomographies performed.

From the above discussions, we showed that the experiments with NMR contain only evolution of free J-couplings and unitaries with single qubit. For the evolutions of the J-couplings, we imposed in the system to be applied free equation of Hamilton (4) for the same time. For local unitaries, we use techniques for Gradient Ascent Pulse Engineering (GRAPE), in order to optimize them (46), (47). The method GRAPE provides 1 ms width of the pulse and above 99.8% fidelity for each local unitary, and furthermore all pulses are corrected by setup of the control of the feedback in NMR spectrometer in order to reduce the contradictions between the ideal and the applied pulses (48)-(50).

Experimental results and analysis of errors. We prepared six input states by varying λ from 0.2 to 0.7 with 0.1 size of the step in the form of equation. After the preparation, we carried out two-qubit full diagnostics of the state for each state, and reconstructed them as p_{0}^{e}, where the exponent means experiment. The fidelity between the theoretical state p_{AB}^{0} and the measured state p_{0}^{e} is above 99.2% for each of the six input states. The fidelity can be attributed to the deficiencies of PPS, GRAPE pulses and the effect of low decoherence. However this fidelity is only used to assess the accuracy of the preparation of our input state. For the last experiments, we only compared the experimental results with p_{0}^{is}, as p_{0}^{is} was the actual state from which we have started the experiment based on the filter.

After the initial preparation of the state and each filter we received the reduced matrix of the density of qubit A and/or B through diagnostics of single qubit in the subspace, where the ancillary qubits are $|00\rangle$ (see "Methods"). The average fidelity between the measured state with single qubit and the expected state, calculated with p_{0}^{e} is about 99.6% ("Additional table" S1), which demonstrates that our filtering operations and tomographies with single qubit are accurate.

The Fidelities between p_{0}^{e} and p_{f}^{e} for various λ-s. P_{0}^{is} is obtained from the diagnostics of two-qubit state immediately after the creation of the input state p_{AB}^{0}, and p_{f}^{e}, of the maximum probability for recreating p_{AB}^{0}, on the basis of the own seven states with single qubit. The bar of the errors comes from the relevant uncertainty when extracting the NMR spectrum in the quantum states. All fidelities are above 93.0%, which means that the initial two-qubit state can be reconstructed only by seven states with single qubit.

With seven states with single qubit, we can recreate the initial two-qubit state p_{0}^{e}. Here was applied the method for maximum probability and we came to the conclusion that p_{f}^{e} is the closest to the experimental raw data. Quite surprisingly, p_{f}^{e} is very similar to p_{0}^{e} and the fidelity between them for each λ is above 92%. The experimental results clearly show that the information of the seven states with single qubit collected at the time of the procedure for the distillation of the entanglement based on filters, facilitates the recreation of the initial two-qubit state. In other words, this proposal based on the filter to universally detect and distill the entanglement is equivalent compared to the conduct of diagnostics with two-qubit state.

Then we calculated the concurrence for each one of the cases with different input two-qubit state. The concurrence is monotone entanglement defined for combined state p of two qubits

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{9}$$

where λ_1, λ_2, λ_3 и λ_4 are the own values of

$$R = \sqrt{\sqrt{p}\left(\sigma_y \otimes \sigma_y\right) p^{*} \left(\sigma_y \otimes \sigma_y\right) \sqrt{p}} \tag{10}$$

in descending order. Obviously, the concurrence is proportional to the λ, as the λ is the weight of the state of the Bell, which is the only thing, which contributes to the entanglement. The blue squares show the concurrence of p_{0}^{e}, the state obtained by the diagnostics of the two-qubit state on the experimentally prepared state. Recall that the preparation of the fidelity is always 98.2% so the blue squares do not deviate much from the brown curve. The red circles show the concurrence of p_{f}^{e}, which in the ideal case must be the same as the blue squares if there are no experimental errors. But in the experiment there are unavoidable errors caused by many factors such as the imprecision of the stage of reading the single qubit, the imperfect application of the filters and the relaxation, and we must take them into account.

For convenience, we assume that the errors come from three main aspects and are additive. One error is caused by the inaccuracy of the procedure of the diagnostics with single qubit. As we used the least-square fitting algorithm to analyze the spectrum of the outcome, and we have converted the data into quantum states, the concurrence caused around 3.00% uncertainty of the final result with single qubit. The second is the error from the application of imperfect filters in the experiment. It originates mainly from errors in the accumulation of GRAPE pulses, which is about 1.59% for each operation of the filter. The third error, to a lesser extent, was around 1.20%, caused by decoherence. Therefore, in total we have concluded that at most 5.79% error may appear in the whole process. We identified them as artificial noise and we've incorporated them in the theoretical input state p_{AB}^{0}. In the simulation, we first discretized λ to 200 values from $\lambda = 0.1$ to $\lambda = 0.8$. For a given λ 2500 states were randomly calculated as deviation from p_{AB}^{0} within the 5.79% range of the noise. For each specified state the concurrence was calculated and projected onto one point. From here, a color space was generated, taking into account the density of the points. All of our experimental results fall within that space, which is consistent with the simulation of the model.

3. Conclusion

We proved the no-go theorem, and that there is no way to detect the entanglement of an arbitrary bipartite state p_{AB} without FDS if only nonadaptive measurements with one copy are allowed. Our conclusion follows from a good geometric picture: It is not possible to detect a projection of the space of the state on a lower hyperplane of the dimension with non-overlapping image for a set of separable states and entangled states. Our method provides a common framework for studying the detection of other interesting quantities for the bipartite quantum state, such as positive partial transposition and *k*-symmetric-extendibility. We also have analyzed the case of the adaptive measurements. It is assumed in (33) that the concurrence in the measurement of the entanglement for two-qubit states can be determined without FDS, through measurements with single copy. In order to apply this protocol, we have developed a method with maintaining of ancillaries to finalize the filters. Practically, the technique can be extended to cover other quantum systems, except the optics in order to apply a channel reducing the amplitude that is of enormous importance in the quantum information. Through the application of this protocol, we show that when the experimental data are collected for a given state p_{AB}, this protocol leads in fact to FDS of p_{AB}. Therefore, this protocol does not lead to universal detection of p_{AB} without FDS.

Concurrence of p_{AB}^{0}, p_{0}^{e} and p_{f}^{e} as a function of the weight of the state of Bell λ. The concurrence is calculated for the theoretical state p_{AB}^{0} and displays the value of the concurrence as a monotonically increasing function of λ. The concurrence of p_{0}^{e}, the state is obtained from the diagnostics of the two-qubit state immediately after the preparation of the input state. In total, we roughly can suppose that this state is truly prepared state, and that the following operations of the filters are always applied to this state as long as we do not neglect the measurement of the errors at the reconstruction of p_{0}^{e}. The concurrence of p_{f}^{e} the state is reproduced with seven states with single qubit. In the ideal case, p_{0}^{e} and $p^{e}{}_{f}$ must be one and the same if there are no experimental errors. The artificial noise is with a strength of 5.89%, which is roughly calculated from the coincidence of the errors up to 3%. The GRAPE imperfection error is 1.49 and the error upon decoherence is 1/21%. We have added this noise to the theoretical state

p_{AB}^{0} and randomly provided 2400 states within the range of the noise for each λ (200 values in [0.1, 0.9]. The defined places thus are based on the density of the projected points of 2400.

This justifies our unchanging results, which indicates that even when at the adaptive measurements, the universal entanglement detection by measurements with single-copy is impossible without FDS.

It is important to underline that the large identity does not arise at any unitary propagator and it is not observed in NMR. So there is only a need for focusing on the part for the deviation |0000>, as the entire system is functioning exactly as it.

In consequence, each component undergoes the entire procedures for filtering and measuring respectively, as the final result is obtained by summarizing all three experiments.

Diagnosis with single qubit after each filter. The procedure for the distillation of the entanglement described to in Ref. (33), includes local filtering operations, which means that each filter depends on the previous result from the measurement with single qubit. In one experiment, we performed diagnostics with single qubit on the system qubits C_2 and C_3, respectively. It requires measurement of the expected values of σx, σy and σz, respectively. In our 4-qubit system, this diagnostics with single qubit (as we assume the measurement of is equal to the measurement $|0\rangle\langle 0| \otimes \sigma\xi, \psi, \zeta \otimes I \otimes |0\rangle\langle 0|$, as we must be focusing on the subspace where the ancilla qubits are $|00\rangle$. In order to obtain the expected values of the observables, a procedure for fitting the spectrum must be applied in order to extract these results from the spectrum of NMR.

So our survey shows the fundamental ratio between the detection of the entanglement and the diagnostics of the quantum state. Namely that universal entanglement detection without FDS is not possible only with single-copy measurements. Naturally occurs also the question of whether the joint measurements on r copies of the state p_{AB}^{0} (*i.e.* $p \otimes r_{AB}$) for $r > 1$ are allowed. In this case, indeed the entanglement may be detected universally for any p_{AB}^{0} without to reconstruct the state, and one example for defining the concurrence of the two-qubit p_{AB}^{0} is given in (22)-(24). But, the Protocol (23) includes joint measurements on 4 copies of p_{AB}^{0} (*i.e.* $p \otimes r_{AB}$), which makes it difficult to be implemented in practice. It would be interesting to find a smaller r such that the joint measurements of c are sufficient for the universal detection of the entanglement of p_{AB} without full diagnostics of the state.

In fact there are also cases in which this is possible even for $r = 2$. For example, we found such a scheme, which reveals the entanglement of an arbitrary two-qubit state p_{AB} without FDS, if we allow joint measurements of two copies. The idea is that p_{AB} is entangled only and only when (23) Det (p_{AB}^{Ta}) <0, where p_{AB}^{Ta} is partial transposition of p_{AB} of the system *A*. Thus we must only design a scheme with measurement of $p \otimes r_{AB}$, which may give the value of Det (p_{AB}^{Ta}). This really can be done without FDS.

Furthermore, if only measurements with single-copy are permitted, the value of Det (p_{AB}^{Ta}), cannot be determined, even with adaptive measurements. Let's suppose that similar adaptive measurements exist. Now we assume that the input state is maximal mixed state $I/4$, after measurement, and thus we can calculate the determinant. Please note that there is at least one traceless R other than zero, which is not measured, which means that these measurements cannot make a distinction between $I/4$ and $I/4 + tr$. Therefore, Det ($I/4 + tR^{r}$) = Det ($I/4$) for small enough $R = 0$.

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On the Theory of Fractional Order Differential Games of Pursuit

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Abstract

This article is devoted to studying of the problem of prosecution described by differential equations of a fractional order. It has received sufficient conditions of a possibility of completion of prosecution for such operated systems.

Keywords

Equations, Control Systems, Function, Derivative Kaputo

1. Introduction

The dynamics of the systems described by the equations of fractional order is the subject of research experts from around the middle of the XX century. The study of dynamical systems with fractional order management is actively developing in the last 5 - 8 years [1] [2]. The growing interest in these areas is due to two main factors. Firstly, by the middle of the last century it has been adequately worked out the mathematical foundations of fractional integro-differential calculus and the theory of differential equations of fractional order. Around the same time, it began to develop a methodology and application of fractional calculus in applications, and we started to develop numerical methods for calculating integrals and derivatives of fractional order. Secondly, in fundamental and applied physics by this time, it had accumulated a considerable amount of results, which showed the need for fractional calculus apparatus for an adequate description of a number of real systems and processes [3]. Examples of real systems will mention electrochemical cells, capacitors fractal electrodes, the viscoelastic medium. These systems have typically not trivial physical properties useful from a practical standpoint [4]-[7]. For example, the irregular structure of the electrodes in capacitors allows them to reach a much larger capacity, and the use of electrical circuits with elements having a transfer characteristic of fractional-

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power type, provides more flexible configuration of fractional order controllers used in modern control systems. For such control systems of fractional order as of today, there are no similar results Pontryagin type [8]-[11].

2. Methods

Let driving of object in a finite-dimensional Euclidean space of R^n be described by a differential equation of a fractional order of a look

$$ {}_0^C D_t^{\alpha} z = Az + Bu - G\upsilon + f(t), \tag{1} $$

where $z \in R^n, n \ge 1$; ${}_0^C D_t^{\alpha_i}$—operator of fractional derivation, $\alpha > 0$, $t \in [0,T]$, A—$n \times n$, B—$p \times n$ and G—$q \times n$ constant matrixes, u, υ—the operating parameters, u—the operating parameter of the pursuing player, $u \in P \subset R^p$, υ—the operating parameter of the running-away player, $\upsilon \in Q \subset R^q$, P and Q-compact, $f(t)$-known measurable vector function. We will understand a fractional derivative as left-side fractional derivative Kaputo [1]-[6]. Let's remind that fractional derivative Kaputo of the random inappropriate order $\alpha > 0$ from function $z(t) \in \mathrm{AC}^{[\alpha]+1}(a,b)$, $a, b \in R^1$, is defined by expression

$$ {}_{a+}^{C} D_t^{\alpha} z(t) = \frac{1}{\Gamma(1-\{\alpha\})} \int_a^t \frac{\mathrm{d}^{[\alpha]+1} z(\xi)}{\mathrm{d}\xi^{[\alpha]+1}} \frac{\mathrm{d}\xi}{(t-\xi)^{\{\alpha\}}}. \tag{2} $$

Besides in space R^n the terminal set *M* is allocated. The running-away player seeks to place the aim of the pursuing player to bring *z* to a set *M*, to it. The problem of prosecution about rapprochement of a trajectory of the conflict operated system (1) with a terminal set *M* for terminating time from the standard initial positions z_0 is considered. Let's say that differential game (1) can be finished from initial situation z_0 during $T = T(z_0)$ if there is such measurable function $u(t) = u(z_0, \upsilon(t)) \in P, t \in [0,T]$ that the solution of the equation

$$ {}_0^C D_t^{\alpha} z = Az + Bu(t) - G\upsilon(t) + f(t), \quad z(0) = z_0 $$

belongs to a set *M* at the time of $t = T$ at any measurable functions $\upsilon(t), \upsilon(t) \in Q$, $0 \le t \le T$.

This work is dedicated to the receipt of sufficient conditions for the completion of the prosecution managed fractional order systems adjacent to the study [12]-[22]. Some results of this paper were announced at the International Labour Conference [16] [17]. In such a setting the pursuit problem was studied in [8]-[11], but it was devoted to the study of control systems of the whole order. In this sense, this paper summarizes these works.

3. Results and Discussion

Let's pass to the formulation of the main results. Everywhere further: 1) the terminal set *M* has an appearance $M = M_0 + M_1$, where the M_0—linear subspace R^n, M_1—subset of a subspace of *L*—orthogonal addition M_0; 2) π—operator of orthogonal projection from R^n on *L*; 3) operation $\overset{*}{-}$ is understood as operation of a geometrical subtraction [8].

Let $e_\alpha^{At} = t^{\alpha-1} \sum_{k=0}^{\infty} A^k \frac{t^{\alpha k}}{\Gamma((k+1)\alpha)}$-matrix α—an exponential curve [1] and $r \ge 0$, $\hat{u}(r) = \pi e_\alpha^{rA} BP$, $\hat{\upsilon}(r) = \pi e_\alpha^{rA} GQ$, $\hat{w}(r) = \hat{u}(r) \overset{*}{-} \hat{\upsilon}(r)$;

$$ W(\tau) = \int_0^{\tau} \hat{w}(r) \mathrm{d}r, \ \tau > 0, \ W_1(\tau) = -M_1 + W(\tau). \tag{3} $$

Theorem 1. If in game (1) at some $\tau = \tau_1$, inclusion is carried out

$$ -\pi z_0 - \int_0^{\tau} \pi e_\alpha^{A(\tau - r)} \left[Az_0 + f(r) \right] \mathrm{d}r \in W_1(\tau) \tag{4} $$

That from initial situation z_0 is possible will finish prosecution during $T = \tau_1$.

Let now the ω-arbitraries splitting a piece $[0,\tau], \omega = \{0 = t_0 < t_1 < \cdots < t_k = \tau\}$, $i = 1, 2, \cdots, k$, и $A_0 = -M_1$,

$$ A_i(M,\tau) = \left(A_{i-1}(M,\tau) + \int_{t_{i-1}}^{t_i} \pi e_\alpha^{rA} BP \mathrm{d}r \right) \overset{*}{-} \int_{t_{i-1}}^{t_i} \pi e_\alpha^{rA} GQ \mathrm{d}r, \ i = 1, 2, \cdots, k, \ W_2(\tau) = \bigcap_{\omega} A_i(M,\tau). \tag{5} $$

Theorem 2. If in game (1) at some $\tau=\tau_2$, inclusion is carried out

$$-\pi z_0-\int_0^{\tau}\pi e_\alpha^{A(\tau-r)}\left[Az_0+f(r)\right]\mathrm{d}r\in W_2(\tau) \tag{6}$$

That from initial situation z_0 is possible will finish prosecution during $T=\tau_2$.

Let's designate through $\hat{w}(r,\tau)$ set $\left[-\frac{1}{\tau}M_1+\hat{u}(r)\right]\overset{*}{-}\hat{\upsilon}(r)$ defined at all $r\geq 0,\tau>0$. Let's consider integral

$$W_3(\tau)=\int_0^{\tau}\hat{w}(r,\tau)\mathrm{d}r. \tag{7}$$

Theorem 3. If in game (1) at some $\tau=\tau_3$, inclusion is carried out

$$-\pi z_0-\int_0^{\tau}\pi e_\alpha^{A(\tau-r)}\left[Az_0+f(r)\right]\mathrm{d}r\in W_3(\tau) \tag{8}$$

that from initial situation z_0 is possible will finish prosecution during $T=\tau_3$.

Proof of the theorem 1. Two cases are possible: 1) $\tau_1=0$; $\tau_1>0$. Case 1) trivial as at $\tau_1=0$ of inclusion (4) we have $-\pi z_0\in -M_1$ or $\pi z_0\in M_1$ that is equivalent to inclusion $z_0\in M$. Let now $\tau_1>0$. After a theorem condition $-\pi z_0-\int_0^{\tau}\pi e_\alpha^{A(\tau-r)}\left[Az_0+f(r)\right]\mathrm{d}r\in W_1(\tau)$, then there will be vectors $d\in M_1$ and $w\in\int_0^{\tau}\hat{w}(r)\mathrm{d}r$ such that (see (3), (4)) $d+w=-\pi z_0-\int_0^{\tau}\pi e_\alpha^{A(\tau-r)}\left[Az_0+f(r)\right]\mathrm{d}r$. Further, according to determination of integral $\int_0^{\tau_1}\hat{w}(r)\mathrm{d}r$ there is a summable function $w(r),0\leq r\leq\tau_1,w(r)\in\hat{w}(r)$ that $w=\int_0^{\tau_1}w(r)\mathrm{d}r$. Considering this equality, we will consider the equation

$$\pi e_\alpha^{A(\tau_1-t)}\left[Bu-G\upsilon\right]=w(\tau_1-t) \tag{9}$$

Relatively $u\in P$ at fixed $t\in[0,\tau_1]$ and $\upsilon\in Q$. As $w(r)\in\hat{w}(r)$, the equation (9) has the decision. We will choose the least in lexicographic sense from all solutions of the equation (9) and we will designate it through $u(t,\upsilon)$. Function $u(t,\upsilon),0\leq t\leq\tau_1,\upsilon\in Q$, is lebegovsk measurable on t and borelevsk measurable on υ [7]. Therefore for any measurable function $\upsilon=\upsilon(t),0\leq t<\infty,\upsilon(t)\in Q$, function $u(t,\upsilon(t)),0\leq t\leq\tau_1$, will be lebegovsk measurable function [7]. Let's put $u(t)=u(t,\upsilon(t)),0\leq t\leq\tau_1$ and we will show that at such way of management of the parameter of *u* the trajectory $z(u(\cdot),\upsilon(\cdot),z_0)$ gets on a set *M* in time, not surpassing $T=\tau_1$.

Really, on (9) for the decision $z(t),0\leq t<\infty$, the equation

${}_0^C D_t^\alpha z=Az+Bu(t)-G\upsilon(t)+f(t),\quad z(0)=z_0,$

we have ([1], p. 414)

$$\begin{aligned}
\pi z(\tau_1)&=\pi z_0+\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Az_0+f(r)\right]\mathrm{d}r+\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Bu(r)-G\upsilon(r)\right]\mathrm{d}r\\
&=\pi z_0+\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Az_0+f(r)\right]\mathrm{d}r+\int_0^{\tau_1}w(\tau_1-r)\mathrm{d}r\\
&=\pi z_0+\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Az_0+f(r)\right]\mathrm{d}r+w\\
&=\pi z_0+\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Az_0+f(r)\right]\mathrm{d}r-\pi z_0-\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Az_0+f(r)\right]\mathrm{d}r-d\\
&=-d.
\end{aligned}$$

As $d+w=-\pi z_0-\int_0^{\tau_1}\pi e_\alpha^{A(\tau_1-r)}\left[Az_0+f(r)\right]dr$. Further we have $\pi z(\tau_1)=-d\in -M_1, d\in M_1$. From here we will receive that $z(\tau_1)\in M$. The theorem is proved completely.

Proof of the theorem 2. In view of a case triviality we will begin $\tau_2=0$ consideration with a case $\tau_2>0$. We have (see (5), (6)) $-\pi z_0-\int_0^{\tau_2}\pi e_\alpha^{A(\tau_2-r)}\left[Az_0+f(r)\right]\mathrm{d}r\in W_2(\tau_2)$. $W_2(\tau_2)$ is alternating integral with an initial set $A_0=-M_1$ [8]. Therefore for it semigroup property [4] is executed

$$W_2(\tau_2)\subset\left(W_2(\tau_2-\varepsilon)+\int_{\tau_2-\varepsilon}^{\tau_2}\pi e_\alpha^{rA}BP\mathrm{d}r\right)\overset{*}{-}\int_{\tau_2-\varepsilon}^{\tau_2}\pi e_\alpha^{rA}GQ\mathrm{d}r, \tag{10}$$

where the ε—arbitraries positive fixed number $0<\varepsilon\le\tau_2$; $\upsilon_0(r)$,the $\tau_2-\varepsilon\le r\le\tau_2$—arbitraries measurable function with values from Q.

Let $\upsilon=\upsilon(t), 0\le t<\infty$,—arbitrary measurable function $\upsilon(t)\in Q$. According to theorem conditions in an instant $t=0$ is known a narrowing $\upsilon(t), 0\le t\le\varepsilon$, functions $\upsilon(t), 0\le t<\infty$, on a piece $[0,\varepsilon]$. Follows from inclusion (10) that for the arbitrary function $\tilde{\upsilon}(\tau_2-r), \tau_2-\varepsilon\le r\le\tau_2, \tilde{\upsilon}(\tau_2-r)\in Q$, we have

$$-\pi z_0-\int_0^{\tau_2}\pi e_\alpha^{A(\tau_2-r)}\left[Az_0+f(r)\right]\mathrm{d}r\in W_2(\tau_2-\varepsilon)+\int_{\tau_2-\varepsilon}^{\tau_2}\pi e_\alpha^{rA}BPdr-\int_{\tau_2-\varepsilon}^{\tau_2}\pi e_\alpha^{rA}G\tilde{\upsilon}(\tau_2-r)\mathrm{d}r. \tag{11}$$

Thus, for the arbitrary function $\tilde{\upsilon}(s), 0\le s\le\varepsilon$, inclusion takes place (12). Therefore, at $\tilde{\upsilon}(s)\equiv\upsilon(s), 0\le s\le\varepsilon$, inclusion is fair (12). From here existence of measurable function $u(s), 0\le s\le\varepsilon$, such follows that $u(s)\in P$ and

$$-\pi z_0-\int_0^{\tau_2}\pi e_\alpha^{A(\tau_2-r)}\left[Az_0+f(r)\right]\mathrm{d}r\in W_2(\tau_2-\varepsilon)+\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}Bu(s)\mathrm{d}s-\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}G\tilde{\upsilon}(s)\mathrm{d}s, \tag{12}$$

Then

$$-\pi z_0-\int_0^{\tau_2}\pi e_\alpha^{A(\tau_2-r)}\left[Az_0+f(r)\right]\mathrm{d}r-\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}Bu(s)\mathrm{d}s+\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}G\tilde{\upsilon}(s)\mathrm{d}s\in W_2(\tau_2-\varepsilon). \tag{13}$$

Further we argue similarly. As

$$W_2(\tau_2-\varepsilon)\subset\left(W_2(\tau_2-2\varepsilon)+\int_{\tau_2-2\varepsilon}^{\tau_2-\varepsilon}\pi e_\alpha^{rA}BP\mathrm{d}r\right)\overset{*}{-}\int_{\tau_2-2\varepsilon}^{\tau_2-\varepsilon}\pi e_\alpha^{rA}GQ\mathrm{d}r, \tag{14}$$

Let's receive

$$\begin{aligned}&-\pi z_0-\int_0^{\tau_2}\pi e_\alpha^{A(\tau_2-r)}\left[Az_0+f(r)\right]dr-\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}Bu(s)\mathrm{d}s+\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}G\tilde{\upsilon}(s)\mathrm{d}s\\&\in W_2(\tau_2-2\varepsilon)+\int_{\tau_2-2\varepsilon}^{\tau_2-\varepsilon}\pi e_\alpha^{rA}BP\mathrm{d}r-\int_{\tau_2-2\varepsilon}^{\tau_2-\varepsilon}\pi e_\alpha^{rA}G\tilde{\tilde{\upsilon}}(\tau_2-r)\mathrm{d}r,\end{aligned} \tag{15}$$

For the arbitrary measurable function $\tilde{\tilde{\upsilon}}(\tau_2-r)$, $\tau_2-2\varepsilon\le r\le\tau_2-\varepsilon, \tilde{\tilde{\upsilon}}(\tau_2-r)\in Q$. Therefore, there is a measurable function $u(s), \varepsilon\le s\le 2\varepsilon$, such that $u(s)\in P$ and

$$\begin{aligned}&-\pi z_0-\int_0^{\tau_2}\pi e_\alpha^{A(\tau_2-r)}\left[Az_0+f(r)\right]\mathrm{d}r-\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}Bu(s)\mathrm{d}s+\int_0^{\varepsilon}\pi e_\alpha^{(\tau_2-s)A}G\tilde{\upsilon}(s)\mathrm{d}s\\&\in W_2(\tau_2-2\varepsilon)+\int_{\varepsilon}^{2\varepsilon}\pi e_\alpha^{rA}Bu(r)\mathrm{d}r-\int_{\varepsilon}^{2\varepsilon}\pi e_\alpha^{rA}G\tilde{\upsilon}(\tau_2-r)\mathrm{d}r,\end{aligned} \tag{16}$$

Follows from a ratio (16) that

$$-\pi z_0 - \int_0^{\tau_2} \pi e_\alpha^{A(\tau_2-r)}\left[Az_0 + f(r)\right]\mathrm{d}r - \int_0^{2\varepsilon} \pi e_\alpha^{(\tau_2-s)A} Bu(s)\,\mathrm{d}s + \int_0^{2\varepsilon} \pi e_\alpha^{(\tau_2-s)A} G\tilde{\upsilon}(s)\,\mathrm{d}s \in W_2(\tau_2 - 2\varepsilon), \tag{17}$$

etc. It is clear, that there is a natural number j it that: 1) $(j-1)\varepsilon < \tau_2 \le j\varepsilon$; 2) on the known function $\upsilon(s), 0 \le s \le \tau_2$, where the $\upsilon(s), 0 \le s \le \tau_2$ narrowing of function $\upsilon(s), 0 \le s < \infty$, on a piece $[0, \tau_2]$, will be the measurable function $u(s)$, $(j-1)\varepsilon < \tau_2 \le \tau_2$, $u(s) \in P$ meeting a condition ((17))

$$W_2(\tau_2 - (j-2)\varepsilon) \subset \left(W_2(\tau_2 - (j-1)\varepsilon) + \int_{\tau_2-(j-2)\varepsilon}^{\tau_2-(j-1)\varepsilon} \pi e_\alpha^{rA} BP\mathrm{d}r \right) \overset{*}{-} \int_{\tau_2-(j-2)\varepsilon}^{\tau_2-(j-1)\varepsilon} \pi e_\alpha^{rA} GQ\mathrm{d}r, \tag{18}$$

But

$$\begin{gathered} -\pi z_0 - \int_0^{\tau_2} \pi e_\alpha^{A(\tau_2-r)}\left[Az_0 + f(r)\right]\mathrm{d}r - \int_0^{\tau_2-(j-1)\varepsilon} \pi e_\alpha^{(\tau_2-s)A} Bu(s)\,\mathrm{d}s + \int_0^{\tau_2-(j-1)\varepsilon} \pi e_\alpha^{(\tau_2-s)A} G\tilde{\upsilon}(s)\,\mathrm{d}s \\ \in W_2(\tau_2 - (j-1)\varepsilon) + \int_{\tau_2-(j-1)\varepsilon}^{\tau_2} \pi e_\alpha^{rA} Bu(r)\,\mathrm{d}r - \int_{\tau_2-(j-1)\varepsilon}^{\tau_2} \pi e_\alpha^{rA} G\tilde{\upsilon}(\tau_2 - r)\,dr. \end{gathered} \tag{19}$$

Therefore ((18), (19))

$$-\pi z_0 - \int_0^{\tau_2} \pi e_\alpha^{A(\tau_2-r)}\left[Az_0 + f(r)\right]\mathrm{d}r - \int_0^{\tau_2} \pi e_\alpha^{(\tau_2-s)A} Bu(s)\,\mathrm{d}s + \int_0^{\tau_2} \pi e_\alpha^{(\tau_2-s)A} G\tilde{\upsilon}(s)\,\mathrm{d}s \in W_2(\tau_2 - (j-1)\varepsilon). \tag{20}$$

Similarly on formulas (18), (19), (20) finally we receive

$-\pi z(\tau_2) \in W_2(\tau_2 - (j-1)\varepsilon) \subset W_2(0) = -M_1,\ -\pi z(\tau_2) \in -M_1,\ \pi z(\tau_2) \in M_1.$

Thus, for any point z_0 we have $z(\tau_2) \in M$, that is the trajectory, left a point z_0, in an instant $t = \tau_2$ turns out M on a set. The theorem is proved completely.

Proof of the theorem 3. Owing to a condition of the theorem (8) we have

$-\pi z_0 - \int_0^{\tau_3} \pi e_\alpha^{A(\tau-r)}\left[Az_0 + f(r)\right]\mathrm{d}r \in W_3(\tau_3)$. Therefore (7), there is such measurable function

$w(r), 0 \le r \le \tau_3, w(r) \in \hat{w}(r)$, that

$$-\pi z_0 - \int_0^{\tau_3} \pi e_\alpha^{A(\tau-r)}\left[Az_0 + f(r)\right]\mathrm{d}r = \int_0^{\tau_3} w(r)\,\mathrm{d}r, w(r) \in \hat{w}(\tau_3 - r, \tau_3). \tag{21}$$

Let $\upsilon = \upsilon(t), 0 \le t \le \tau_3, \upsilon(t) \in Q$ arbitrary measurable function, by definition of subtraction operation $\overset{*}{-}$ we will receive

$$w(r) + \pi e_\alpha^{A(\tau_3-r)} G\upsilon(r) \in -\frac{1}{\tau_3} M_1 + \hat{u}(\tau_3 - r),\ 0 \le r \le \tau_3. \tag{22}$$

From here owing to a condition of measurability existence of the measurable functions $d(r), u(r)$, defined on a piece $0 \le r \le \tau_3$ follows and

$$d(t) \in -\frac{1}{\tau_3} M_1,\ u(r) \in \hat{u}(\tau_3 - r),\ w(r) + \pi e_\alpha^{A(\tau_3-r)} G\upsilon(r) = d(t) + u(r),\ 0 \le r \le \tau_3. \tag{23}$$

We will determine function by the found measurable function $u(r)$

$$u(r) = \pi e_\alpha^{A(\tau_3-r)} G\upsilon(r) \in -\frac{1}{\tau_3} M_1 + \hat{u}(\tau_3 - r),\ 0 \le r \le \tau_3. \tag{24}$$

For the decision $z(t), 0 \le t \le \tau_3$, corresponding to functions $u(t), \upsilon(t)$, we have (21)-(24)

$$-\pi z(\tau_3) = -\pi z_0 - \int_0^{\tau_3} \pi e_\alpha^{A(\tau_3-r)}\left[Az_0 + f(r)\right]\mathrm{d}r - \int_0^{\tau_3} \pi e_\alpha^{(\tau_3-s)A} Bu(s)\,\mathrm{d}s + \int_0^{\tau_3} \pi e_\alpha^{(\tau_3-s)A} G\tilde{\upsilon}(s)\,\mathrm{d}s = \int_0^{\tau_3} d(\tau)\,\mathrm{d}\tau \in -M_1.$$

From here $\pi z(\tau_3) \in M_1,\ z(\tau_3) \in M$, that is the trajectory which left a point z_0 in an instant $t = \tau_3$ turns out M on a set. The theorem is proved completely.

4. Conclusion

Summarizing the results, we conclude that the differential game of pursuit of fractional order (1), starting from the position can be completed in time, respectively. Thus, to solve the game problem kind of persecution (1), we used a derivative of fractional order Caputo, which is determined by the expression (2). Many (3) analogue of the so-called first integral Pontryagin, including (4) gives the first sufficient condition for the possibility of the persecution of the task. Many (5)—an analog of the second integral Pontryagin, inclusion (6) gives the second sufficient condition for the possibility of the persecution of the task. Lots (7)—analogue N. Satimova third method, and the inclusion (8) gives a sufficient condition for the third opportunity to end the game. In Theorems 1 - 3, we obtain sufficient conditions for the solution of relevant problems in this form.

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Spline Solution for the Nonlinear Schrödinger Equation

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Abstract

We develop an exponential spline interpolation method to solve the nonlinear Schrödinger equation. The truncation error and stability analysis of the method are investigated and the method is shown to be unconditionally stable. The conservation quantities are computed to determine the conservation properties of the problem. We will describe the method and present numerical tests by two problems. The numerical simulations results demonstrate the well performance of the proposed method.

Keywords

Nonlinear Schrödinger Equation, Exponential Spline Interpolation, Gross-Pitaevskii Equation, Mass and Energy Conservation

1. Introduction

Consider the following nonlinear Schrödinger equation

$$mu_t + \lambda_1 u_{xx} + \lambda_2 |u|^2 u + \varepsilon(x,t) u = 0, \tag{1}$$

With the boundary conditions

$$u(c,t) = \beta_1(t), u(d,t) = \beta_2(t), t \geq 0 \tag{2}$$

And the initial condition

$$u(x,0) = u^0(x), x \in \mathbb{R}, \tag{3}$$

where $m = \sqrt{-1}$, $u(x,t)$ is the complex-valued wave function. λ_1 and λ_2 are constant, $\varepsilon(x,t)$ is a

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bounded real function. This equation plays important roles in nonlinear physics. It can describe many nonlinear phenomena including plasma physics [1], hydrodynamics [1] [2], self-focusing in laser pulses [3], propagation of heat pulses in crystals, models of protein dynamics [4], quantum mechanics [5], models of energy transfer in molecular systems [6] and quantum mechanics and optical communication [7]-[9] and so on.

In the past few years a great deal of efforts has been expended to solve NLS equations. It is more difficult to find the analytical solutions of the NLS equation, so the study of the numerical solution of NLS equation in the theory and application is important. Its numerical solutions have been researched by many authors. For example, finite difference method [10] [11], quasi-interpolation scheme [12], quadratic B-spline finite element scheme [13], compact split-step finite difference method and pseudo-spectral collocation method [14] [15], exponential spline method [16], spline methods [17] [18], split-step orthogonal spline collocation method [19], a high-order and accurate method [20], linearly implicit conservative scheme [21].

The aim of this paper is to give an exponential spline interpolation method for the NLS equation. The paper is organized as follows. In Section 2, construction of the method is presented. The stability analysis of the scheme is investigated in Section 3. In Section 4, the computation of conserved quantities and error norms are given. In Section 5, two numerical examples are presented to demonstrate our theoretical results. The last section is a brief conclusion.

2. Construction of Exponential Spline Interpolation Method

We set up a grid in the x,t plane with grid points $\left(x_i,t_j\right)$ and uniform grid spacing h and k, where $x_i = a + ih, h_{i+1} = x_{i+1} - x_i, i = 0,1,2,\cdots,N$ and $t_j = jk, j = 0,1,2,\cdots$.

In the interval $\left[x_i, x_{i+1}\right]$, a exponential spline function $S_i\left(x,t_j\right)$ is given by

$$S_i\left(x,t_j\right) = c_{1i}^j + c_{2i}^j\left(x-x_i\right) + c_{3i}^j\psi_i\left(x-x_i\right) + c_{4i}^j\phi_i\left(x-x_i\right), \tag{4}$$

where $c_{1i}, c_{2i}, c_{3i}, c_{4i}$ are coefficients to be determined, ψ_i and ϕ_i are the auxiliary functions which contain a stiffness parameter p_{i+1} which will be used to raise the accuracy of the method, on the support $\left[x_i, x_{i+1}\right]$ and are given by

$$\psi_i\left(x\right) = 2\left[\cosh\left(p_{i+1}\left(x-x_i\right)\right) - 1\right] / p_{i+1}^2, \tag{5}$$

$$\phi_i\left(x\right) = 6\left[\sinh\left(p_{i+1}\left(x-x_i\right)\right) - p_{i+1}\left(x-x_i\right)\right] / p_{i+1}^2, \tag{6}$$

Since the Taylor series expansions of the hyperbolic functions are

$$\sinh\left(px\right) = px + \frac{\left(px\right)^3}{3!} + \frac{\left(px\right)^5}{5!} + \cdots, \tag{7}$$

$$\cosh\left(px\right) = 1 + \frac{\left(px\right)^2}{2!} + \frac{\left(px\right)^4}{4!} + \cdots, \tag{8}$$

We note that ψ_i and ϕ_i tend to $\left(x-x_i\right)^2$ and $\left(x-x_i\right)^3$ in the limit of p tending to zero, and in the opposite limit of p tending to infinity the nonlinear terms in ψ_i and ϕ_i vanish as $1/p$.

So the exponential spline defined above share a number of interesting properties:

(1) When $p \to 0$, $S_i\left(x,t_j\right)$ reduces to cubic spline; when $p \to \infty$, $S_i\left(x,t_j\right)$ reduces to linear spline.

(2) A change of character of the exponential spline function is from linear to third order polynomial on adjacent support intervals.

(3) In the general case the stiffness parameters p are different on every interval which provides the extremely high flexibility of the exponential spline function.

We wish to find c_{ni}^j in Equation (4), $n = 1,2,3,4$, Letting $M_i^j = S_\Delta^{(2)}\left(x,t_j\right)$ be the unknown second derivative of the exponential spline of interpolation at the grid points, we can obtain the following representation for $S_\Delta\left(x,t_j\right)$ on $\left[x_i, x_{i+1}\right]$ in terms of the known interpolation data u_i^j, u_{i+1}^j and the unknown spline second derivatives M_i^j, M_{i+1}^j

$$S_{\Delta}\left(x,t_{j}\right)=\frac{x-x_{i+1}}{x_{i}-x_{i+1}}u_{i}^{j}+\frac{x-x_{i}}{x_{i+1}-x_{i}}u_{i+1}^{j}-\frac{M_{i}^{j}}{p_{i+1}^{2}}\left[\frac{\sinh\left(p_{i+1}\left(x-x_{i+1}\right)\right)}{\sinh\left(p_{i+1}\left(x_{i}-x_{i+1}\right)\right)}-\frac{x-x_{i+1}}{x_{i}-x_{i+1}}\right]$$
$$+\frac{M_{i+1}^{j}}{p_{i+1}^{2}}\left[\frac{\sinh\left(p_{i+1}\left(x-x_{i}\right)\right)}{\sinh\left(p_{i+1}\left(x_{i+1}-x_{i}\right)\right)}-\frac{x-x_{i}}{x_{i+1}-x_{i}}\right],x\in\left[x_{i},x_{i+1}\right], \tag{9}$$

The terms involving the values u_i^j and u_{i+1}^j represent the linear interpolation part of $S_{\Delta}\left(x,t_j\right)$. The terms involving the second derivatives M_i^j and M_{i+1}^j introduce the curvature.

The function $S_{\Delta}\left(x,t_j\right)$ on the interval $\left[x_{i-1},x_i\right]$ is obtained with $i-1$ replacing *i* in Equation (9).

The continuity requirement for the first derivative $S_{\Delta}^{(1)}\left(x,t_j\right)$ at the point x_i yields the following equation:

$$A_i M_{i-1}^j+\left(B_i+B_{i+1}\right)M_i^j+A_{i+1}M_{i+1}^j=\frac{u_{i+1}^j-u_i^j}{h_{i+1}}-\frac{u_i^j-u_{i-1}^j}{h_i}, \tag{10}$$

where $A_i=h_i\dfrac{\sinh\left(p_ih_i\right)-p_ih_i}{p_i^2\sinh\left(p_ih_i\right)},B_i=h_i\dfrac{p_ih_i\cosh\left(p_ih_i\right)-\sinh\left(p_ih_i\right)}{p_i^2\sinh\left(p_ih_i\right)}$,

Remark 1.

(1) By expanding Equation (10) in Taylor series, the truncation error for Equation (10) is of the form

$$\begin{aligned}T_i^j&=\frac{u_{i+1}^j-u_i^j}{h_{i+1}}-\frac{u_i^j-u_{i-1}^j}{h_i}-A_iD_x^2u_{i-1}^j-\left(B_i+B_{i+1}\right)D_x^2u_i^j-A_{i+1}D_x^2u_{i+1}^j\\&=\left[\frac{h_i}{2}\left(\sigma_i+1\right)-A_i-A_{i+1}-B_i-B_{i+1}\right]\left(u_{2x}\right)_i^j+\left[\frac{h_i}{6}\left(\sigma_i^2-1\right)+A_i-A_{i+1}\sigma_i\right]h_i\left(u_{3x}\right)_i^j\\&\quad+\left[\frac{h_i}{12}\left(\sigma_i^3+1\right)-A_i-A_{i+1}\sigma_i^2\right]\frac{h_i^2}{2}\left(u_{4x}\right)_i^j+\left[\frac{h_i}{20}\left(\sigma_i^4-1\right)+A_i-A_{i+1}\sigma_i^3\right]\frac{h_i^3}{6}\left(u_{5x}\right)_i^j\\&\quad+\left[\frac{h_i}{30}\left(\sigma_i^5+1\right)-A_i-A_{i+1}\sigma_i^4\right]\frac{h_i^4}{24}\left(u_{6x}\right)_i^j+O\left(h_i^5\right),\end{aligned} \tag{11}$$

where $\sigma_i=h_{i+1}/h_i,h_{i+1}=x_{i+1}-x_i$.

For $A_i=\dfrac{h}{12}\left(-\sigma_i^2+\sigma_i+1\right),A_{i+1}=\dfrac{h}{12\sigma_i}\left(\sigma_i^2+\sigma_i-1\right)$, $B_i+B_{i+1}=\dfrac{h_i}{12\sigma_i}\left(\sigma_i^3+4\sigma_i^2+4\sigma_i+1\right)$, the truncation error in space of the relation (10) is of $O\left(h^4\right)$.

From Equation (10), we can obtain

$$A_iM_{i-1}^j+\left(B_i+B_{i+1}\right)M_i^j+A_{i+1}M_{i+1}^j=\frac{\sigma_iu_{i-1}^j-\left(\sigma_i+1\right)u_i^j+u_{i+1}^j}{\sigma_ih_i}, \tag{12}$$

Or

$$A_iM_{i-1}^{j+\frac{1}{2}}+\left(B_i+B_{i+1}\right)M_i^{j+\frac{1}{2}}+A_{i+1}M_{i+1}^{j+\frac{1}{2}}=\frac{\sigma_iu_{i-1}^{j+\frac{1}{2}}-\left(\sigma_i+1\right)u_i^{j+\frac{1}{2}}+u_{i+1}^{j+\frac{1}{2}}}{\sigma_ih_i}, \tag{13}$$

Further, when $\sigma_i=1$, then $h=h_i=h_{i+1}$, $A_i=A_{i+1}=\dfrac{h}{12},B_i+B_{i+1}=\dfrac{10h}{12}$, the truncation error in space of the relation (10) is of $O\left(h^5\right)$, Equation (2.7) can be rewritten as

$$M_{i+1}^j+10M_i^j+M_{i-1}^j=\frac{12}{h^2}\left(u_{i+1}^j-2u_i^j+u_{i-1}^j\right), \tag{14}$$

$$M_{i+1}^{j+\frac{1}{2}}+10M_i^{j+\frac{1}{2}}+M_{i-1}^{j+\frac{1}{2}}=\frac{12}{h^2}\left(u_{i+1}^{j+\frac{1}{2}}-2u_i^{j+\frac{1}{2}}+u_{i-1}^{j+\frac{1}{2}}\right), \tag{15}$$

In order to get the error estimates of Equation (10), we put $E=\mathrm{e}^{hD}$ in Equation (12), where *E* and *D* are the shift and differential operators respectively, and expand them in powers of hD, we have

$$M_i^j = \frac{12}{h^2}\frac{E-2I+E^{-1}}{E+10I+E^{-1}}u_i^j = u_{xx}\left(x_i,t_j\right)+O\left(h^4\right), i=1,2,\cdots,N. \tag{16}$$

Or

$$\left(u_{xx}\right)_i^j = M_i^j + O\left(h^4\right), i=1,2,\cdots,N. \tag{17}$$

At the grid point $\left(x_i,t_j\right)$, Equation (1) can be discretized by

$$m\frac{u_i^{j+1}-u_i^j}{k}+\lambda_1\left(u_{xx}\right)_i^{j+\frac{1}{2}}+\varepsilon_i^{j+\frac{1}{2}}u_i^{j+\frac{1}{2}}+\lambda_2\frac{\left|u_i^{j+1}\right|^2+\left|u_i^j\right|^2}{2}u_i^{j+\frac{1}{2}}+O\left(k^2\right)=0, \tag{18}$$

From Equation (18), we have

$$M_{i-1}^{j+\frac{1}{2}} = -m\frac{u_{i-1}^{j+1}-u_{i-1}^j}{k\lambda_1}-\frac{1}{\lambda_1}\left(\varepsilon_{i-1}^{j+\frac{1}{2}}+\lambda_2\frac{\left|u_{i-1}^{j+1}\right|^2+\left|u_{i-1}^j\right|^2}{2}\right)u_{i-1}^{j+\frac{1}{2}}+O\left(k^2\right), \tag{19}$$

$$M_{i}^{j+\frac{1}{2}} = -m\frac{u_{i}^{j+1}-u_{i}^j}{k\lambda_1}-\frac{1}{\lambda_1}\left(\varepsilon_{i}^{j+\frac{1}{2}}+\lambda_2\frac{\left|u_{i}^{j+1}\right|^2+\left|u_{i}^j\right|^2}{2}\right)u_{i}^{j+\frac{1}{2}}+O\left(k^2\right), \tag{20}$$

$$M_{i+1}^{j+\frac{1}{2}} = -m\frac{u_{i+1}^{j+1}-u_{i+1}^j}{k\lambda_1}-\frac{1}{\lambda_1}\left(\varepsilon_{i+1}^{j+\frac{1}{2}}+\lambda_2\frac{\left|u_{i+1}^{j+1}\right|^2+\left|u_{i+1}^j\right|^2}{2}\right)u_{i+1}^{j+\frac{1}{2}}+O\left(k^2\right), \tag{21}$$

Substituting Equation (19), Equation (20) and Equation (21) into Equation (15) and after some simplifications, we obtain

$$F_{1i}u_{i-1}^{j+1}+F_{2i}u_i^{j+1}+F_{3i}u_{i+1}^{j+1} = F_{1i}^*u_{i-1}^j+F_{2i}^*u_i^j+F_{3i}^*u_{i+1}^j \tag{22}$$

where $i=1,2,\cdots,N,\ j=1,2,\cdots,\delta_i^j=\varepsilon_i^{j+\frac{1}{2}}+\lambda_2\frac{\left|u_i^{j+1}\right|^2+\left|u_i^j\right|^2}{2}$,

$$F_{1i}=\frac{A_i}{\lambda_1}\left(2m/k+\delta_{i-1}^j\right)+\frac{1}{h^2}, F_{2i}=\frac{B_i+B_{i+1}}{\lambda_1}\left(2m/k+\delta_i^j\right)-\frac{\sigma_i+1}{\sigma_i h_i},$$
$$F_{3i}=\frac{A_{i+1}}{\lambda_1}\left(2m/k+\delta_{i+1}^j\right)+\frac{1}{\sigma_i h_i}, F_{1i}^*=\frac{A_i}{\lambda_1}\left(-2m/k+\delta_{i-1}^j\right)+\frac{12}{h^2},$$
$$F_{2i}^*=\frac{B_i+B_{i+1}}{\lambda_1}\left(-2m/k+\delta_i^j\right)-\frac{24}{h^2}, F_{3i}^*=\frac{A_i}{\lambda_1}\left(-2m/k+\delta_{i+1}^j\right)+\frac{12}{h^2}.$$

The local truncation error of the relation (22) is of $O\left(k^2+h^4\right)$.

The boundary conditions (2) and the system given in the Equation (22) consists of $N+2$ equations in $N+2$ unknown. We can write this system in a matrix form as follows:

$$FU^{j+1}=F^*U^j, \tag{23}$$

where $U^j=\left(u_0^j,u_1^j,\cdots,u_N^j,u_{N+1}^j\right)^{\mathrm{T}}$,

Once the vectors U^0 are computed, $U^n, n=1,2,3,\cdots$, unknown vectors can be found repeatedly by solving the recurrence relation (23).

3. Stability Analysis

Following the von Neumann technique, we first linearize the nonlinear term in Equation (18) by making the quantity δ_i^j as locally constant δ and assume that the numerical solution can be expressed by means of a

Fourier series

$$u_i^j = \eta^j \exp(m\varphi ih) \tag{24}$$

where $m=\sqrt{-1}$, η^j is the amplitude at time level j, φ is the wave number and h is the element size. Substituting Equation (24) into Equation (22), the amplification factor can be written as

$$\eta = \frac{F_{1i}^* e^{-m\varphi h} + F_{2i}^* + F_{3i}^* e^{m\varphi h}}{F_{1i} e^{-m\varphi h} + F_{2i} + F_{3i} e^{m\varphi h}} \tag{25}$$

Using Eulers formula, we have

$$\eta = \frac{X_1 + mY_1}{X_2 + mY_2},$$

where $X_1 = X_2 = \left(\frac{\delta}{\lambda_1} + \frac{12}{h^2}\right)\cos(\varphi h) + \frac{10\delta}{\lambda_1} - \frac{24}{h^2}, Y_2 = -Y_1 = \frac{4}{k\lambda_1}\cos(\varphi h) + \frac{40}{k\lambda_1}$,

Since

$$|\eta| = \sqrt{\frac{X_1^2 + Y_1^2}{X_2^2 + Y_2^2}} = 1,$$

Thus this method is unconditionally stable.

4. Computation of Conserved Quantities and Error Norms

The nonlinear Schrödinger equation possesses two conservation quantities:

(1) Mass conservation:

$$C_1^{exact} = \int_a^b |u(x,t)|^2 \,\mathrm{d}x, \tag{26}$$

Calculated by

$$C_1 = h\sum_{j=0}^{N} |u_j^n|^2, \tag{27}$$

(2) Energy conservation: If $\lambda_1(t)$ and $\lambda_2(t)$ are independent of t, then

$$C_2^{exact} = \int_a^b \left(\lambda_1 |u_x(x,t)|^2 - \varepsilon(x)|u(x,t)|^2 - \frac{\lambda_2}{2}|u(x,t)|^4\right)\mathrm{d}x, \tag{28}$$

Calculated by

$$C_2 = h\sum_{j=0}^{N}\left[\lambda_1 \left|(u_x)_j^n\right|^2 - \varepsilon_i |u_j^n|^2 - \frac{\lambda_2}{2}|u_j^n|^4\right], \tag{29}$$

where u_j^n and u are the approximate solution at n-th time step at j-th node and exact solution, respectively.

The maximum error norm L_∞ and discrete root mean square error norm L_2 will be calculated

$$L_\infty(h,k) = \|u - u^n\|_\infty = \max_{0\le i\le N} |u(x_i) - u_i^n| \tag{30}$$

$$L_2(h,k) = \|u - u^n\|_2 = \sqrt{h\sum_{i=0}^{N} |u(x_i) - u_i^n|^2} \tag{31}$$

The relative error of numerical solution is defined as

$$E^r = \frac{\sqrt{\sum_{i=1}^{N} |u(x_i) - u_i^n|^2}}{\sqrt{\sum_{i=1}^{N} |u_i^n|^2}} \tag{32}$$

5. Numerical Results

In the section, we present the results of our numerical experiments for the proposed scheme described in the previous section.

Example 1. Consider the one dimensional Gross-Pitaevskii equation

$$mu_t + \frac{1}{2}u_{xx} - \cos^2(x)u - |u|^2 u = 0, x \in [0, 2\pi], t \ge 0 \tag{33}$$

With the analytical solution

$$u(x,t) = \mathrm{e}^{-m\frac{3t}{2}} \sin(x), \tag{34}$$

Conserved quantities and error norms at various times are recorded in **Table 1**. The real and imaginary parts of the numerical and exact solutions are tabulated in **Table 2**, the numerical results reveal the accuracy of the proposed method.

The absolute error at different space step sizes h at time $t = 1$ are shown in **Figure 1**, it can be seen that the absolute errors becomes smaller as decreasing h.

Example 2. Consider the equation (1) with $\lambda_1 = -1, \lambda_2 = 1$,

$$\varepsilon(x,t) = 4(x-2t)^2 - \mathrm{e}^{-2(x-2t)^2}, \tag{35}$$

The exact solution of this problem is

$$u(x,t) = \mathrm{e}^{-(x-2t)^2 + m(-x+3t)}, \tag{36}$$

Table 1. Conserved quantities and error norms at various times for example 1 with $k = 0.01, h = \frac{2\pi}{64}, a = 0, b = 2\pi$.

t	C_1	C_2	L_∞	L_2	E^r
5.0	3.14159265358952	5.00720563249462	1.4158e−004	2.5096e−004	1.4158e−004
10	3.14159265358946	5.00720563249418	2.8317e−004	5.0191e−004	2.8317e−004
20	3.14159265358965	5.00720563249524	5.6635e−004	1.0038e−003	5.6635e−004
30	3.14159265358984	5.00720563234957	8.4953e−004	1.5057e−003	8.4953e−004

$C_1^{0exact} = C_1^0 = 3.14159265358979$

Table 2. The real and imaginary parts of the numerical and exact solutions for Example 1 with $k = 0.001, h = \frac{2\pi}{64},$. $a = 0, b = 2\pi, t = 1$.

x_i	Real parts			Imaginary parts		
	Exact solution	Approximation	Absolute error	Exact solution	Approximation	Absolute error
$\frac{\pi}{4}$	0.05001875498139	0.05001908991577	3.35e-007	−0.70533546922731	−0.70533544547538	2.37e−008
$\frac{\pi}{2}$	0.07073720166770	0.07073767533643	4.73e-007	−0.99749498660405	−0.99749495301379	3.35e−008
$\frac{3\pi}{4}$	0.05001875498139	0.05001908991578	3.35e-007	−0.70533546922731	−0.70533544547537	2.37e−008
$\frac{5\pi}{4}$	−0.05001875498139	−0.05001908991578	3.35e-007	0.70533546922731	0.70533544547538	2.37e−008
$\frac{6\pi}{4}$	−0.07073720166770	−0.07073767533646	4.73e-007	0.99749498660405	0.99749495301376	3.36e−008
$\frac{7\pi}{4}$	−0.05001875498139	−0.05001908991577	3.35e-007	0.70533546922731	0.70533544547537	2.37e−008

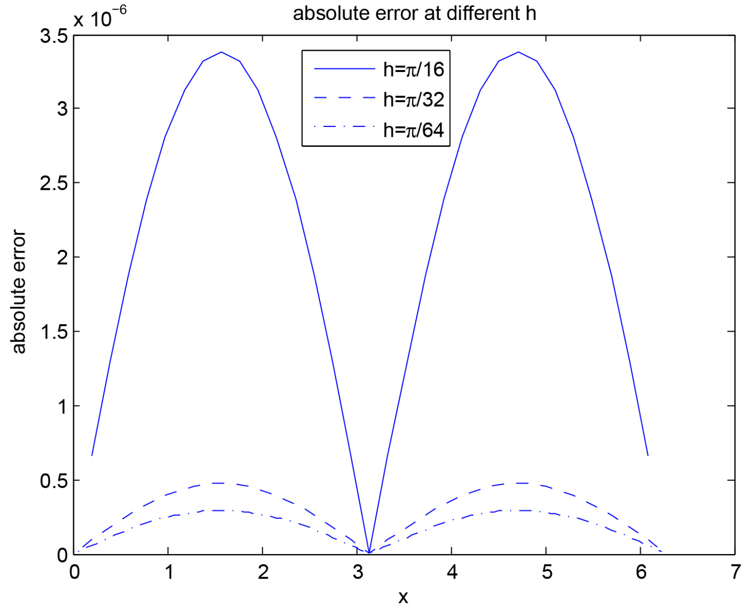

Figure 1. The absolute error at different h for example 1 with $k = 0.001, t = 1$.

Table 3. Conserved quantities and error norms at various times for example 2 with $k = 0.01, h = 0.1, a = -10, b = 20$.

t	C_1	C_2	L_∞	L_2	E^r
1	1.25331413731550	−3.31870453365852	2.1940e-003	2.8448e-003	2.5411e-003
2	1.25331413731550	−3.33237042327679	2.3980e-004	3.2407e-003	2.8948e-003
3	1.25331413731550	−3.32685242595849	7.7694e-004	9.4702e-004	8.4592e-004
4	1.25331435202165	−3.31733074605112	2.2382e-003	2.5906e-003	2.3140e-003

$C_1^{0exact} = C_1^0 = 1.25331413731550, C_2^{0exact} = C_2^0 = -3.32358067657703$

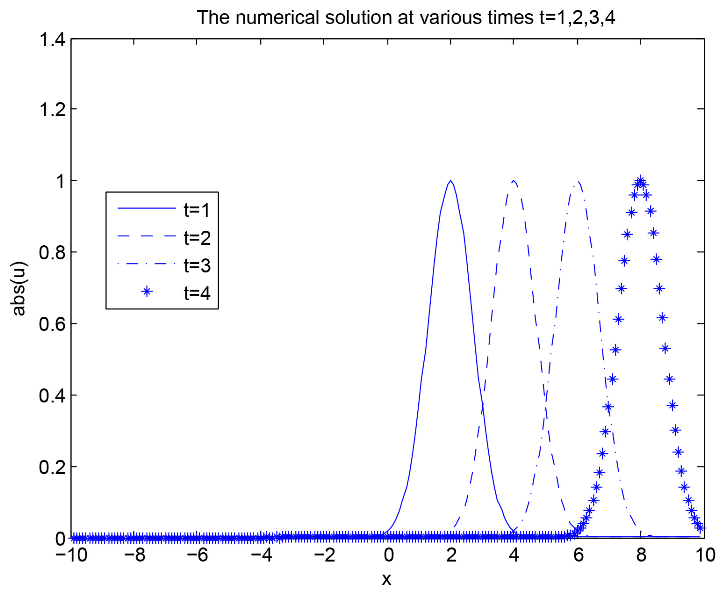

Figure 2. The numerical solution at various times t = 1, 2, 3, 4 with $k = 0.01, h = 0.1$.

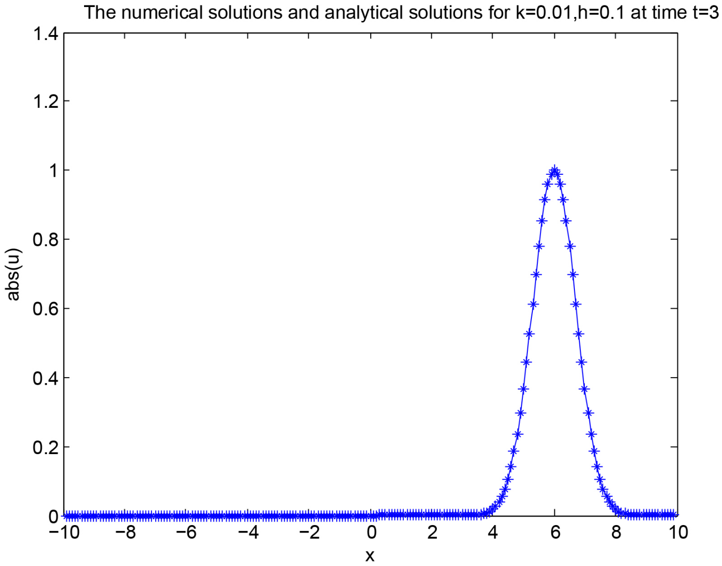

Figure 3. The numerical solutions and analytical solutions for $k = 0.01$, $h = 0.1$ at time $t = 3$.

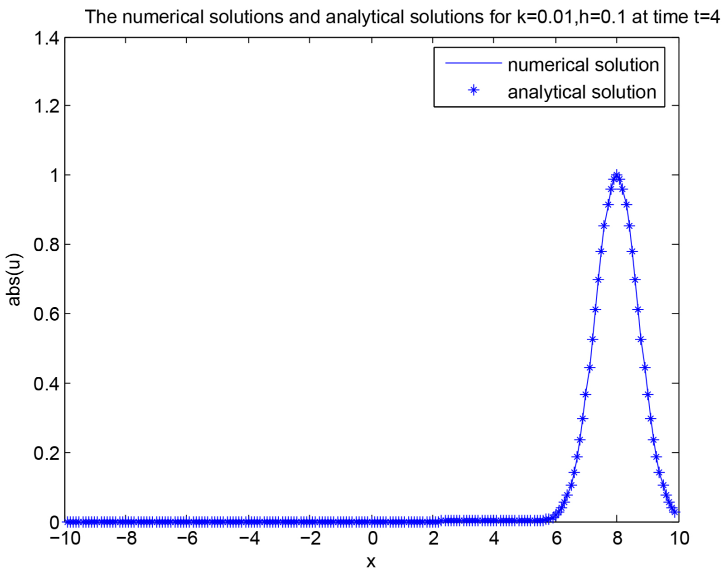

Figure 4 The numerical solutions and analytical solutions for $k = 0.01$, $h = 0.1$ at time $t = 4$.

Conserved quantities and error norms at various times are presented in **Table 3**. The numerical results reveal that the values of C_1 is almost constant while the values of C_2 differ slightly and the errors are very small.

The numerical solutions at various times are given in **Figure 2**. The numerical solutions and analytical solutions at time $t = 3$ and $t = 4$ are shown in **Figure 3** and **Figure 4**, respectively. The absolute error at time $t = 3$ and $t = 4$ are plotted in **Figure 5** and **Figure 6**, respectively. It observed that (1) the propagation of solitary wave is rightward while preserving unchanged shape; (2) our method gives a good approximation compared with the exact solutions.

6. Conclusion

A numerical method based on exponential spline interpolation function is applied to study a class of nonlinear Schrödinger equation. We use exponential spline collocation method, which results in tri-diagonal systems of

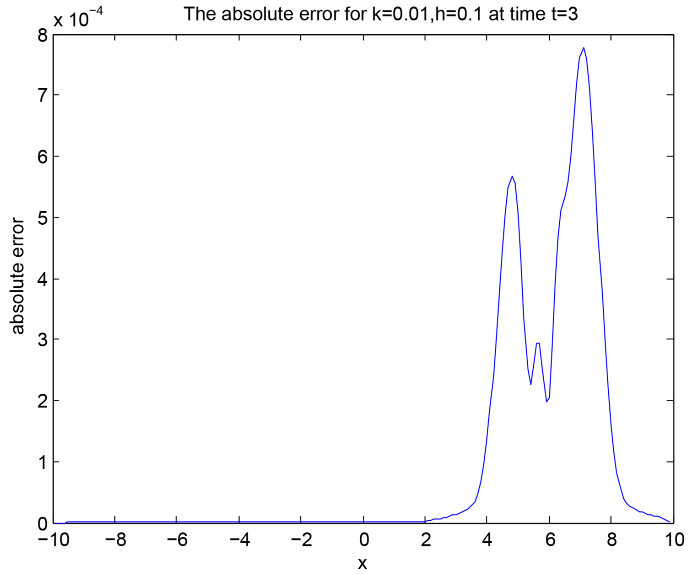

Figure 5. The absolute error for $k = 0.01$, $h = 0.1$ at time $t = 3$.

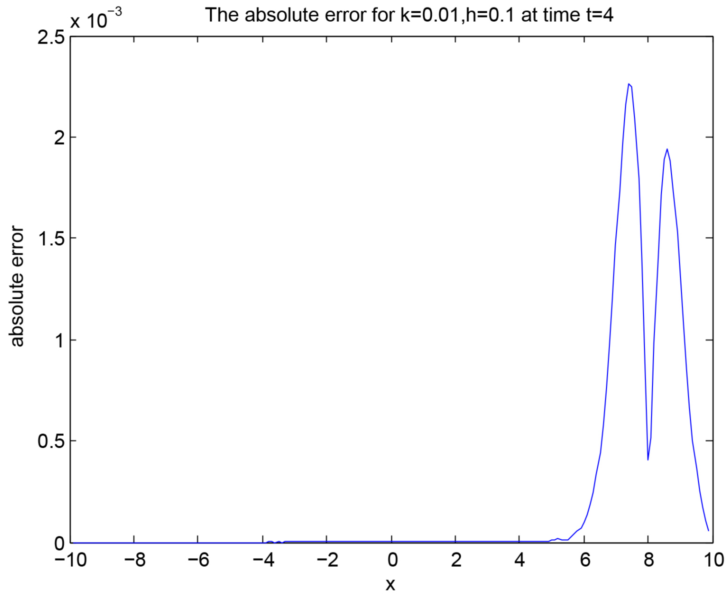

Figure 6. The absolute error for $k = 0.01$, $h = 0.1$ at time $t = 4$.

equations that can be solved efficiently by the Thomas algorithm. The numerical simulations confirm and demonstrate the reliability and efficiency of the schemes and tell us that the method is applicable technique, relatively simple and approximates the exact solution very well.

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Information Theory Model for Radiation

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Abstract

Information based models for radiation emitted by a Black Body which passes through a scattering medium are analyzed. In the limit, when there is no scattering this model reverts to the Black Body Radiation Law. The advantage of this mathematical model is that it includes the effect of the scattering of the radiation between source and detector. In the case when the exact form of the scattering mechanism is not known a model using a single scattering parameter is derived. A simple version of this model is derived which is useful for analyzing large data.

Keywords

Information Theory, Maximum Information, Entropy, Scattering, Conditional Entropy

1. Introduction

An **I**nformation **T**heory **R**adiation **M**odel (ITRM) for radiation emitted by a black body passing through a scattering medium is analyzed. A variational method is used to derive three equations similar to the **B**lack **B**ody **R**adiation (BBR) law. But, these equations include the effect of scattering. In the limit when there is no scattering these models revert to the BBR law. The advantage of this mathematical model is that it includes the effect of the scattering of the radiation between source and detector. Three equations are derived. One equation describes the effect of the scattering on the radiation when the form of the scattering mechanism is known. The second result is an equation for the case when the forms of the scattering mechanisms are not known. In this case the model contains a single scattering parameter. The third equation is a simplified version of the case when the form of the scattering is not known. This is useful for the analysis of large data. The derivation of the case when the form of the scattering mechanism is not known is similar to one I presented in a previous publication [1]. It was formulated for the analysis of the Cosmic Background Radiation. A variational method using Information Theory was used to derive this model. To obtain the three equations the Information was maximized, subject to what is known about the system.

The basis of Information Theory was developed by C. E. Shannon [2] at Bell Laboratories. The information theoretical model facilitates the inclusion of the effect of information propagating through noisy channels [2]. Here the effect of the thermal radiation being scattered on the way from the hot Black Body source to the detector is analyzed.

One phenomenon to which one can apply this model is to the statistics of light quanta. In the Quantum Mechanical model of nature the energy of the electromagnetic radiation oscillating with any given frequency is divided into energy quanta or photons. The different quantum mechanical energy states are energy packets containing different numbers of photons. The information transmitted at any frequency by the hot body is encoded in the number of photons radiated. Different numbers of photons radiated represent different information, see **Figure 1**. Thus, an energy state represents an amount of information. The amount of information in each photon packet or energy state is not equal to the number of photons, but to a function of the number of photons. This is illustrated in **Figure 1**. The photon packets are represented by mail bags in **Figure 1**. The information in each mail bag is displayed on the tags. The unit of information used in this schematic representation is in binary bits. However, the information used in this article is in a unit of Joules per degree Kelvin.

The photons travel through space where some are absorbed or scattered. Maybe, the radiation when passing through an ionized cloud is even amplified by stimulated emission. This can occur by generating additional photons of the same frequency as the incoming radiation neutralizes some of the charges.

The model derived here can be used both when the details of the scattering mechanism is known and in a simple case when the details of the scattering mechanism is unknown. In the case when the scattering mechanism is known, the scattering mechanism is represented here by a stochastic model. It is represented by conditional probabilities that one number of photons were radiated provided another number of photons were received. The conditional probabilities have to be constructed to represent the known scattering model.

In the case when the detail forms of the scattering mechanisms are not known, the forms of the conditional probabilities can not be specified. In this case, the information formed by the conditional probabilities can be approximated by a function that includes a single average scattering parameter. Like the BBR, the ITRM depends on the absolute temperature.

In the case when the radiation data is known, the source temperature and the average scattering parameter can be determined by comparing the data to calculated ITRM values. This is the case, for example, for the Cosmic Microwave Background radiation.

One result of including the effect of scattering is a blue shift of the distribution of photons, see **Figure 2**. Unlike in the case of the Doppler effect the wavelength of the individual photons do not change. However, the distribution changes to more short wave photons. The total number of photons can also change to fewer or more photons.

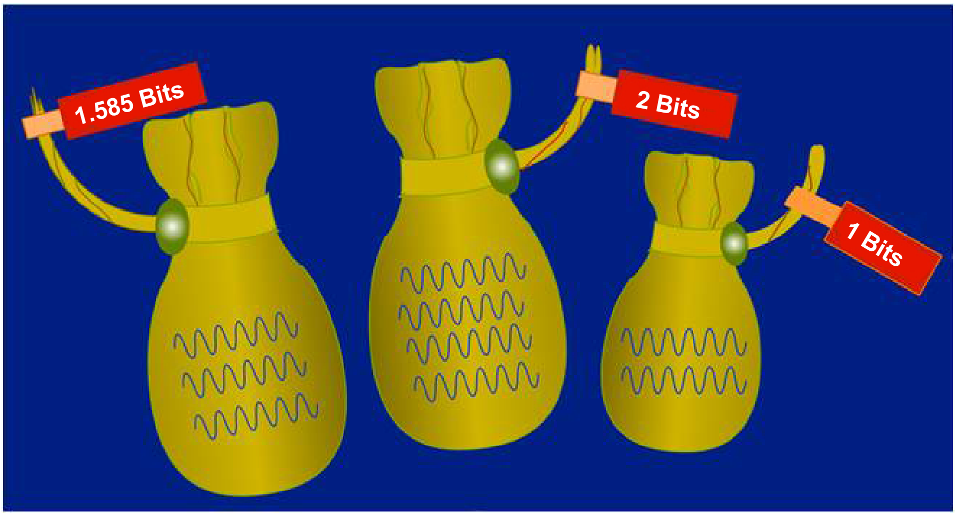

Figure 1. Schematic representation of the encoding of the information. The information is encoded in the number of photons transmitted. The amount of information in each photon packet is a function of the number of photons in the packet. The photon packets are represented by mail bags here. The information in each mail bag is displayed on its tag.

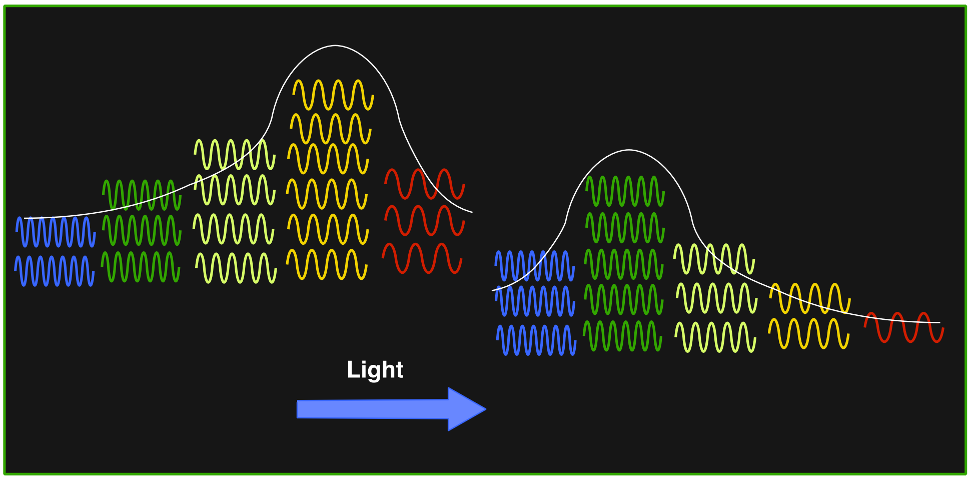

Figure 2. Schematic representation of a blue shifted photon distribution. Unlike in the case of the Doppler effect, the wavelength of the individual photons do not change. However, the distribution changes to more short wave photons.

2. Information Model

The ITRM for thermal radiation through a scattering medium is derived below. Some of the derivation is similar to the derivation in my Cosmic Background Radiation analysis paper [1]. The concepts used here can be found in a paper by C. E. Shannon [2] and many Probability texts [3]-[5]. The information is encoded in the number n of photons radiated by the hot object, see **Figure 1**. The value of the information g_n in each information packet is a function of the number n of photons.

$$g_n = -k \ln P_n \tag{2.1}$$

where k is Boltzmann constant and P_n is the probability that a signal of n photons is being received. This is similar to the famous Boltzmann equation engraved on his tombstone. The equation on the tombstone is for uniform probabilities $P_n = \frac{1}{N}$. The information here is in a unit of Joules per degree Kelvin. Note that the probabilities P_n are less or equal to one. Therefore, the logarithm of the probability is negative and the information g_n is positive. The information packets are shown schematically as mail bags in **Figure 1**. The probabilities P_n are normalized.

$$\sum_{n=0}^{n=\infty} P_n - 1 = 0 \tag{2.2}$$

The average detected Shannon information is equal to the average value $\boldsymbol{H}$ of all the information packets.

$$a)\ \boldsymbol{H} = \sum_{n=0}^{n=\infty} g_n P_n \qquad b)\ \boldsymbol{H} = -k \sum_{n=0}^{n=\infty} P_n \ln P_n \tag{2.3}$$

The propagation of the photons from the hot Black Body source to the detector is modeled by conditional probabilities P (m photons radiated | provided n Photons received) that m photons are radiated provided n photons were received. Associated with the conditional probabilities having the same condition of n photons being received are conditional entropies $\boldsymbol{h}(S|n)$. Here S is the set of all the different numbers m of radiated photons and n is the number of photons that were received [5].

$$\boldsymbol{h}(S \mid n) = -k \sum_{m=0}^{m=\infty} P(m \mid n) \ln P(m \mid n) \tag{2.4}$$

Here k is Boltzmann constant and n is the number of photons that were received. The average value $\boldsymbol{N}$ of the conditional information is also known as the noise:

$$\boldsymbol{N} = \sum_{n=0}^{n=\infty} \boldsymbol{h}(S \mid n) P_n \tag{2.5}$$

The information $\boldsymbol{I}$ is equal to the difference between the received Shannon information $\boldsymbol{H}$ of Equation (2.3) and the noise $\boldsymbol{N}$ of Equation (2.5)

$$\boldsymbol{I} = \boldsymbol{H} - \boldsymbol{N} \tag{2.6}$$

The temperature T is the change of the light energy U with the information $\boldsymbol{H}$ carried by the photons. For this derivation the temperature T of the radiation at the receiver is assumed to be known.

$$T = \frac{\partial U}{\partial \boldsymbol{H}} \tag{2.7}$$

where the average energy U of the received photons is given by:

$$\sum_{n=0}^{n=\infty} \hbar \omega n P_n - U = 0 \tag{2.8}$$

Its value, at this point, is not known. Here $\hbar$ is Plank's constant divided by 2π and ω is the oscillating frequency of the radiation.

A variational method is used to calculate the values of the Probabilities P_n. The probabilities P_n can be derived by finding an extremum value of the information $\boldsymbol{I}$ subject to what is known about the system. In this case the temperature T at the receiver and the fact that the probabilities are normalized are known about the system. However, the Equation (2.7) for the temperature, is not in the form of a constraint equation like Equations (2.2) and (2.8). Therefore, it can not be used in this process directly. Instead one has to use the average energy U first. By multiplying the two constraint equations, Equations (2.2) and (2.8) by convenient constants $-\alpha k$ and $-\beta k$ and adding them to the equation for the information $\boldsymbol{I}$ one obtains:

$$\boldsymbol{I} = -k \sum_{n=0}^{n=\infty} \left[P_n \ln P_n + \frac{\boldsymbol{h}(S \mid n)}{k} P_n + \alpha P_n + \beta \hbar \omega n P_n \right] + \alpha k + \beta k U \tag{2.9}$$

The information $\boldsymbol{I}$ will have an extremum value when all its derivatives with respect to the probabilities P_n are equal to zero. By taking the derivative of the information $\boldsymbol{I}$ with respect to one of the probabilities P_n, setting the result equal to zero and solving for the probability P_n one obtains:

$$P_n = \exp(-1-\alpha) \exp\left[-\frac{h(S \mid n)}{k} - \beta \hbar \omega n \right] \tag{2.10}$$

The values of the constants α and β are not known at this point. In order to evaluate the constant α one substitutes the probability of Equation (2.10) into the first constraint equation, which is Equation (2.2). One obtains for $\exp(-1-\alpha)$:

$$\exp(-1-\alpha) = \frac{1}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - \beta \hbar \omega m \right]} \tag{2.11}$$

The constant α can be eliminated by substituting Equation (2.11) into Equation (2.10).

$$P_n = \frac{\exp\left[-\frac{\boldsymbol{h}(S \mid n)}{k} - \beta \hbar \omega n \right]}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - \beta \hbar \omega m \right]} \tag{2.12}$$

The constant β has yet to be evaluated. In order to accomplish this, one must first calculate the information $-k \ln(P_n)$ of receiving n photons by taking the logarithm of the probability P_n of Equation (2.12) and multiplying the result by minus the Boltzmann constant.

$$-k \ln(P_n) = \boldsymbol{h}(S \mid n) + \beta k \hbar \omega n + k \ln \sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - \beta \hbar \omega m \right] \tag{2.13}$$

By substituting the information associated with receiving n photons, Equation (2.13), into the average Shannon information of Equation (2.3) one obtains:

$$\boldsymbol{H} = \sum_{n=0}^{n=\infty} \boldsymbol{h}(S \mid m) P_n + \beta k U + k \ln \sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - \beta \hbar \omega m\right] \tag{2.14}$$

where Equation (2.8) was used for the average energy U. By solving Equation (2.14) for the average energy U, substituting the resulting expression into Equation (2.7) and solving for β one obtains the well known expression:

$$\beta = \frac{1}{kT} \tag{2.15}$$

The probability P_n of receiving n photons can now be completely specified by substituting Equation (2.15) for the constant β into Equation (2.12).

$$P_n = \frac{\exp\left[-\frac{\boldsymbol{h}(S \mid n)}{k} - \frac{\hbar \omega n}{kT}\right]}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - \frac{\hbar \omega m}{kT}\right]} \tag{2.16}$$

The average energy of a one dimensional radiating system where the radiation passes through a scattering medium is derived by substituting the probabilities of Equation (2.16) into the equation for the average received energy U, Equation (2.8).

$$U = \frac{\hbar \omega \sum_{n=0}^{n=\infty} n \exp\left[-\frac{\boldsymbol{h}(S \mid n)}{k} - xn\right]}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - xm\right]} \tag{2.17}$$

where the normalized frequency x is given by:

$$x = \frac{\hbar \omega}{kT} \tag{2.18}$$

Finally, by multiplying Equation (2.17) by an appropriate density of states constant one obtains for the change $\frac{\mathrm{d}u}{\mathrm{d}\omega}$ of the average received energy density u with radiation frequency ω of a three dimensional system radiating through a scattering medium:

$$\frac{\mathrm{d}u}{\mathrm{d}\omega} = \left(\frac{4\pi k^3 T^3}{c^3 \hbar^2}\right) \frac{x^3 \sum_{n=0}^{n=\infty} n \exp\left[-\frac{\boldsymbol{h}(S \mid n)}{k} - xn\right]}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S \mid m)}{k} - xm\right]} \tag{2.19}$$

where $\boldsymbol{h}(S|n)$ is given by Equation (2.4) and where c is the speed of light in free space. **This is the first result.** It is the Black Body Radiation law for systems radiating through a scattering medium. It can be used when the form of the conditional probabilities that describe the scattering mechanism are known. Note that the temperature T is the observed temperature at the receiver. For the case when there is no scattering, when $\boldsymbol{h}(S|n)$ is equal to zero, Equation (2.19) reverts to the standard Black Body Radiation law.

$$a)\ \frac{\mathrm{d}u}{\mathrm{d}\omega} = \left(\frac{4\pi k^3 T^3}{c^3 \hbar^2}\right) \frac{x^3 \sum_{n=0}^{n=\infty} n \exp[-xn]}{\sum_{m=0}^{m=\infty} \exp[-xm]} \qquad b)\ \frac{\mathrm{d}u}{\mathrm{d}\omega} = \left(\frac{4\pi k^3 T^3}{c^3 \hbar^2}\right) \frac{x^3}{\exp(x) - 1} \tag{2.20}$$

For the case when the details of the scattering models are not known the conditional probabilities can not be specified. However, one can postulate a simple model for the conditional information or conditional entropies [3] [4] $\boldsymbol{h}(S|n)$.

$$\boldsymbol{h}(S\,|\,n) \approx \begin{cases} k\rho \ln(n) & \text{for } n \neq 0 \\ 0 & \text{for } n = 0 \end{cases} \tag{2.21}$$

where ρ is an average scattering parameter. By substituting Equation (2.21) into Equation (2.19) one obtains for the change $\frac{\mathrm{d}u}{\mathrm{d}\omega}$ of the average received energy density u with radiation frequency ω for the case when the scattering mechanisms are not known:

$$\frac{\mathrm{d}u}{\mathrm{d}\omega} = \left(\frac{4\pi k^3 T^3}{c^3 \hbar^2}\right) \frac{x^3 \sum_{n=0}^{n=\infty} n \exp\left[-\rho \ln(n) - xn\right]}{\sum_{m=1}^{m=\infty} \exp\left[-\rho \ln(m) - xm\right] + 1} \tag{2.22}$$

and where $\lim_{n \to 0} n \ln(n) = 0$. Since the scattering parameter ρ is an Entropy Amplitude it must always be positive. **This is the second result.** This result was also used in my Cosmic Background Radiation analysis paper [1]. It is applicable when the form of the scattering mechanism is not known. It describes the scattering process in terms of a single scattering parameter ρ. This shifts the peak of the distribution to larger values of the normalized frequency $x = \frac{\hbar\omega}{kT}$ see **Figure 2**.

Equation (2.22) can be expressed as follows for large values of $\frac{\hbar\omega}{kT}$:

$$\frac{\mathrm{d}u}{\mathrm{d}\omega} \approx \left(\frac{4\pi k^3 T^3}{c^3 \hbar^2}\right) \frac{x^3 \left(y + 2^{1-\rho} y^2 + 3^{1-\rho} y^3 + \cdots\right)}{1 + y + \frac{y^2}{2^\rho} + \frac{y^3}{3^\rho} + \cdots} \tag{2.23}$$

where

$$y = \exp\left(-\frac{\hbar\omega}{kT}\right) \tag{2.24}$$

Equation 2.23 is the third result. This equation is especially useful when large data is to be analyzed.

For completeness, the input information $\boldsymbol{I}$ is calculated by subtracting the noise $\boldsymbol{N}$ from the Shannon information $\boldsymbol{H}$ of Equation (2.3). The noise $\boldsymbol{N}$ is calculated in Equation (2.5). Equation (2.16) is used for the probabilities P_n.

$$\boldsymbol{I} = \sum_{n=0}^{n=\infty} \boldsymbol{h}(S\,|\,n) P_n + \frac{U}{T} + k \ln \sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S\,|\,m)}{k} - xm\right] - \sum_{n=0}^{n=\infty} \boldsymbol{h}(S\,|\,n) P_n \tag{2.25}$$

Note that the first and last terms of Equation (2.25) cancel. By making use of Equation (2.17) for the average energy U one obtains:

$$\boldsymbol{I} = \frac{\frac{\hbar\omega}{T} \sum_{n=0}^{n=\infty} n \exp\left[-\frac{\boldsymbol{h}(S\,|\,n)}{k} - xn\right]}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S\,|\,m)}{k} - xm\right]} + k \ln \sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S\,|\,m)}{k} - xm\right] \tag{2.26}$$

By multiplying Equation (2.26) by the same density of states as was used in Equation (2.19) one obtains:

$$\frac{4\pi\omega^2}{c^3} \boldsymbol{I} = \left(\frac{4\pi k^3 T^2}{c^3 \hbar^2}\right) \left\{ \frac{x^3 \sum_{n=0}^{n=\infty} n \exp\left[-\frac{\boldsymbol{h}(S\,|\,n)}{k} - xn\right]}{\sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S\,|\,m)}{k} - xm\right]} + x^2 \ln \sum_{m=0}^{m=\infty} \exp\left[-\frac{\boldsymbol{h}(S\,|\,m)}{k}(m) - xm\right] \right\} \tag{2.27}$$

Equation (2.27) is in $\mathrm{Joules}/\mathrm{m}^3 \cdot \mathrm{Hertz} \cdot {}^0\mathrm{K}$. The explanation given here of the shift of the photon distribution to higher frequencies is due to the effect of the scattering process as discussed above.

3. Conclusion

An **I**nformation **T**heory **R**adiation **M**odel (ITRM) for radiation passing through a scattering medium radiated by a black body has been derived. The result of this analysis is given by Equations (2.19), (2.22) and (2.23). Equation (2.19) is the ITRM for the case when the conditional probabilities that describe the scattering mechanisms are known. Equation 2.22 is an approximation of the ITRM for the case when the form of the scattering mechanisms is not known. In this case a single average scattering parameter is used to characterize the scattering process. Equation (2.23) is an approximation of Equation (2.22). It is an equation for the case when the form of the scattering mechanisms are not known. It has a simpler form than Equation (2.22), but it is only valid for large values of $\frac{\hbar\omega}{kT}$. This equation is especially useful when large sets of data are analyzed. In the limit of no scattering the ITRM reverts to the Black Body Radiation law.

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Razumikhin-Type Theorems on General Decay Stability of Impulsive Stochastic Functional Differential Systems with Markovian Switching

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Abstract

In this paper, the Razumikhin approach is applied to the study of both *p*-th moment and almost sure stability on a general decay for a class of impulsive stochastic functional differential systems with Markovian switching. Based on the Lyapunov-Razumikhin methods, some sufficient conditions are derived to check the stability of impulsive stochastic functional differential systems with Markovian switching. One numerical example is provided to demonstrate the effectiveness of the results.

Keywords

Impulsive Stochastic Functional Differential System, *p*-th Moment Stability, Almost Sure Stability, Lyapunov-Razumikhin Approach

1. Introduction

Impulsive stochastic systems with Markovian switching is a class of hybrid dynamical systems, which is composed of both the logical switching rule of continuous-time finite-state Markovian process and the state represented by a stochastic differential system [1]. Because of the presence of both continuous dynamics and discrete events, these types of models are capable of describing many practical systems in many areas, including social science, physical science, finance, control engineering, mechanical and industry. So this kind of systems have received much attention, recently (for instance, see [2]-[5]).

It is well-known that stability is the major issue in the study of control theory, one of the most important

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techniques applied in the investigation of stability for various classes of stochastic differential systems is based on a stochastic version of the Lyapunov direct method. However, the so-called Razumikhin technique combined with Lyapunov functions has also been a powerful and effective method in the study of stability. Recalled that Razumikhin developed this technique to study the stability of deterministic systems with delay in [6] [7], then, Mao extended this technique to stochastic functional differential systems [8]. This technique has become very popular in recent years since it is extensively applied to investigate many phenomena in physics, biology, finance, etc.

Mao incorporated the Razumikhin approach in stochastic functional differential equations [9] and in neutral stochastic functional differential equations [10] to investigate both p-th moment and almost sure exponential stability of these systems (see also [11]-[13], for instance). Later, this technique was appropriately developed and extended to some other stochastic functional differential systems, especially important in applications, such as stochastic functional differential systems with infinite delay [14]-[16], hybrid stochastic delay interval systems [17] and impulsive stochastic delay differential systems [18]-[20]. Recently, some researchers have introduced ψ-type function and extended the stability results to the general decay stability, including the exponential stability as a special case in [21]-[23], which has a wide applicability.

In the above cited papers, both the p-th moment and almost sure stability on a general decay are investigated, but mostly used in stochastic differential equations. And As far as I know, a little work has been done on the impulsive stochastic differential equations or systems. In this paper, we will close this gap by extending the general decay stability to the impulsive stochastic differential systems. To the best of our knowledge, there are no results based on the general decay stability of impulsive stochastic delay differential systems with Markovian switching. And the main aim of the present paper is attempt to investigate the p-th moment and almost sure stability on a general decay of impulsive stochastic delay differential systems with Markovian switching. Since the delay phenomenon and the Markovian switching exists among impulsive stochastic systems, the whole systems become more complex and may oscillate or be not stable, we introduce Razumikhin-type theorems and Lyapunov methods to give the conditions that make the systems stable. By the aid of Lyapunov-Razumikhin approach, we obtain the p-th moment general decay stability of impulsive stochastic delay differential systems with Markovian. In order to establish the criterion on almost surely general decay stability of impulsive stochastic delay differential systems with Markovian, the Holder inequality, Burkholder-Davis-Gundy inequality and Borel-Cantelli's lemma are utilized in this paper.

The paper is organized as follows. Firstly, the problem formulations, definitions of general dacay stability and some lemmas are given in Section 2. In Section 3, the main results on p-th moment and almost sure stability on a general decay of impulsive stochastic delay differential systems with Markovian switching are obtained with Lyapunov-Razumikhin methods. An example is presented to illustrate the main results in Section 4. In the last section the conclusions are given.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (*i.e.*, the filtration is increasing and right continuous while $\mathcal{F}_0$ contain all P-null sets). Let $B=(B(t), t\geq 0)$ be an m-dimensional $\mathcal{F}_t$-adapted Brownian motion.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ denotes the $n\times m$ real matrix space; $\mathbb{R}_+$ is the set of all non-negative real numbers; $PC([-\tau,0];\mathbb{R}^n)$ denotes the family of continuous functions $\varphi:[-\tau,0]\to\mathbb{R}^n$ with the norm $\|\varphi\|=\sup_{\theta\leq 0}|\varphi(\theta)|$; $|\cdot|$ denotes the standard Euclidean norm for vectors; let $p\geq 1$, $t\geq 0$, $PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$) denotes the family of $\mathcal{F}_t$-measurable $PC([-\tau,0];\mathbb{R}^n)$-valued random variables $\varphi=\{\varphi(\theta):-\tau\leq\theta\leq 0\}$ such that $\sup_{\theta\leq 0} E|\varphi(\theta)|^p<\infty$ and $PC^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ be the $\mathcal{F}_0$-measurable $PC([-\tau,0];\mathbb{R}^n)$-valued random variables; $E[\cdot]$ means the expectation of a stochastic process; $\mathbb{N}=1,2,\cdots,N$ is a discrete index set, where N is a finite positive integer.

Let $r(t), t\geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\Gamma=1,2,\cdots,N$ with generator $\Pi=(\pi_{ij}), i,j\in\Gamma$ given by

$$P\{r(t+\Delta)=j\,|\,r(t)=i\}=\begin{cases}\pi_{ij}\Delta+o(\Delta) & i\neq j\\ 1+\pi_{ij}\Delta+o(\Delta) & i=j\end{cases}$$

where $\Delta > 0$, $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta} = 0$ and $\pi_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\pi_{ij} = -\sum_{j\ne i}\pi_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple in any subinterval if $[0,\infty)$. In other words, there exist a sequence of stopping times $0 = t_0 < t_1 < \cdots < t_k \to \infty$ almost surely such that $r(t)$ is a constant in every interval $[t_{k-1}, t_k)$ for any $k \ge 1$, *i.e.*

$$r(t) = r(t_{k-1}), \quad \forall t \in [t_{k-1}, t_k), k \ge 1.$$

In this paper, we consider the following impulsive stochastic delay differential systems with Markovian switching

$$\begin{cases} \mathrm{d}x(t) = f(t, x_t, r(t))\mathrm{d}t + g(t, x_t, r(t))\mathrm{d}B(t), \ t \ne t_k, t \ge t_0 \\ \Delta x(t_k) = I_k\left(x(t_k^-), t_k\right), \ k \in \mathbb{N} \\ x_{t_0} = \xi \end{cases} \tag{1}$$

where the initial value $\xi \in PC^b([-\tau,0];\mathbb{R}^n)$, $x(t) = (x_1(t),\cdots,x_n(t))^{\mathrm{T}}$, $x_t = x(t+\theta) \in PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$, $f: PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}_n$, $g: PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}^{n\times d}$, $I_k\left(x(t_k^-), t_k\right): \mathbb{R}^n\times\mathbb{R}_+ \to \mathbb{R}^n$ represents the impulsive perturbation of x at time t_k. The fixed moments of impulse times t_k satisfy $0 \le t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty$ (as $k \to \infty$), $\Delta x(t_k) = x(t_k) - x(t_k^-)$.

For the existence and uniqueness of the solution we impose a hypothesis:

Assumption (H): For $f(\varphi,t)$ and $g(\varphi,t)$ satisfy the local Lipschitz condition and the linear growth condition. That is, there exist a constant $L > 0$ such that

$$\left|f(\varphi,t) - f(\phi,t)\right|^2 \vee \left|g(\varphi,t) - g(\phi,t)\right|^2 \le L\|\varphi-\phi\|^2$$

For all $t \ge 0$, and $\varphi,\phi \in PC([-\tau,0];\mathbb{R}^n)$, and, moreover, there are a constant $K > 0$ such that

$$\left|f(\varphi,t)\right|^2 \vee \left|g(\varphi,t)\right|^2 \le K\left(1+\|\varphi\|^2\right)$$

For all $t \ge 0$, and $\varphi \in PC([-\tau,0];R^n)$.

Definition 1 $\psi(t): \psi(t) \in C^1([-\tau,\infty];\mathbb{R}_+)$ *is said to be ψ-type function, if it satisfies the following conditions*:

(1) It is continuous, monotone increasing and differentiable;

(2) $\psi(0) = 1$ and $\psi(\infty) = \infty$;

(3) $\psi_1(t) = \psi'(t)/\psi(t) > 0$.

(4) for any $t, s \ge 0, \psi(t) \le \psi(s)\psi(t-s)$.

Definition 2 *For* $p > 0$, *impulsive stochastic delay differential systems with Markovian switching* (1) *is said to be p-th moment stable with decay* $\psi(t)$ *of order* γ, *if there exist positive constants* γ *and function* $\psi(\cdot)$, *such that*

$$\limsup_{t\to\infty}\frac{\ln E\left|x(t,\xi)\right|^p}{\ln\psi(t)} \le -\gamma. \tag{2}$$

when $p = 2$, we say that it is ψ^γ stable in mean square, when $\psi(t) = \mathrm{e}^t$, we say that it is p-th moment exponential stable, when $\psi(t) = 1+t$, we say that it is *p*-th moment polynomial stable.

Definition 3 *impulsive stochastic delay differential systems with Markovian switching* (1) *is said to be almost surely stable with decay* $\psi(t)$ *of order* γ, *if there exist positive constant* γ *and function* $\psi(\cdot)$, *such that*

$$\limsup_{t\to\infty}\frac{\ln\left|x(t,\xi)\right|}{\ln\psi(t)} \le -\gamma, \quad a.s. \tag{3}$$

when $\psi(t) = e^t$, we say that it is almost surely exponential stable, when $\psi(t) = 1 + t$, we say that it is almost surely polynomial stable.

Let $\mathcal{C}^{1,2}\left(\mathbb{R}^n \times [t_0 - \tau, \infty) \times \Gamma\right)$ denote the family of all nonnegative functions $V(x,t,i)$ on $\mathbb{R}^n \times [t_0 - \tau, \infty) \times \Gamma$ that are continuously once differentiable in t and twice in x. For each $V \in \mathcal{C}^{1,2}$ define an operator $\mathcal{L}V : PC^p_{\mathcal{F}_t}\left([-\tau, 0]; \mathbb{R}^n\right) \times [t_0 - \tau, \infty) \times \Gamma \to \mathbb{R}^+$ for system (1) by

$$\mathcal{L}V(x_t,t,i) = V_t(x,t,i) + V_x(x,t,i) f(x_t,t,i) + \frac{1}{2} trace\left[g^{\mathrm{T}}(x_t,t,i) V_{xx}(x,t,i) g(x_t,t,i)\right] + \sum_{j=1}^{N} \pi_{ij} V(x_t,t,j)$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\right), V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\right)_{n\times n}$$

Lemma 1 (*Burkholder-Davis-Cundy inequality*) *Let* $g \in L^2\left([0,T]; R^{n\times m}\right)$, $0 < p < \infty$, *there exist positive constants* c_p *and* C_p, *such that*

$$c_p E\left[\int_0^T |g(t)|^2 \mathrm{d}t\right]^{\frac{p}{2}} \le E\left[\sup_{0\le t\le T}\left|\int_0^t g(s)\mathrm{d}w(s)\right|^p\right] \le C_p E\left[\int_0^T |g(t)|^2 \mathrm{d}t\right]^{\frac{p}{2}}$$

where

$$c_p = \left(\frac{p}{2}\right)^p, C_p = \left(\frac{32}{p}\right)^{\frac{p}{2}}, \text{ if } p \in (0,2);$$

$$c_p = 1, C_p = 4, \text{ if } p = 2;$$

$$c_p = (2p)^{\frac{p}{2}}, C_p = \left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}, \text{ if } p > 2.$$

Lemma 2 (*Borel-Cantelli's lemma*)

(1) If $\{A_k\} \subset \mathcal{F}$ and $\sum_{k=1}^{\infty} P(A_k) < \infty$, then

$$P\left(\limsup_{t\to\infty} A_k\right) = 0.$$

That is, there exist a set $\Omega_o \in \mathcal{F}$ with $P(\Omega_o) = 1$ and an integer valued random variable k_o such that for every $\omega \in \Omega_o$ we have $\omega \notin A_k$ whenever $k \ge k_o(\omega)$.

(2) If the sequence $\{A_k\} \subset \mathcal{F}$ is independent and $\sum_{k=1}^{\infty} P(A_k) = \infty$, then

$$P\left(\limsup_{t\to\infty} A_k\right) = 1.$$

That is, there exists a set $\Omega_\theta \in \mathcal{F}$ with $P(\Omega_\theta) = 1$, such that for every $\omega \in \Omega_\theta$, there exists a sub-sequence $\{A_{k_i}\}$ such that the ω belongs to every A_{k_i}.

3. Main Results

In this section, we shall establish some criteria on the p-th moment exponential stability and almost exponential stability for system (1) by using the Razumikhin technique and Lyapunov functions.

Theorem 1 *For systems* (1), *let* (*H*) *hold, and* ψ *is a* ψ*-type function, Assume that there exist a function* $V \in \mathcal{C}^{1,2}\left(\mathbb{R}^n \times [t_0 - \tau, \infty); \mathbb{R}_+\right)$, *positive constants* $c_1, c_2, \gamma > 0, \mu \ge \gamma$ *and* $\lambda > 1$ *such that*

(H_1) For all $(x,t,i) \in \mathbb{R}^n \times [-\tau, \infty) \times \Gamma$

$$c_1 |x|^p \le V(x,t,i) \le c_2 |x|^p \tag{4}$$

(H_2) For all $t \in [t_k - 1, t_k), k \in \mathbb{N}$

$$E\left[\max_{i\in\Gamma}\mathcal{L}V(\varphi,t,i)\right]\le-\mu\psi_1(t)E\left[\min_{i\in\Gamma}V(\varphi(0),t,i)\right] \tag{5}$$

For all $t\ge0,\theta\in[-\tau,0]$ and those $\varphi\in PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ satisfying

$$E\left[\min_{i\in\Gamma}V(\varphi(t+\theta),t+\theta,i)\right]\le qE\left[\max_{i\in\Gamma}V(\varphi(0),t,i)\right] \tag{6}$$

where $q\ge\lambda\psi^\gamma(-\theta)$.

(H_3) For all $k\in\mathbb{N}$ and $x\in PC^p_{\mathcal{F}_t}(\omega;\mathbb{R}^n)$

$$EV(x+I_k(x,t_k),t_k,r(t_k))\le\beta_kEV(x(t_k^-),t_k^-,r(t_k^-)) \tag{7}$$

where $0\le\beta_k\le\psi^{-\gamma}(t_{k+1}-t_k)$ and $\lambda\beta_k\ge1$.

Then, for any initial $\xi\in PC^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, there exists a solution $x(t)=x(t,\xi)$ on $[t_0,\infty]$ to system (1). Moreover, the system (1) is *p*-th moment exponentially stable with decay $\psi(t)$ of order γ.

Proof. Fix the initial data $\xi\in PC^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ arbitrarily and write $x(t,\xi)=x(t)$ simply. When μ is replaced by γ, if we can prove that the system (1) is *p*-th moment exponentially stable with decay $\psi(t)$ of order γ for all $\gamma\in(0,\mu)$, then the desired result is obtained. Choose $M>0$ satisfying $0<c_2\psi^\gamma(t_1-t_0)\le M<c_2\lambda\psi^\gamma(-\theta)$, and thus we can have the following fact:

$$0<c_2\|\xi\|^p\le M\|\xi\|^p\psi^{-\gamma}(t_1-t_0).$$

Then it follows from condition (H_1) that

$$EV(x(t),t,r(t))\le c_2\|\xi\|^p\le M\|\xi\|^p\psi^{-\gamma}(t_1-t_0),\quad t\in[t_0-\tau,t_0].$$

In the following, we will use the mathematical induction method to show that

$$EV(x(t),t,r(t))\le M\|\xi\|^p\psi^{-\gamma}(t_k-t_0),\quad t\in[t_{k-1},t_k),k\in\mathbb{N}. \tag{8}$$

In order to do so, we first prove that

$$EV(x(t),t,r(t))\le M\|\xi\|^p\psi^{-\gamma}(t_1-t_0),\quad t\in[t_0,t_1). \tag{9}$$

This can be verified by a contradiction. Hence, suppose that inequality (9) is not true, than there exist some $t\in[t_0,t_1)$ such that $EV(x(t),t,r(t))>M\|\xi\|^p\psi^{-\gamma}(t_1-t_0)$. Set $t^*=\inf\left\{t\in[t_0,t_1):EV(x(t),t,r(t))>M\|\xi\|^p\psi^{-\gamma}(t_1-t_0)\right\}$. By using the continuity of $EV(x(t),t,r(t))$ in the interval $[t_0,t_1)$, then $t^*\in(t_0,t_1)$ and

$$EV(x(t^*),t^*,r(t^*))=M\|\xi\|^p\psi^{-\gamma}(t_1-t_0) \tag{10}$$

$$EV(x(t),t,r(t))<M\|\xi\|^p\psi^{-\gamma}(t_1-t_0),\quad t\in[t_0-\tau,t^*). \tag{11}$$

Define $t^{**}=\sup\left\{t\in[t_0-\tau,t^*]:EV(x(t),t,r(t))\le c_2\|\xi\|^p\right\}$, then $t^{**}\in[t_0,t^*)$ and

$$EV(x(t^{**}),t^{**},r(t^{**}))=c_2\|\xi\|^p \tag{12}$$

$$EV(x(t),t,r(t))>c_2\|\xi\|^p,\quad t\in(t^{**},t^*]. \tag{13}$$

Consequently, for all $t\in[t^{**},t^*]$, we have

$$\begin{aligned}&EV(x(t+\theta),t+\theta,r(t+\theta))\\&\le M\|\xi\|^p\psi^{-\gamma}(t_1-t_0)<c_2\lambda\psi^\gamma(-\theta)\|\xi\|^p\psi^{-\gamma}(t_1-t_0)<c_2\lambda\psi^\gamma(-\theta)\|\xi\|^p\\&=\lambda\psi^\gamma(-\theta)EV(x(t^{**}),t^{**},r(t^{**}))\le qEV(x(t^{**}),t^{**},r(t^{**}))\le qEV(x(t),t,r(t)),\end{aligned}$$

And so

$$E\left[\min_{i\in\Gamma} V\left(\varphi(\theta),t+\theta,i\right)\right] < qE\left[\max_{i\in\Gamma} V\left(\varphi(0),t,i\right)\right].$$

By condition (H_2) we have

$$E\left[\max_{i\in\Gamma} \mathcal{L}V\left(\varphi,t,i\right)\right] \le -\mu\psi_1(t)E\left[\min_{i\in\Gamma} V\left(\varphi(0),t,i\right)\right], \quad t\in\left[t^{**},t^*\right].$$

Consequently,

$$E\mathcal{L}V\left(\varphi,t,i\right) \le -\mu\psi_1(t)EV\left(\varphi(0),t,i\right), \quad t\in\left[t^{**},t^*\right]. \tag{14}$$

Applying the *Itô* formula to $\psi^{\gamma}(t)EV\left(x(t),t,r(t)\right)$ yields

$$\begin{aligned}
&\psi^{\gamma}\left(t^*\right)EV\left(x\left(t^*\right),t^*,r\left(t^*\right)\right)-\psi^{\gamma}\left(t^{**}\right)EV\left(x\left(t^{**}\right),t^{**},r\left(t^{**}\right)\right)\\
&=\int_{t^{**}}^{t^*}E\mathcal{L}\left[\psi^{\gamma}(t)V\left(x(t),t,r(t)\right)\right]\mathrm{d}t\\
&=\int_{t^{**}}^{t^*}\gamma\psi^{\gamma-1}(t)\left(\psi^{\gamma}(t)\right)' EV\left(x(t),t,r(t)\right)+\psi^{\gamma}(t)E\mathcal{L}LV\left(x(t),t,r(t)\right)\\
&=\int_{t^{**}}^{t^*}\psi^{\gamma}(t)\left[\gamma\psi_1(t)EV\left(x(t),t,r(t)\right)+E\mathcal{L}V\left(x(t),t,r(t)\right)\right]\mathrm{d}t.
\end{aligned} \tag{15}$$

By condition (14), we obtain

$$\psi^{\gamma}\left(t^*\right)EV\left(x\left(t^*\right),t^*,r\left(t^*\right)\right)-\psi^{\gamma}\left(t^{**}\right)EV\left(x\left(t^{**}\right),t^{**},r\left(t^{**}\right)\right)<0. \tag{16}$$

On the other hand, a direct computation yields

$$\begin{aligned}
&\psi^{\gamma}\left(t^*\right)EV\left(x\left(t^*\right),t^*,r\left(t^*\right)\right)=\psi^{\gamma}\left(t^*\right)M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_1-t_0\right)\\
&\ge c_2\psi^{\gamma}\left(t_1-t_0\right)\psi^{\gamma}\left(t^{**}\right)\left\|\xi\right\|^p\psi^{-\gamma}\left(t_1-t_0\right)\\
&=c_2\psi^{\gamma}\left(t^{**}\right)\left\|\xi\right\|^p=\psi^{\gamma}\left(t^{**}\right)EV\left(x\left(t^{**}\right),t^{**},r\left(t^{**}\right)\right),
\end{aligned}$$

that is

$$\psi^{\gamma}\left(t^*\right)EV\left(x\left(t^*\right),t^*,r\left(t^*\right)\right)-\psi^{\gamma}\left(t^{**}\right)EV\left(x\left(t^{**}\right),t^{**},r\left(t^{**}\right)\right)>0,$$

which is a contradiction. So inequality (9) holds and (8) is true for $k=1$. Now we assume that (8) is satisfied for $k=1,2,\cdots,m\left(m\ge 1\right)$, *i.e.* for every $t\in\left[t_{k-1},t_k\right), k=1,2,\cdots,m$,

$$EV\left(x(t),t,r(t)\right)\le M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_k-t_0\right). \tag{17}$$

Then, we will prove that (8) holds for $k=m+1$,

$$EV\left(x(t),t,r(t)\right)\le M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_{m+1}-t_0\right). \tag{18}$$

Suppose (18) is not true, *i.e.* there exist some $t\in\left[t_m,t_{m+1}\right)$ such that

$$EV\left(x(t),t,r(t)\right)> M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_{m+1}-t_0\right). \tag{19}$$

Then, it follows from the condition (H_3) and (17) that

$$\begin{aligned}
EV\left(x\left(t_m\right),t_m,r\left(t_m\right)\right)&=EV\left(x+I_m\left(x\left(t_m^-\right)\right),t_m,r\left(t_m\right)\right)\le\beta_m EV\left(x\left(t_m^-\right),t_m^-,r\left(t_m^-\right)\right)\\
&\le\psi^{-\gamma}\left(t_{m+1}-t_m\right)M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_m-t_0\right)\\
&\le M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_{m+1}-t_m+t_m-t_0\right)=M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_{m+1}-t_0\right),
\end{aligned}$$

which implies that the t_m dose not satisfy the inequality (19). And from this, set $t^*=\inf\left\{t\in\left(t_m,t_{m+1}\right):EV\left(x(t),t,r(t)\right)>M\left\|\xi\right\|^p\psi^{-\gamma}\left(t_{m+1}-t_0\right)\right\}$. By the continuity of $EV\left(x(t),t,r(t)\right)$ in the interval (t_m, t_{m+1}), we know that $t^* \in (t_m, t_{m+1})$ and

$$EV\left(x\left(t^*\right), t^*, r\left(t^*\right)\right) = M\|\xi\|^p \psi^{-\gamma}\left(t_{m+1} - t_0\right), \tag{20}$$

$$EV\left(x(t), t, r(t)\right) < M\|\xi\|^p \psi^{-\gamma}\left(t_{m+1} - t_0\right), \quad t \in \left[t_m, t^*\right). \tag{21}$$

Define $t^{**} = \sup\left\{t \in \left[t_m - \tau, t^*\right] : EV\left(x(t), t, r(t)\right) \le \beta_m M\|\xi\|^p \psi^{-\gamma}\left(t_{m+1} - t_0\right)\right\}$, then $t^{**} \in \left[t_m, t^*\right)$ and

$$EV\left(x\left(t^{**}\right), t^{**}, r\left(t^{**}\right)\right) = \beta_m M\|\xi\|^p \psi^{-\gamma}\left(t_m - t_0\right), \tag{22}$$

$$EV\left(x(t), t, r(t)\right) > \beta_m M\|\xi\|^p \psi^{-\gamma}\left(t_m - t_0\right), \quad t \in \left(t^{**}, t^*\right]. \tag{23}$$

Fix any $t \in \left[t^{**}, t^*\right]$, when $t + \theta \ge t_m$ for all $\theta \in [-\tau, 0]$, then (20)-(22) imply that

$$\begin{aligned}
&EV\left(x(t+\theta), t+\theta, r(t+\theta)\right) \\
&\le M\|\xi\|^p \psi^{-\gamma}\left(t_{m+1} - t_0\right) < M\|\xi\|^p \psi^{-\gamma}\left(t + \theta - t_0\right) \\
&\le \lambda\beta_m M\|\xi\|^p \psi^{-\gamma}\left(t + \theta - t_0\right) \le \lambda\beta_m \|\xi\|^p \psi^{\gamma}(-\theta)\psi^{-\gamma}\left(t_m - t_0\right) \\
&\le qEV\left(x\left(t^{**}\right), t^{**}, r\left(t^{**}\right)\right) \le qEV\left(x(t), t, r(t)\right)
\end{aligned}$$

If $t + \theta < t_m$ for some $\theta \in [-\tau, 0)$, we assume that, without loss of generality, $t + \theta \in \left[t_{s-1}, t_s\right)$, for some $s \in \mathbb{N}, s < m$, then from (17) and (20)-(22), we obtain

$$\begin{aligned}
&EV\left(x(t+\theta), t+\theta, r(t+\theta)\right) \\
&\le M\|\xi\|^p \psi^{-\gamma}\left(t_s - t_0\right) < M\|\xi\|^p \psi^{-\gamma}\left(t + \theta - t_0\right) \le M\|\xi\|^p \psi^{-\gamma}\left(t - t_0\right)\psi^{\gamma}(-\theta) \\
&\le M\|\xi\|^p \psi^{-\gamma}\left(t_m - t_0\right)\psi^{\gamma}(\alpha) \le \lambda\beta_m \psi^{\gamma}(-\theta) M\|\xi\|^p \psi^{-\gamma}\left(t_m - t_0\right) \\
&= \lambda\psi^{\gamma}(-\theta) EV\left(x\left(t^{**}\right), t^{**}, r\left(t^{**}\right)\right) \le qEV\left(x\left(t^{**}\right), t^{**}, r\left(t^{**}\right)\right) \le qEV\left(x(t), t, r(t)\right)
\end{aligned} \tag{24}$$

Therefore,

$$E\left[\min_{i\in\Gamma} V\left(\varphi(\theta), t+\theta, i\right)\right] < qE\left[\max_{i\in\Gamma} V\left(\varphi(0), t, i\right)\right], \quad t \in \left[t^{**}, t^*\right], \theta \in [-\tau, 0],$$

by condition (H_2) we have

$$E\left[\max_{i\in\Gamma} \mathcal{L}V(\varphi, t, i)\right] \le -\mu\psi_1(t) E\left[\min_{i\in\Gamma} V\left(\varphi(0), t, i\right)\right], \quad t \in \left[t^{**}, t^*\right], \theta \in [-\tau, 0].$$

Consequently,

$$E\mathcal{L}V(\varphi, t, i) \le -\mu\psi_1(t) EV\left(\varphi(0), t, i\right), \quad t \in \left[t^{**}, t^*\right]. \tag{25}$$

Similar to (15), applying the *Itô* formula to $\psi^{\gamma}(t) EV\left(x(t), t, r(t)\right)$ yields

$$\begin{aligned}
&\psi^{\gamma}\left(t^*\right) EV\left(x\left(t^*\right), t^*, r\left(t^*\right)\right) - \psi^{\gamma}\left(t^{**}\right) EV\left(x\left(t^{**}\right), t^{**}, r\left(t^{**}\right)\right) \\
&= \int_{t^{**}}^{t^*} \psi^{\gamma}(t)\left[\gamma\psi_1(t) EV\left(x(t), t, r(t)\right) + E\mathcal{L}V\left(x(t), t, r(t)\right)\right] \mathrm{d}t.
\end{aligned}$$

By condition (25), we obtain

$$\psi^{\gamma}\left(t^*\right) EV\left(x\left(t^*\right), t^*, r\left(t^*\right)\right) - \psi^{\gamma}\left(t^{**}\right) EV\left(x\left(t^{**}\right), t^{**}, r\left(t^{**}\right)\right) < 0.$$

On the other hand, by (20) and (22), we have

$$\begin{aligned}
&\psi^{\gamma}\left(t^{*}\right)EV\left(x\left(t^{*}\right),t^{*},r\left(t^{*}\right)\right)\\
&=\psi^{\gamma}\left(t^{*}\right)M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(t_{m+1}-t_{0}\right)\geq\psi^{\gamma}\left(t^{*}\right)M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(t_{m+1}-t_{m}\right)\psi^{-\gamma}\left(t_{m}-t_{0}\right)\\
&\geq\psi^{\gamma}\left(t^{*}\right)M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(\alpha\right)\psi^{-\gamma}\left(t_{m}-t_{0}\right)\geq\beta_{m}\psi^{\gamma}\left(t^{**}\right)M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(t_{m}-t_{0}\right)\\
&=\psi^{\gamma}\left(t^{**}\right)EV\left(x\left(t^{**}\right),t^{**},r\left(t^{**}\right)\right),
\end{aligned}$$

that is

$$\psi^{\gamma}\left(t^{*}\right)EV\left(x\left(t^{*}\right),t^{*},r\left(t^{*}\right)\right)-\psi^{\gamma}\left(t^{**}\right)EV\left(x\left(t^{**}\right),t^{**},r\left(t^{**}\right)\right)>0.$$

which is a contradiction. So inequality (18) holds. Therefore, by mathematical induction, we obtain (8) holds for all $k\in\mathbb{N}$. Then from condition (H_1), we have

$$E\left|x(t)\right|^{p}\leq\frac{c_2}{c_1}M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(t_k-t_0\right)\leq\frac{c_2}{c_1}M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(t-t_0\right),\quad t\in\left[t_{k-1},t_k\right),k\in\mathbb{N},$$

which implies

$$E\left|x(t)\right|^{P}\leq\frac{c_2}{c_1}M\left\|\xi\right\|^{p}\psi^{-\gamma}\left(t-t_0\right),\quad t\geq t_0$$

i.e., system (1) is pth moment exponentially stable with decay $\psi(t)$ of order γ. The proof is complete. □

Theorem 2 *For system* (1), *suppose all of the conditions of Theorem 1 are satisfied. Let* $p\geq1$, *assume that there exist constants* $K>0$, *such that for all* $t\geq0$ *and* $\varphi\in PC^{p}_{\mathcal{F}_t}\left(\left[-\tau,0\right];\mathbb{R}^{n}\right)$,

$$E\left|f\left(t,\varphi\right)\right|^{p}+E\left|g\left(t,\varphi\right)\right|^{p}\leq K\sup_{\theta\leq0}\psi^{-\mu}\left(-\theta\right)E\left|\varphi\left(\theta\right)\right|^{p}. \tag{26}$$

Then, for any initial $\xi\in PC^{b}_{\mathcal{F}_0}\left(\left[-\tau,0\right];\mathbb{R}^{n}\right)$ and for any $\gamma\in\left(0,\mu\right)$, there exists a solution $x(t)=x(t,\xi)$ on $\left[t_0,\infty\right)$ to stochastic delay nonlinear system (1). Moreover, the system (1) is almost surely stable with decay $\psi(t)$ of order γ and

$$\limsup_{t\to\infty}\frac{\ln\left|x\left(t,\xi\right)\right|}{\ln\psi\left(t\right)}\leq-\frac{\gamma}{p}.\quad a.s. \tag{27}$$

Proof. Fix the initial data $\xi\in PC^{b}_{\mathcal{F}_0}\left(\left[-\tau,0\right];R^{n}\right)$ arbitrarily and write $x(t,\xi)=x(t)$ simply. We claim that

$$E\left[\sup_{t_{k-1}\leq t\leq t_k}\left|x(t)\right|^{p}\right]\leq H\psi^{\gamma}\left(t_{k-1}\right),\quad k\in N \tag{28}$$

where

$$H=3^{p-1}\left(k_\delta+1\right)\left(M\frac{c_2}{c_1}+\frac{K\left(c_1+Mc_2\right)\left(1+C_p\right)}{c_1}\right)\left\|\xi\right\|^{p}.$$

Choose δ sufficiently small and $0<\delta<1\wedge t_k-t_{k-1}$, for the fixed δ, let $k_\delta=\left[\frac{t_k-t_{k-1}}{\delta}\right]\in N$, where $[x]$ is the maximum integer not more than x. Then for any $t\in\left[t_{k-1},t_k\right)$, there exist positive integer i, $1\leq i\leq k_\delta+1$, such that $t_{k-1}+\left(i-1\right)\delta\leq t\leq t_{k-1}+i\delta$. So, for any $t\in\left[t_{k-1},t_k\right)$, $k\in\mathbb{N}$, we obtain

$$E\left[\sup_{t_{k-1}\leq t\leq t_k}\left|x(t)\right|^{p}\right]\leq\sum_{i=1}^{k_\delta+1}E\left[\sup_{t_{k-1}+(i-1)\delta\leq t\leq t_{k-1}+i\delta}\left|x(t)\right|^{p}\right] \tag{29}$$

For each i when $1 \le i \le k_\delta + 1$, $k \in \mathbb{N}$, we obtain

$$\begin{aligned} & E\left[\sup_{t_{k-1}+(i-1)\delta \le t \le t_{k-1}+i\delta} \left|x(t)\right|^p\right] \\ & \le 3^{p-1} E\left|x\left(t_{k-1}+(i-1)\delta\right)\right|^p + 3^{p-1} E\left|\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} f\left(x_s, s, r(s)\right)\mathrm{d}s\right|^p \\ & \quad + 3^{p-1} E\left[\sup_{t_{k-1}+(i-1)\delta \le t \le t_{k-1}+i\delta} \left|\int_{t_{k-1}+(i-1)\delta}^{t} g\left(x_s, s, r(s)\right)\mathrm{d}B(s)\right|^p\right]. \end{aligned} \tag{30}$$

By Theorem 1, we have

$$E\left|x\left(t_{k-1}+(i-1)\delta\right)\right|^p \le \frac{c_2}{c_1} M \|\xi\|^p \psi^{-\mu}\left(t_{k-1}+(i-1)\delta\right) \tag{31}$$

By Holder inequality, condition (26) and Theorem 1, we derives that

$$\begin{aligned} & E\left|\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} f\left(x_s, s, r(s)\right)\mathrm{d}s\right|^p \le \int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} E\left|f\left(x_s, s, r(s)\right)\right|^p \mathrm{d}s \\ & \le K\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} \sup_{\theta\le 0} \psi^{-\mu}(-\theta) E\left|x(s+\theta)\right|^p \mathrm{d}s \\ & \le K\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} \left[\sup_{\theta\le -s} \psi^{-\mu}(-\theta) E\left|x(s+\theta)\right|^p + \sup_{-s\le\theta\le 0} \psi^{-\mu}(-\theta) E\left|x(s+\theta)\right|^p\right]\mathrm{d}s \\ & \le K\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} \left[\psi^{-\mu}(s)\|\xi\|^p + \sup_{-s\le\theta\le 0} \psi^{-\mu}(-\theta)\psi^{-\mu}(s+\theta)\frac{c_2}{c_1}\|\xi\|^p\right]\mathrm{d}s \\ & \le K\left(1+M\frac{c_2}{c_1}\right)\|\xi\|^p \int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} \psi^{-\mu}(s)\mathrm{d}s \\ & \le \delta K\left(1+M\frac{c_2}{c_1}\right)\|\xi\|^p \psi^{-\mu}\left(t_{k-1}+(i-1)\delta\right) \\ & \le K\left(1+M\frac{c_2}{c_1}\right)\|\xi\|^p \psi^{-\mu}\left(t_{k-1}+(i-1)\delta\right). \end{aligned} \tag{32}$$

Similarly, by the Lemma 1 and (32), we obtain

$$\begin{aligned} & E\left[\sup_{t_{k-1}+(i-1)\delta \le t \le t_{k-1}+i\delta} \left|\int_{t_{k-1}+(i-1)\delta}^{t} \left(x_s, s, r(s)\right)\mathrm{d}B(s)\right|^p\right] \\ & \le C_p E\left[\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} \left|g\left(x_s, s, r(s)\right)\right|^2 \mathrm{d}s\right]^{\frac{p}{2}} \\ & \le C_p E\left[\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta} \left|g\left(x_s, s, r(s)\right)\right|^p \mathrm{d}s\right] \\ & \le C_p K\left(1+M\frac{c_2}{c_1}\right)\|\xi\|^p \psi^{-\mu}\left(t_{k-1}+(i-1)\delta\right), \end{aligned} \tag{33}$$

where C_p is a positive constant dependent on p only.

Substituting (31), (32) and (33) into (30) yields

$$E\left[\sup_{t_{k-1}+(i-1)\delta \le t \le t_{k-1}+i\delta} \left|x(t)\right|^p\right] \le 3^{p-1}\left[M\frac{c_2}{c_1} + \left(1+C_p\right)K\left(1+M\frac{c_2}{c_1}\right)\right]\|\xi\|^p \psi^{-\mu}\left(t_{k-1}+(i-1)\delta\right). \tag{34}$$

Thus, it follows from (29) and (34), we obtain

$$E\left[\sup_{t_{k-1}\le t\le t_k}|x(t)|^p\right]\le 3^{p-1}(k_\delta+1)\left[M\frac{c_2}{c_1}+(1+C_p)K\left(1+M\frac{c_2}{c_1}\right)\right]\|\xi\|^p\psi^{-\mu}(t_{k-1})=H\psi^{-\mu}(t_{k-1}).$$

Using Chebyshev inequality, we have

$$\begin{aligned}P\left\{\sup_{t_{k-1}\le t\le t_k}|x(t)|^p\ge\psi^{-\gamma}(t_k)\right\}&\le\psi^{\gamma}(t_k)E\left[\sup_{t_{k-1}\le t\le t_k}|x(t)|^p\right]\le\psi^{\gamma}(t_k)H\psi^{-\mu}(t_{k-1})\\&\le H\psi^{\gamma}(t_k-t_{k-1})\psi^{-(\mu-\gamma)}(t_{k-1})\end{aligned}$$

Since $\sum_{k=1}^{\infty}\psi^{-(\mu-\gamma)}(t_k-t_{k-1})<\infty$, by Lemma 2, when $t_k\to\infty$, $t_{k-1}\le t\le t_k$, we obtain

$$|x(t)|^p\le\psi^{-\gamma}(t_k)\le\psi^{-\gamma}(t),\quad a.s.$$

That is

$$\limsup_{t\to\infty}\frac{\ln|x(t,\xi)|}{\ln\psi(t)}\le-\frac{\gamma}{p}.\quad a.s.$$

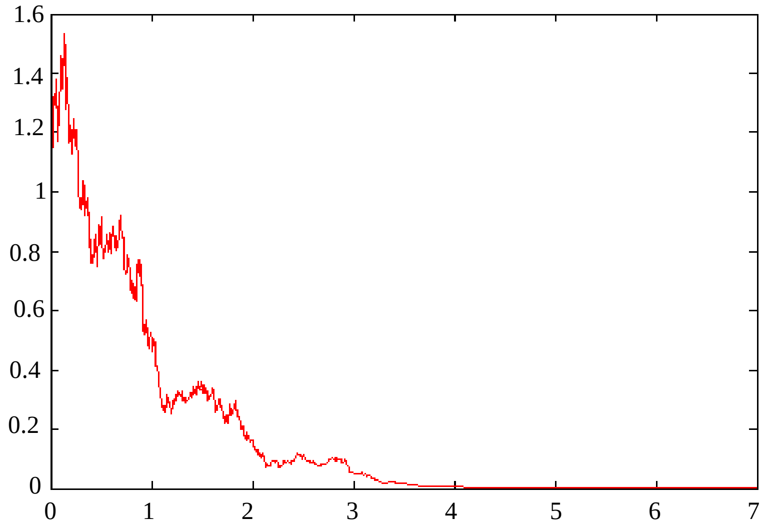

Figure 1. State of the example.

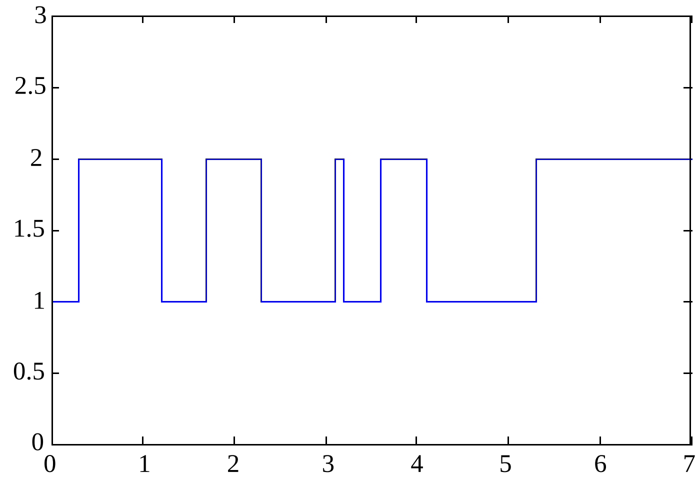

Figure 2. Markovian switching of the example.

Thus, the system (1) is almost surely stable with decay $\psi(t)$ of order γ.

4. Examples

In this section, a numerical example is given to illustrate the effectiveness of the main results established in Section 3 as follows. Consider an impulsive stochastic delay system with Markovian switching as follows

$$\begin{cases} \mathrm{d}x(t) = f(t, x_t, r(t))\mathrm{d}t + g(t, x_t, r(t))\mathrm{d}B(t), & t \neq t_k, t \geq t_0 \\ \Delta x(t_k) = \frac{1}{k^2}\left(x(t_k^-)\right), & k \in \mathbb{N} \end{cases} \tag{35}$$

where $r(t)$ is a right-continuous Markov chain taking values in $\{1,2\}$ with generator

$$\Pi = \begin{bmatrix} -1.5 & 1.5 \\ 1 & -1 \end{bmatrix}$$

And independent of the scalar Brownian motion $B(t)$, $f(x_t, t, 1) = -5x(t) + 0.5x(t-0.2)$, $f(x_t, t, 2) = -8x(t) + 6x(t-0.2)$, $g(x_t, t, 1) = \frac{\sqrt{10}}{4}x(t-0.2)$, $g(x_t, 2) = 0.5x(t-0.2)$, $t_k - t_{k-1} = 0.4, k \in \mathbb{N}$.

Choosing $p = 2, V(t, x, 1) = V(t, x, 2) = x^2, \tau = 0.2, q = 1.9$, $\gamma = 2$, $\max\{t_{k+1} - t_k\} \leq 0.4, k \in \mathbb{N}$, $\psi(t) = \mathrm{e}^{t^2}(1 + 0.5t^2), t > 0$, then $\psi(0) = 1, \psi(\infty) = \infty$, $\psi'(t) = t\mathrm{e}^{t^2}(3 + t^2)$, $\psi_1(t) = \frac{\psi'(t)}{\psi(t)} = 2 + \frac{2}{2 + t^2}, 2 \leq \psi_1(t) \leq 3$, then we have

$$\begin{aligned} E\mathcal{L}V_1(x_t, t, r(t)) &= -10Ex|t|^2 + Ex(t)x(t-0.2) + 1.25E|x(t-0.2)|^2 \\ &\leq -10E|x(t)|^2 + 0.5E|x(t)|^2 + 0.5E|x(t-0.2)|^2 + 1.25E|x(t-0.2)|^2 \\ &\leq -10E|x(t)|^2 + 0.5E|x(t)|^2 + 1.75qE|x(t)|^2 \\ &\leq -9E|x(t)|^2 + 3E|x(t)|^2 \leq -2\psi_1(t) \end{aligned}$$

and

$$\begin{aligned} E\mathcal{L}V_1((x_t, t, r(t)) &= -16Ex|t|^2 + 3Ex(t)x(t-0.2) + 0.5E|x(t-0.2)|^2 \\ &\leq -16E|x(t)|^2 + 3E|x(t)|^2 + 3E|x(t-0.2)|^2 + 3.5E|x(t-0.2)|^2 \\ &\leq -16E|x(t)|^2 + 3E|x(t)|^2 + 3.5qE|x(t)|^2 \\ &\leq -16E|x(t)|^2 + 10E|x(t)|^2 \leq -2\psi_1(t) \end{aligned}$$

By Theorem 1, we know that $E\left[\max_{i\in\Gamma} \mathcal{L}V(\varphi, t, i)\right] \leq -\gamma\psi_1(t)E\left[\min_{i\in\Gamma} V(\varphi(0), t, i)\right]$, which means that the conditions of Theorem 1 are satisfied. So the impulsive stochastic delay system with Markovian switching is p-th moment stable with decay $\mathrm{e}^{t^2}(1 + 0.5t^2)$ of order 2. The simulation result of system (35) is shown in **Figure 1**, and the Markovian switching of system (35) is described in **Figure 2**.

5. Conclusion

In this paper, p-th moment and almost surely stability on a general decay have been investigated for a class of impulsive stochastic delay systems with Markovian switching. Some sufficient conditions have been derived to check the stability criteria by using the Lyapunov-Razumikhin methods. A numerical example is provided to verify the effectiveness of the main results.

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Forward (Δ) and Backward (∇) Difference Operators Basic Sets of Polynomials in $\mathbb{C}^n$ and Their Effectiveness in Reinhardt and Hyperelliptic Domains

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Abstract

We generate, from a given basic set of polynomials in several complex variables $\{P_m(z)\}_{m\geq 0}$, new basic sets of polynomials $\{\tilde{P}_m(z)\}_{m\geq 0}$ and $\{\tilde{Q}_m(z)\}_{m\geq 0}$ generated by the application of the ∇ and Δ operators to the set $\{P_m(z)\}_{m\geq 0}$. All relevant properties relating to the effectiveness in Reinhardt and hyperelliptic domains of these new sets are properly deduced. The case of classical orthogonal polynomials is investigated in details and the results are given in a table. Notations are also provided at the end of a table.

Keywords

Effectiveness, Cannon Condition, Cannon Sum, Cannon Function, Reinhardt Domain, Hyperelliptic Domain

1. Introduction

Recently, there has been an upsurge of interest in the investigations of the basic sets of polynomials [1]-[27]. The inspiration has been the need to understand the common properties satisfied by these polynomials, crucial to gaining insights into the theory of polynomials. For instance, in numerical analysis, the knowledge of basic sets of polynomials gives information about the region of convergence of the series of these polynomials in a given domain. Namely, for a particular differential equation admitting a polynomial solution, one can deduce the range

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of convergence of the polynomials set. This is an advantage in numerical analysis which can be exploited to reduce the computational time. Besides, if the basic set of polynomials satisfies the Cannon condition, then their fast convergence is guaranteed. The problem of derived and integrated sets of basic sets of polynomials in several variables has been recently treated by A. El-Sayed Ahmed and Kishka [1]. In their work, complex variables in complete Reinhardt domains and hyperelliptical regions were considered for effectiveness of the basic set. Also, recently the problem of effectiveness of the difference sets of one and several variables in disc D(R) and polydisc $\prod_{i=1}^{m}(R_i)$ has been treated by A. Anjorin and M.N Hounkonnou [27].

In this paper, we investigate the effectiveness, in Reinhardt and hyperelliptic domains, of the set of polynomials generated by the forward (Δ) and backward (∇) difference operators on basic sets. These operators are very important as they involve the discrete scheme used in numerical analysis. Furthermore, their composition operators form the most of second order difference equations of Mathematical Physics, the solutions of which are orthogonal polynomials [25] [26].

Let us first examine here some basic definitions and properties of basic sets, useful in the sequel.

Definition 1.1 *Let* $z=(z_1,z_2,\cdots,z_n)$ *be an element of the space of several complex variables* $\mathbb{C}^n$*. The hyperelliptic region of radii* $r_s>0, s\in I=\{1,2,\cdots,n\}$ *), is denoted by* $E_{[r]}$ *and its closure by* $\bar{E}_{[r]}$ *where*

$$E_{[r]}=\{w\in\mathbb{C}:|w|<1\},\ \bar{E}_{[r]}=\{w\in\mathbb{C}:|w|\le 1\}$$

And

$$w=\{w_1,w_2,\cdots,w_n\},\ w_s=\frac{z_s}{r_s};s\in I.$$

Definition 1.2 *An open complete Reinhardt domain of radii* $\rho_s>0, s\in I$ *is denoted by* $\Gamma_{(\rho)}$ *and its closure by* $\bar{\Gamma}_{(\rho)}$*, where*

$$\Gamma_{(\rho)}=\Gamma_{(\rho_1,\rho_2,\cdots,\rho_n)}=\{z\in\mathbb{C}^n:|z_s|<\rho_s:s\in I\},$$
$$\bar{\Gamma}_{(\rho)}=\bar{\Gamma}_{(\rho_1,\rho_2,\cdots,\rho_n)}=\{z\in\mathbb{C}^n:|z_s|\le\rho_s:s\in I\}.$$

The unspecified domains $D(\bar{\Gamma}_{(\rho)})$ and $D(\bar{E}_{(r)})$ are considered for both the Reinhardt and hyperelliptic domains. These domains are of radii $\rho_s^\star>\rho_s, r_s^\star>r_s, s\in I$. Making a contraction of this domain, we get the domain $D((\rho)^+)=D((\rho_1^+,\rho_2^+,\cdots,\rho_n^+))$ where ρ_s^+ stands for the right-limits of $\rho_s^\star, s\in I$:

$$D(\bar{\Gamma}_{(\rho)})\equiv D((\rho)^+)=D((\rho_1^+,\rho_2^+,\cdots,\rho_n^+))=\{z\in\mathbb{C}^n:|z_s|\le\rho_s^*:s\in I\}.$$

Thus, the function $F(z)$ of the complex variables $z_s, s\in I$, which is regular in $\bar{E}_{(r)}$ can be represented by the power series

$$F(z)=\sum_{m=0}^{\infty}a_m z^m=\sum_{(m_1,m_2,\cdots,m_n)}^{\infty}a_{(m_1,m_2,\cdots,m_n)}\left(z_1^{m_1},z_2^{m_2},\cdots,z_n^{m_n}\right),\tag{1}$$

where $m=(m_1,m_2,\cdots,m_n)$ represents the mutli indicies of non-negative integers for the function F(z). We have [1]

$$M(F(z),[r])=M(F(z):(r_1,r_2,\cdots,r_n))=\max_{\bar{E}_{(r)}}|F(z)|\tag{2}$$

where $r=(r_1,r_2,\cdots,r_n)$ is the radius of the considered domain. Then for hyperelliptic domains $\bar{E}_{(\Gamma)}$ [1]

$$\sigma_m=\inf_{|t|=1}\frac{1}{t^m}\frac{\{\langle m\rangle\}^{\langle m\rangle/2}}{\prod_{s=1}^{n}m_s^{m_s/2}},$$

t being the radius of convergence in the domain, $\langle m\rangle=m_1+m_2+\cdots+m_n$ assuming $1\le\sigma_m\le\left(\sqrt{n}\right)^{\langle m\rangle}$ and $m_s^{m_s/2}=1$, whenever $m_s=0;s\in I$. Since $\omega_s=\frac{z_s}{r_s};s\in I$, we have (1)

$$\overline{\lim}_{\langle m\rangle\to\infty}\left(\frac{|a_m|}{\sigma_m\prod_{s=1}^{n}(r_s)^{\langle m\rangle-m_s}}\right)^{\frac{1}{\langle m\rangle}}=\overline{\lim}_{\langle m\rangle\to\infty}\left\{\frac{|a_m|^{\frac{1}{\langle m\rangle}}}{\sigma_m^{\frac{1}{\langle m\rangle}}\left(\prod_{s=1}^{n}(r_s)\right)^{\frac{\langle m\rangle-m_s}{\langle m\rangle}}}\right\}\le\frac{1}{\prod_{s=1}^{n}r_s}$$

where also, using the above function $F(z)$ of the complex variables $z_s, s\in I$, which is regular in $\overline{\Gamma}_{(\rho)}$ and can be represented by the power series above (1), then we obtain

$$|a_m|\le\frac{M\left(F(z),(\rho'_m)\right)}{\rho'_m}, m_s\ge 0; s\in I,$$

$\rho'_m\in\left(\rho_{m_1},\rho_{m_2},\cdots,\rho_{m_n}\right)$ and

$$M\left(F(z),(\rho')\right)=\max_{\overline{\Gamma}_{(\rho')}}|F(z)|. \tag{3}$$

Hence, we have for the series $F(z)$

$$\sum_{m=0}^{\infty}a_m z^m=\sum_{m=0}^{\infty}a_m\left(\sum_{k=0}^{m}\pi_{m,k}P_k(z)\right)\equiv\sum_{m=0}^{\infty}a_m\left(\sum_{k=0}^{m}\pi_{m,k}z^k\right),$$

$k=(k_1,k_2,\cdots,k_n), m=(m_1,m_2,\cdots,m_n), \pi_{m,k}=\begin{pmatrix}m\\k\end{pmatrix}, \max_{\overline{\Gamma}_{(\rho')}}|F(z)|=M\left(F(z),(\rho')\right)$. Since ρ'_s can be taken arbitrary near to $\rho_s, s\in I$, we conclude that

$$\overline{\lim}_{\langle m\rangle\to\infty}\left\{|a_m|\prod_{s=1}^{n}\rho_s^{(m_s-\langle m\rangle)}\right\}^{\frac{1}{\langle m\rangle}}\le\frac{1}{\prod_{s=1}^{n}\rho_s},$$

With $\overline{\lim}=\limsup$ and $(\langle m\rangle)\gg m_s$.

Definition 1.3 *A set of polynomials* $\{P_m(z)\}_{m\ge0}=\{P_0(z),P_1(z),\cdots\}_{m\ge0}$ *is said to be basic when every polynomial in the complex variables* $z_s; s\in I$ *can be uniquely expressed as a finite linear combination of the elements of the basic set* $\{P_m(z)\}_{m\ge0}$.

Thus, according to [4], the set $\{P_m(z)\}_{m\ge0}$ will be basic if and only if there exists a unique row-finite-matrix $\overline{P}$ such that $\overline{P}P=P\overline{P}=\mathbb{1}$, where $P=\left(\overline{P}_{m,h}\right)$ is a matrix of coefficients of the set $\{P_m(z)\}_{m\ge0}$; $h=(h_1,h_2,\cdots,h_n)$ are multi indices of nonegative integers, $\overline{P}$ is the matrix of operators deduced from the associated set of the set $\{P_m(z)\}_{m\ge0}$ and $\mathbb{1}$ is the infinite unit matrix of the basic set $\{P_m(z)\}_{m\ge0}$, the inverse of which is $\{\overline{P}_m(z)\}_{m\ge0}$. We have

$$P_m(z)=\sum_{h=0}^{m}P_{m,h}z^h z^m=\sum_{h=0}^{m}\overline{P}_{m,h}P_h(z)=\sum_{h=0}^{m}P_{m,h}\overline{P}_m(z)\overline{P}_m(z)=\sum_{h=0}^{m}\overline{P}_{m,h}z^h. \tag{4}$$

Thus, for the function $F(z)$ given in (1), we get $F(z)=\sum_{m=0}^{\infty}\pi_{m,h}P_m(z)$ where

$\pi_{m,h}=\sum_{h=0}^{m}\overline{P}_{m,h}a_h=\sum_{h=0}^{m}\overline{P}_{m,h}\dfrac{f^{(h)}(0)}{h_s!}\ne\pi_{m,k}=\begin{pmatrix}m\\k\end{pmatrix}$, $h!=h(h-1)(h-2)\cdots3\cdot2\cdot1$. The series $\sum_{m=0}^{\infty}\pi_{m,h}P_h(z)$ is an associated basic series of F(z). Let $N_m=N_{m_1},N_{m_2},\cdots,N_{m_n}$ be the number of non zero coefficients $\overline{P}_{m,h}$ in the representation (4).

Definition 1.4 *A basic set satisfying the condition*

$$\overline{\lim}_{\langle m\rangle\to\infty}N_m^{\frac{1}{\langle m\rangle}}=1 \tag{5}$$

Is called a Cannon basic set. If

$$\overline{\lim}_{\langle m\rangle\to\infty}N_m^{\frac{1}{\langle m\rangle}}=a>1,$$

Then the set is called a general basic set.

Now, let $D_m = D_{m_1,m_2,m_3,\cdots,m_n}$ be the degree of polynomials of the highest degree in the representation (4). That is to say $D_h = D_{h_1,h_2,h_3,\cdots,h_n}$ is the degree of the polynomial $P_h(z)$; the $D_n \le D_m \; \forall n_s \le m_s : s \in I$ and since the element of basic set are linearly independent [6], then $N_m \le 1+2+\cdots+(D_m+1) < \lambda D_m$, where λ is a constant. Therefore the condition (5) for a basic set to be a Cannon set implies the following condition [6]

$$\overline{\lim}_{\langle m\rangle\to\infty} D_m^{\frac{1}{\langle m\rangle}} = 1. \tag{6}$$

For any function $F(z)$ of several complex variables there is formally an associated basic series $\sum_{h=0}^{\infty} P_h(z)$. When the associated basic series converges uniformly to $F(z)$ in some domain, in other words as in classical terminology of Whittaker (see [5]) the basic set of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which they are represented. To study the convergence property of such basic sets of polynomials in complete Reinhardt domains and in hyperelliptic regions, we consider the following notations for Cannon sum

$$\mu\left(P_m(z),(\rho)\right) = \prod_{s=1}^{n} \rho_s^{\langle m\rangle} \sum_{h=0}^{m} \left|\bar{P}_{m,h}\right| M\left(P_h(z),(\rho)\right) \tag{7}$$

For Reinhardt domains [24],

$$\sigma_m \prod_{s=1}^{n} (r_s)^{\langle m\rangle} \sum_{h=0}^{m} \left|\bar{P}_{m,h}\right| M\left(P_h(z),(r)\right) = \Omega\left(P_m(z),(r)\right) \tag{8}$$

For hyperelliptic regions [1].

2. Basic Sets of Polynomials in $\mathbb{C}^n$ Generated by ∇ and Δ Operators

Now, we define the forward difference operator Δ acting on the monomial z^m such that

$$J(\Delta) = \Delta^n z^m$$

$$\Delta^n \equiv \left(E + (-\mathbb{1})\right)^n = \sum_{k=0}^{n} \binom{n}{k} E^{n-k} (-\mathbb{1})^k \text{ by binomial expansion.}$$

where E is the shift operator and $\mathbb{1}$-the identity operator. Then

$$\Delta^n P_m(z) = \sum_{k=0}^{n} \binom{n}{k} (-\mathbb{1})^k E^{n-k} P_m(z) = \sum_{k=0}^{n} \binom{n}{k} (-\mathbb{1})^k P_m(z+n-k).$$

So, considering the monomial z^m

$$\Delta^n z^m = \sum_{k=0}^{n} \binom{n}{k} (-\mathbb{1})^k (z+n-k)^m = \sum_{k=0}^{n} \binom{n}{k} (-\mathbb{1})^k (z+\alpha)^m, \quad \alpha = n-k.$$

$$(z+\alpha)^m = \sum_{j=0}^{m} \binom{m}{j} z^{m-j} \alpha^j.$$

Hence

$$\Delta^n z^m = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} (-\mathbb{1})^k \alpha^j z^{m-j}.$$

Since $\alpha = n-k$,

$$\alpha^j = (n-k)^j = \sum_{t=0}^{j} \binom{j}{t} n^{j-t} (-k)^t = \sum_{k=0}^{j} \binom{j}{t} n^{j-t} (-1)^t (k)^t.$$

Hence

$$\Delta^n z^m = \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j}\sum_{s=0}^{m-j}\binom{n}{k}\binom{m}{j}\binom{j}{t}\times\zeta(-1\!\!1)^{t+k} n^{j-t}(k)^t z^s,$$

where $\zeta=\binom{m-j}{s}$ and $z^{m-j}=\sum_{s=0}^{m-j}\zeta z^s$ by definition. Similarly, we define the backward difference operator ∇ acting on the monomial z^m such that

$$J(\nabla)=\nabla^n z^m, \nabla = 1\!\!1 - E^{-1}. \quad (9)$$

Equivalently, in terms of lag operator L defined as $LF(z)=f(z-1)$, we get $\nabla=1\!\!1-L$. Remark that the advantage which comes from defining polynomials in the lag operator stems from the fact that they are isomorphic to the set of ordinary algebraic polynomials. Thus, we can rely upon what we know about ordinary polynomials to treat problems concerning lag-operator polynomials. So,

$$\nabla^n = \left(1\!\!1-E^{-1}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(-E^{-1}\right)^k = \sum_{k=0}^{n}\binom{n}{k}(-1\!\!1)^k E^{-k}. \quad (10)$$

The Cannon functions for the basic sets of polynomils in complete Reinhardt domain and in hyperelliptical regions [1], are defined as follows, respectively:

$$\mu(P(z),(\rho)) = \overline{\lim}_{\langle m\rangle\to\infty}\left\{\mu(P_m(z),(\rho))\right\}^{\frac{1}{\langle m\rangle}}$$

$$\Omega(P(z),(r)) = \overline{\lim}_{\langle m\rangle\to\infty}\left\{\Omega(P_m(z),(r))\right\}^{\frac{1}{\langle m\rangle}}.$$

Concerning the effectiveness of the basic set $\{P_m(z)\}_{m\geq 0}$ in complete Reinhardt domain we have the following results:

Theorem 2.1 *A necessary and sufficient condition* [7] *for a Cannon set* $\{P_m(z)\}_{m\geq 0}$ *to be*:

1. effective in $\bar{\Gamma}_{(\rho)}$ is that $\mu(P(z),(\rho))=\prod_{s=1}^{n}\rho_s$;

2. effective in $D(\bar{\Gamma}_{(\rho)})$ is that $\mu(P(z),(\rho^+))=\prod_{s=1}^{n}\rho_s$.

Theorem 2.2 *The necessary and sufficient condition for the Cannon basic set* $\{P_m(z)\}_{m\geq 0}$ *of polynomials of several complex variables to be effective* [1] *in the closed hyperelliptic* $\bar{E}_{(r)}$ *is that* $\Omega(P(z),(r))=\prod_{s=1}^{n}r_s$ *where* $r=(r_1,r_2,\cdots,r_n)$.

The Cannon basic set $\{P_m(z)\}_{m\geq 0}$ of polynomials of several complex variables will be effective in $D(\bar{E}_{(r)})$ if and only if $\Omega(P(z),(r^+))=\prod_{s=1}^{n}r_s$. See also [1]. We also get for a given polynomial set $\{Q_m(z)\}_{m\geq 0}$:

$$\nabla^n Q_m(z)=\sum_{k=0}^{n}\binom{n}{k}(-1\!\!1)^k E^{-k}Q_m(z)\nabla^n Q_m(z)=\sum_{k=0}^{n}\binom{n}{k}(-1\!\!1)^k Q_m(z-k).$$

So, considering the monomial z^m,

$$\nabla^n z^m=\sum_{k=0}^{n}\binom{n}{k}(-1\!\!1)^k (z-k)^m=\sum_{s=0}^{m-j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\times\zeta(-1\!\!1)^{j+k}k^j z^s.$$

Let's prove the following statement:

Theorem 2.3 *The set of polynomials* $\{\tilde{P}_m(z)\}_{m\geq 0}$ *and* $\{\tilde{Q}_m(z)\}_{m\geq 0}$

$$\{P_m(J(\Delta)(z))\}_{m\geq 0}=\{\tilde{P}_m(z)\}_{m\geq 0}$$

$$\{Q_m(J(\nabla)(z))\}_{m\geq 0}=\{\tilde{Q}_m(z)\}_{m\geq 0}$$

Are basic.

Proof: To prove the first part of this theorem, it is sufficient to to show that the initial sets of polynomials

$\{P_m(z)\}_{m\geq 0}$ and $\{Q_m(z)\}_{m\geq 0}$, from which $\{\tilde{P}_m(z)\}_{m\geq 0}$ and $\{\tilde{Q}_m(z)\}_{m\geq 0}$ are generated, are linearly independent. Suppose there exists a linear relation of the form

$$\sum_{i=0}^{m} c_i P_{m,i}(z) \equiv 0, c_i \neq 0 \tag{11}$$

For at least one i, $i \in I$. Then

$$\left(P_m\left(J(\Delta)\right)\right)\left(\sum_{i=0}^{m} c_i P_{m,i}(z)\right) \equiv 0.$$

Hence, it follows that $\sum_{i=0}^{m} c_i P_{m,i}(z) \equiv 0$. This means that $\{P_m(z)\}_{m\geq 0}$ would not be linearly independent. Then the set would not be basic. Consequently (11) is impossible. Since $1, z^1, z^2, \cdots, z^n$ are polynomials, each of them can be represented in the form $z^{\langle m\rangle} = \sum \pi_{\langle m\rangle,\langle k\rangle} P_k(z)$. Hence, we write

$$\begin{aligned}
&m_1 = 0, \quad 1 = \sum_k \pi_{0,k} P_k(z)\\
&m_2 = 1, \quad z = \sum_k \pi_{1,k} P_k(z)\\
&m_3 = 2, \quad z = \sum_k \pi_{2,k} P_k(z)\\
&\qquad\vdots\\
&m_n = n, \quad z^n = \sum_k \pi_{n,k} P_k(z)\\
&\text{with } z = (z_1, z_2, \cdots, z_n), k = (k_1, k_2, \cdots, k_n).
\end{aligned}$$

In general, given any polynomial $P_m(z) = \sum_{i=0}^{m} c_i z^i$ and using

$$\begin{aligned}
P_m(z) &= \sum_{i=0}^{m} c_i \left(\sum_{j=0}^{i} \pi_{i,j} P_j(z)\right)\\
&= \sum_{i=0}^{m} c_i \left(\pi_{i,0} P_0(z) + \cdots\right)\\
&= \sum_{i=0} c_i \pi_{i,0} P_0(z) + c_i \pi_{i,1} P_1(z) + \cdots\\
&= \sum_{m_n = m_1}^{m} \tilde{c}_{m_n} P_{m_n}(z); \quad m = (m_1, m_2, \cdots, m_n); z = (z_1, z_2, \cdots, z_n).
\end{aligned}$$

Hence the representation is unique. So, the set $\{\tilde{P}_m(z)\}_{m\geq 0}$ is a basic set. Changing Δ to ∇ leads to the same conclusion. We obtain the following result.

Theorem 2.4 *The Cannon set* $\{\tilde{P}_m(z)\}_{m\geq 0}$ *of polynomials in several complex variables* $z_s; s \in I$ *is Effective*
in the closed complete Reinhardt domain $\Gamma_{(\rho)}$ *and in the closed Reinhardt region* $D\left(\bar{\Gamma}_{(\rho)}\right)$.

Proof: In a complete Reinhardt domain for the forward difference operator Δ, the Cannon sum of the monomial z^m is given by

$$\mu\left(\tilde{P}_m(z), (\rho)\right) = \frac{\prod_{s=1}^{n} (\rho_s)^{\langle m\rangle - m_s} \sum_h \left|\bar{P}_{m,h}\right| M\left(\tilde{P}_{m,h}(z), (\rho_s)\right)}{\sum_{s=0}^{m-j} \sum_{k=0}^{n} \sum_{j=0}^{m} \sum_{t=0}^{j} \binom{n}{k} \binom{m}{j} \binom{j}{t} \varsigma (-1)^{t+k} n^{j-t} k^t}.$$

Then

$$M\left(\tilde{P}_{m,h}(z), (\rho)\right) = \max_{\Gamma_{(\rho)}} \left|\tilde{P}(z)\right| \leq N_m M\left(P_{m,h}(z), (\rho)\right)$$

where N_m is a constant. Therefore,

$$\mu\left(\tilde{P}_{m,h}(z),(\rho)\right) \le \frac{N_m\left(\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle - m_s}\sum_h \left|\bar{P}_{m,h}\right| M\left(P_{m,h}(z),(\rho)\right)\right)}{\sum_{s=0}^{m-j}\sum_{t=0}^{j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta(-1)^{t+k} n^{j-t}k^t}$$

which implies that

$$\mu\left(\tilde{P}_m(z),(\rho)\right) \le \frac{N_m\left(\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle - m_s}\mu\left(P_m(z),(\rho)\right)\right)}{\sum_{s}^{m-j}\sum_{t=0}^{j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta(-1)^{t+k} n^{j-t}k^t}.$$

Then the Cannon function

$$\mu\left(\tilde{P}(z),(\rho)\right) \le \overline{\lim}_{\langle m\rangle\to\infty}\mu\left(P_m(z),(\rho)\right)^{\frac{1}{\langle m\rangle}}$$

$$\mu\left(\tilde{P}(z),(\rho)\right) \le \prod_{s=1}^{n}\rho_s; s \in I.$$

But $\mu\left(\tilde{P}(z),(\rho)\right) \ge 0$. Hence

$$\mu\left(\tilde{P}(z),(\rho)\right) = \overline{\lim}_{\langle m\rangle\to\infty}\mu\left(P_m(z),(\rho)\right)^{\frac{1}{\langle m\rangle}}$$

$$\mu\left(\tilde{P}(z),(\rho)\right) = \prod_{s=1}^{n}\rho_s; s \in I.$$

Similarly, for the backward difference operator ∇, the Cannon sum

$$\mu\left(\tilde{Q}_m(z),(\rho)\right) = \frac{\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle - m_s}\sum_j \left|\bar{Q}_{m,j}\right| M\left(Q_{m,j}(z),(\rho)\right)}{\sum_{s}^{m-j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\zeta(-1)^{k+j}k^j}$$

$$\mu\left(\tilde{Q}_m(z),(\rho)\right) = \frac{\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle - m_s}\sum_j \left|\bar{Q}_{m,j}\right| M\left(\tilde{Q}_{m,j}(z),(\rho)\right)}{\sum_{s}^{m-j}\sum_{k=0}^{n}\sum_{j=0}^{m}(-1)^j\binom{n}{k}\binom{m}{j}\zeta(-1)^k k^j}.$$

Then

$$\mu\left(\tilde{Q}_m(z),(\rho)\right) \le \frac{\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle - m_s}\left(K_m\mu\left(Q_m(z),(\rho)\right)\right)}{\sum_{s=0}^{m-j}\sum_{k=0}^{n}\sum_{j=0}^{m}(-1)^{j+k}\binom{n}{k}\binom{m}{j}\zeta k^j}.$$

where $K_m = \left|\nabla^m\right| = \text{constant}$ as ∇ is bounded for the Reinhardt domain is complete. Thus,

$$\mu\left(\tilde{Q}(z),(\rho)\right) = \overline{\lim}_{\langle m\rangle\to\infty}\mu\left(\tilde{Q}_m(z),(\rho)\right)^{\frac{1}{\langle m\rangle}}$$
$$\le \overline{\lim}_{\langle m\rangle\to\infty}\mu\left(Q_m(z),(\rho)\right)^{\frac{1}{\langle m\rangle}} \le \prod_{s=1}^{n}\rho_s.$$

But

$$\mu\left(\tilde{Q}(z),[\rho]\right) \ge \prod_{s=1}^{n}\rho_s.$$

Hence, we deduce that $\mu\left(Q_m(z),(\rho)\right)=\prod_{s=1}^{n}\rho_s$.

Theorem 2.5 *If the Cannon basic set* $\{P_m(z)\}_{\geq 0}$ *(resp.* $\{Q_m(z)\}_{\geq 0}$*) of polynomials of the several complex variables* $z_s, s\in I$ *for which the condition* (5) *is satisfied, is effective in* $\bar{E}_{(r)}$*, then the* (Δ) *and* (∇)*-set* $\{\tilde{P}_m(z)\}_{m\geq 0}$ *(resp.* $\{\tilde{Q}_m(z)\}_{m\geq 0}$*) of polynomials associated with the set* $\{P_m(z)\}_{m\geq 0}$ *(resp.* $\{Q_m(z)\}_{m\geq 0}$*) will be effective in* $\bar{E}_{(r)}$.

The Cannon sum $\tilde{\Omega}\left(\tilde{P}(z),(r)\right)$ of the forward difference operator Δ of the set $\{\tilde{P}_m(z)\}_{\geq 0}$ in $\bar{E}_{(r)}$ will have the form

$$\tilde{\Omega}\left(\tilde{P}(z),(r)\right)=\frac{\sigma_m\prod_{s=1}^{n}(r_s)^{\langle m\rangle-m_s}\sum_{h=0}^{m}\left|\bar{P}_{m,h}\right|M\left(\tilde{P}_{m,h}(z),(r)\right)}{\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta^*(-1)^{k+t}n^{j-k}k^t}$$

where $\zeta^*=\sum_{s=0}^{m-j}\times\zeta$ and

$$M\left(\tilde{P}_{m,h}(z),(r)\right)=\max_{E_{(r)}}\left|\tilde{P}_h(z)\right|\leq\sum\left|\bar{P}_{m,h}\right|\frac{\prod_{s=1}^{n}r^{h_s}}{\sigma^h}$$

$$\leq\left|\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j}(-1)^{k+t}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta^*n^{j-t}k^t\right|\times\frac{\sigma^hM\left(P_{m,h}(z),(r)\right)\prod_{s=1}^{n}r^{h_s}}{\prod_{s=1}^{n}r^{h_s}\sigma^h}$$

$$\leq K_1N_mD_m^2M\left(P_{m,h}(z),(r)\right)\leq K_1D_m^{m+2}M\left(P_{m,h}(z),(r)\right)$$

where K_1 is a constant. Then

$$\tilde{\Omega}\left(\tilde{P}_m[z],[r]\right)\leq\frac{K_1\sigma_mD_m^{n+2}\prod_{s=1}^{n}[r_s]^{\langle m\rangle-m_s}\sum_{h=0}^{m}\left|\bar{P}_{m,h}\right|}{\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta^*(-1)^{k+t}n^{j-t}k^t}$$

$$=K_2\prod_{s=1}^{n}[r_s]^{\langle m\rangle-m_s}\sum_{h=0}^{m}\left|\bar{P}_{m,h}\right|$$

where

$$K_2=\frac{K_1\sigma_mD_m^{n+2}}{\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j}\binom{n}{k}\binom{m}{j}\binom{j}{t}\times\zeta^*(-1)^{t+k}n^{j-t}}$$

So, by similar argument as in the case of Reinhardt domain we obtain

$$\tilde{\Omega}\left(\tilde{P}(z),(r)\right)=\overline{\lim}_{\langle m\rangle\to\infty}\left(\tilde{\Omega}\left(\tilde{P}_m(z),(r)\right)\right)^{\frac{1}{\langle m\rangle}}$$

$$=\overline{\lim}_{\langle m\rangle\to\infty}\left(K_2\prod_{s=1}^{n}(r_s)^{\langle m\rangle-m_s}\alpha\right)^{\frac{1}{\langle m\rangle}}=\prod_{s=1}^{n}r_s,$$

where $\alpha=\sum_h\left|\bar{P}_{m,h}\right|$, since the Cannon function is such that [1] $\tilde{\Omega}\left(\tilde{P}(z),(r)\right)\geq\prod_{s=1}^{n}r_s$. Similarly, for the backward difference operator

$$\tilde{\Omega}\left(\tilde{Q}_m(z),(r)\right)=\frac{\sigma_m\prod_{s=1}^{n}r_s^{\langle m\rangle-m_s}\sum_{h=0}^{m}\left|\bar{Q}_{m,h}\right|M\left(\tilde{Q}_{m,h}(z),(r)\right)}{\sum_{s=0}^{m-j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\zeta(-1)^{j+k}k^j}$$

Such that the Cannon function writes as

$$\tilde{\Omega}\left(\tilde{Q}(z),(r)\right) \le \overline{\lim}_{\langle m\rangle\to\infty}\left(\tilde{\Omega}\left(\tilde{Q}_m(z),(r)\right)\right)^{\frac{1}{\langle m\rangle}} = \prod_{s=1}^{n} r_s .$$

But

$$\tilde{\Omega}\left(\tilde{Q}(z),(r)\right) \ge \prod_{s=1}^{n} r_s$$

Since the Cannon function is non-negative. Hence

$$\tilde{\Omega}\left(\tilde{Q}(z),(r)\right) = \prod_{s=1}^{n} r_s .$$

3. Examples

Let us illustrate the effectiveness in Reinhardt and hyperelliptic domains, taking some examples. First, suppose that the set of polynomials $\{P_m(z)\}_{m\ge0}$ is given by

$$P_0(z) = 1$$
$$P_m(z) = z^m + 2^{m^2} \quad \text{for } m \ge 1.$$

Then

$$z^m = P_m(z) - 2^{m^2 \cdot 1}$$
$$M_m(\rho) = \sup_{|z|<\rho_s} \left|P_k(z)\right| = \rho_s^m + 2^{m^2}, \ \ s \in I$$
$$M_0\left(P_0(z),(\rho)\right) = \sup_{|z|\le\rho} \left|P_0(z)\right| = 2^{m^2}.$$

Hence

$$\mu\left(P_m(z),(\rho)\right) = \rho_s^m + 2\cdot 2^{m^2} = \sum_{h=0}^{m} \left|P_{m,h}\right| M\left(P_m(z),(\rho)\right)$$

which implies

$$\mu\left(P(z),(\rho)\right) = \overline{\lim}_{\langle m\rangle\to\infty}\left\{\mu\left(P_m(z),(\rho)\right)\right\}^{\frac{1}{\langle m\rangle}}$$
$$= \overline{\lim}_{\langle m\rangle\to\infty}\left\{\rho_s^m + 2\cdot 2^{m^2}\right\}^{\frac{1}{\langle m\rangle}}$$

for $\rho_s < 2$; $\mu\left(P(z),(\rho)\right) = \infty$.

Now consider the new polynomial from the polynomial defined above:

$$\Delta^n z^m = \sum_{k=0}^{n} \binom{n}{k} (-11)^k (z+n-k)^m$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j} \binom{n}{k}\binom{m}{j}\binom{j}{t} \zeta^* (-11)^{k+t} n^{j-t} (k)^t z^s .$$

Hence by Theorem 2.4,

$$\mu\left(\tilde{P}_m(z),(\rho)\right) = \frac{\prod_{s=1}^{n} (\rho_s)^{\langle m\rangle - m_s} \sum_{h=0}^{n} \left|\overline{P}_{m,h}\right| M\left(\tilde{P}_{m,h}(z),(\rho)\right)}{\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j} \binom{n}{k}\binom{m}{j}\binom{j}{t} \zeta^* (-1)^{k+t} n^{j-t} k^t}.$$

where

$$M\left(\tilde{P}_{m,h}(z),(\rho)\right)=\max_{\Gamma_{(\rho)}}\left|\tilde{P}(z)\right|\le N_m M\left(P_{m,h}(z),(\rho)\right)$$

where N_m is a constant. Hence,

$$\mu\left(\tilde{P}_{m,h}(z),(\rho)\right)\le\frac{N_m\left(\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle-m_s}\sum_h\left|\bar{P}_{m,h}\right|M\left(P_{m,h}(z),(\rho)\right)\right)}{\sum_{t=0}^{j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta^*(-1)^{k+t}n^{j-t}k^t}$$

The Cannon function

$$\mu\left(\tilde{P}(z),(\rho)\right)\le\lim_{\langle m\rangle\to\infty}\frac{\left\{N_m\left(\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle-m_s})\sum_{h=0}^{n}\left|\tilde{P}_{m,h}\right|M\left(P_{m,h}(z),(\rho)\right)\right)\right\}^{\frac{1}{\langle m\rangle}}}{\left\{\sum_{t=0}^{j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta^*(-1)^{k+t}n^{j-t}k^t\right\}^{\frac{1}{\langle m\rangle}}}.$$

which implies

$$\mu\left(\tilde{P}(z),(\rho)\right)=\left(K_1\mu\left(\tilde{P}_m(z),(\rho)\right)\right)^{\frac{1}{\langle m\rangle}}=\infty\quad\text{as } m\to\infty$$

where

$$K_1=\frac{N_m^{\frac{1}{\langle m\rangle}}\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle-m_s}}{\sum_{t=0}^{j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\binom{j}{t}\zeta^*(-1)^{k+t}n^{j-t}k^t}$$

and

$$\mu\left(\tilde{P}_m(z),(\rho)\right)=\sum_{h=0}^{m}\left|\tilde{P}_{m,h}\right|M\left(P_{m,h}(z),(\rho)\right).$$

Hence

$$\mu\left(P(z),(\rho)\right)=\infty.\quad\text{Then }\mu\left(\tilde{P}(z),(\rho)\right)=\infty.$$

Similarly, for the operator ∇, we have

$$\mu\left(\tilde{Q}(z),(\rho)\right)\le\frac{N_m^{\frac{1}{\langle m\rangle}}\left(\prod_{s=1}^{n}(\rho_s)^{\langle m\rangle-m_s}\right)^{\frac{1}{\langle m\rangle}}\left(\sum_{h=0}\left|\bar{Q}_{m,h}\right|M\left(Q_m(z),(\rho)\right)\right)^{\frac{1}{\langle m\rangle}}}{\left(\sum_{s=0}^{m-j}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\zeta(-1)^{j+k}k^j\right)^{\frac{1}{\langle m\rangle}}}$$

Since

$$\sum\left|\tilde{Q}_{m,h}\right|^{\frac{1}{\langle m\rangle}}\left(M\left(Q_{m,h}(z),(\rho)\right)\right)^{\frac{1}{\langle m\rangle}}=\mu\left(\left(Q_m(z),(\rho)\right)\right)^{\frac{1}{\langle m\rangle}}=\infty.$$

Then

$$\mu\left(\tilde{Q}(z),(\rho)\right)=\infty;\quad m\to\infty$$

Table 1. Region of effectiveness: (1) Disc $\lambda(R)=\prod_{j=1}^{n} R_j$; (2) Hyperelliptic $\lambda(R)=\prod_{s=1}^{n} r_s$; Reinhardt domain $\lambda(R)=\prod_{j=1}^{n} \rho_{\hat{s}}$.

Polynomials (P_m)	$\Delta^n P_m[z]$	$\nabla^n P_m[z]$
Monomials		
Z^m	$\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{t=0}^{j}\sum_{s=0}^{m-j}\zeta(-1)^{k+t}k^t$ $\times {}^nC_k\,{}^mC_j\,{}^jC_t(n)^{j-t}Z^s$	$\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{s=0}^{m-j}{}^nC_k\,{}^mC_j\,\zeta k^j(-1)^{k+j}Z^s$
Chebyshev (first kind)		
$P_m(Z)$ $=\sum_{\mu=0}^{[m/2]}{}^mC_{2\mu}Z^{m-2\mu}(Z^2-1)^{\mu}$	$\sum_{\mu=0}^{[m/2]}\sum_{k=0}^{n}\sum_{s=0}^{\mu}\sum_{j=0}^{m-2\mu}\eta_1\times{}^mC_{2\mu}\,{}^{m-2\mu}C_j\,{}^nC_k\,{}^{\mu}C_s$ $\times(-1)^{s+k}(n-k)^{s+t}Z^q$	$\sum_{\mu=0}^{[m/2]}\sum_{k=0}^{n}\sum_{j=0}^{m-2\mu}\eta_2$ $\times{}^nC_k\,{}^mC_{2\mu}\,{}^{m-2\mu}C_j\times(-1)^{q+v+j}Z^q$
Chebyshev (second kind)		
$\sum_{\mu=0}^{[m/2]}(-1)^{\mu}\,{}^{m-\mu}C_{\mu}(2Z)^{m-2\mu}$	$\sum_{\mu=0}^{[m/2]}\sum_{k=0}^{n}\sum_{q=0}^{m-2\mu}\times{}^nC_k\,{}^{m-2\mu}C_q\,{}^{m-\mu}C_{\mu}$ $\times\eta_3 2^{m-2\mu}(-1)^k(n-k)^t Z^s$	$\sum_{\mu=0}^{[m/2]}\sum_{k=0}^{n}\sum_{s=0}^{m-2\mu}{}^nC_k\,{}^{m-\mu}C_{\mu}(-1)^k$ $\times{}^{m-2\mu}C_s\eta_4(-1)^{s+k}2^{m-2\mu}Z^s$
Hermite		
$\sum_{k=0}^{[m/2]}(-1)^k k!(2z)^{m-2k}(m-2k)!$	$\sum_{j=0}^{n}\sum_{k=0}^{[m/2]}\sum_{t=0}^{m-2k}\eta_5$ $\times{}^nC_j\,{}^{m-2k}C_t(-1)^j 2^{m-2k}k!(m-2k)!\times(n-j)^s Z^s$	$\sum_{s=0}^{m-2k-t}\sum_{j=0}^{n}\sum_{t=0}^{m-2k}\sum_{k=0}^{[m/2]}{}^nC_j(-1)^{j+t}\,{}^{m-2k}C_t 2^{m-2k}k!(m-2k)!$ $\times j^t Z^s$

Implication: The new sets are nowhere effective since the parents sets are nowhere effective. By changing $\prod_{s=1}^{n}\rho_s^{\langle m\rangle-m_s}$ in Reinhart domain to $\sigma_m\prod_{s=1}^{n}[r_s]^{\langle m\rangle-m_s}$, where $\sigma_m=\inf_{|t|=1}\frac{1}{t^m}\frac{\{\langle m\rangle\}^{\langle m\rangle/2}}{\prod_{s=1}^{n}m_s^{m_S/2}}$, we obtain the same condition of effectiveness as in Reinhart domain for both operators Δ and ∇ in the hyperelliptic domain.

The following notations are relevant to the table below.

$$\eta_1=\sum_{t=0}^{2(\mu-s)}\sum_{q=0}^{m-2s-t-j}\binom{2\mu-2s}{t}\binom{m-2s-t-j}{q} \tag{12}$$

$$\eta_2=\sum_{q=0}^{\mu}\sum_{s=0}^{2(\mu-q)}\sum_{v=0}^{m-2q-j}(-1)^{q+j+v},\ \eta_3=\sum_{s=0}^{m-2\mu-q}\binom{m-2\mu-q}{s}, \tag{13}$$

$$\eta_4=\sum_{q=0}^{m-2\mu-s}\binom{m-2\mu-s}{q}. \tag{14}$$

$$\eta_5=\sum_{s=0}^{m-2k-t-s}\binom{m-2k-t-s}{s} \tag{15}$$

Finally, for the classical orthogonal polynomials, the explicit results of computation are given in a **Table 1** below.

Thus, in this paper, we have provided new sets of polynomials in C, generated by ∇ and Δ operators, which satisfy all properties of basic sets related to their effectiveness in specified regions such as in hyperelliptic and Reinhardt domains. Namely, the new basic sets are effective in complete Reinhardt domain as well as in closed Reinhardt domain. Furthermore, we have proved that if the Cannon basic set $\{P_m(z)\}_{m\geq 0}$ is effective in hyperelliptic domain, then the new set $\{\tilde{P}_m(z)\}_{m\geq 0}$ is also effective in the hiperelliptic domain.

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Appendix

Key Notations

1) $\mu\left(\tilde{P}_{m,h}[z],[\rho]\right)$ = Cannon sum of the new Δ-set in Reinhardt domain.

2) $\mu\left(\tilde{Q}_{m,h}[z],[\rho]\right)$ = Cannon sum of the new ∇-set in Reinhardt domain.

3) $\mu\left(\tilde{P}_{m,h}[z],[r]\right)$ = Cannon sum of the new Δ-set in Hyperelliptic domain.

4) $\mu\left(\tilde{Q}_{m,h}[z],[r]\right)$ = Cannon sum of the new ∇-set in Hyperelliptic domain.

5) $\mu\left(\tilde{P}[z],[\rho]\right)$ = Cannon function of the new Δ-set in Reinhardt domain.

6) $\mu\left(\tilde{Q}[z],[\rho]\right)$ = Cannon function of the new ∇-set in Reinhardt domain.

7) $\mu\left(\tilde{P}_{m,h}[z],[r]\right)$ = Cannon sum of the new Δ-set in Hyperelliptic domain.

8) $\mu\left(\tilde{Q}_{m,h}[z],[r]\right)$ = Cannon sum of the new ∇-set in Hyperelliptic domain.

9) $M\left(F(z),[r']\right)=\max_{\bar{\Gamma}_{[r']}}\left|F(z)\right|$.

10) $M\left(F(z),[\rho']\right)=\max_{\bar{\Gamma}_{[\rho']}}\left|F(z)\right|$.

11) $M\left(\tilde{P}_{m,h}[z],[r]\right)=\max_{\Gamma_{[r]}}\left|\tilde{P}[z]\right|\le N_m M\left(P_{m,h}[z],[r]\right)$

$$M\left(\tilde{P}_{m,h}[z],[\rho]\right)=\max_{\Gamma_{[\rho]}}\left|\tilde{P}[z]\right|\le N_m M\left(P_{m,h}[z],[\rho]\right)$$

$$M\left(\tilde{Q}_{m,h}[z],[r]\right)=\max_{\Gamma_{[r]}}\left|\tilde{Q}[z]\right|\le N_m M\left(Q_{m,h}[z],[r]\right)$$

$$M\left(\tilde{Q}_{m,h}[z],[\rho]\right)=\max_{\Gamma_{[\rho]}}\left|\tilde{Q}[z]\right|\le N_m M\left(Q_{m,h}[z],[\rho]\right)$$

where N_m is a constant. $Q_{m,h}=Q\left(\binom{m}{h}\right)$ is a coefficient corresponding to polynomials set $\left\{Q_m(z)\right\}_{m\ge0}$, $P_{m,h}=P\left(\binom{m}{h}\right)$ is a coefficient corresponding to polynomial set $\left\{P_m(z)\right\}_{m\ge0}$. We should note that $Q_{m,h}\ne P_{m,h}$ or $Q_{m,h}=P_{m,h}$.

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Bending and Vibrations of a Thick Plate with Consideration of Bimoments

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Abstract

The paper is dedicated to the development of the theory of orthotropic thick plates with consideration of internal forces, moments and bimoments. The equations of motion of a plate are described by two systems of six equations. New equations of motion of the plate and the boundary conditions relative to displacements, forces, moments, and bimoments are given. As an example, the problems of free and forced oscillations of a thick plate are considered under the effect of sinusoidal periodic load. The problem is solved by Finite Difference Method. Eigenfrequencies of the plate are determined, numeric maximum values of displacements, forces and moments of the plate are obtained depending on the frequency of external force. It is shown that when the value of the frequency of external effect approaches the eigenfrequency, there occurs an increase in displacement, force and moment values; that testifies a gradual transition of the motion of plate points into the resonant mode.

Keywords

Plate, Orthotropic, Isotropic, Displacement, Stress, Moment, Bimoment, Bending, Vibrations

1. Introduction

Theory of plates and shells has a special place in design of structural elements. Specified theories of plates are built by many authors. All existing specified theories of plates are developed on the basis of a number of simplifying hypotheses. An overview of the main statements and common methods of constructing an improved theory of plates and shells can be found in the works of S. A. Ambartsumyan [1], K. Z. Galimov [2], Sh. K. Galimov [3], Kh. M. Mushtari [4] and others. Static problem of the bending of a thick isotropic plate in three-dimensional theory of elasticity is considered by B.F. Vlasov in [5], which gives an exact analytical solution in

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trigonometric series. Monograph by E.N. Baida [6] is devoted to solving the problem of bending of orthotropic plates in trigonometric series. Numerical results of displacements and stresses are obtained.

The authors in [7]-[10] deal with dynamic problems of plates with anisotropic properties. Karamooz Ravari M.R., Forouzan M.R. [7] have studied the problems of plates oscillations. Frequency equations of orthotropic circular ring plate were obtained for general boundary conditions in oscillation plane. In [8] the solution of transition oscillations of rectangular viscous-elastic orthotropic plate are given for concrete strain models according to Flugge and Timoshenko-Mindlin's theories. The paper [9] is devoted to analytical solution of the problem of forced steady-state vibrations of orthotropic plate. By the method of superposition the problem is reduced to a quasi-regular infinite system of linear equations. In [10] an analytical method of solution of spatial problem of bending of orthotropic elastic plates subjected to external loads on upper and lower edges is developed. In [11] a problem is considered of a bending of orthotropic rectangular plate laying on two-parameter elastic foundation. Research in the field of thick plates has shown that in the case of spatial deformation of a plate along its thickness there occurs the nonlinear laws of displacements distribution and the hypothesis of plane sections is violated. In the cross-sections of the plate except for the tensile and shear forces, bending and torsional moments, there appear the additional force factors, the so-called bimoments. The author of the article addresses the problem of bending and vibrations of thick plates based on bimoment theory of plates [12]-[15], built as a part of three-dimensional theory of elasticity, using the method of displacements decomposition in one of the spatial coordinates in Maclaurin infinite series.

This paper gives a brief description of the technique of constructing a theory of plates with consideration of bimoments generated due to displacements distribution of cross-section points by a non-linear law. Here the equations of bimoments are built with the equation of three-dimensional dynamic theory of elasticity, described on face surfaces of the plate. The bimoments are introduced in stress dimensions and are characterized by the intensity of generated bimoments. We would use the designations and determinant correlations of forces, moments, bimoments and equations of motion relative to these force factors.

Unlike bimoment theory in [14] and [15], here the bimoment equations are built with the equation of three-dimensional dynamic theory of elasticity, described on face surfaces of the plate. Bimoments are introduced in stress dimensions, and they characterize the intensity of generated bimoments.

Determinant relationships of forces, moments, bimoments and equations of motion relative to these force factors are given.

2. Statement of the Problem

Consider an orthotropic thick plate of constant thickness $H = 2h$ and dimensions a, b in plane. Introduce the designations: E_1, E_2, E_3—elasticity moduli; G_{12}, G_{13}, G_{23}—shear moduli; $\nu_{12}, \nu_{13}, \nu_{23}$—Poisson ratio of plate material.

When building an equation of motion the plate is considered as a three-dimensional body and all components of stress and strain tensors: $\sigma_{ij}, \varepsilon_{ij}, (i, j = 1,3)$ are taken into consideration. The components of displacement vector are the functions of three spatial coordinates and time $u_1(x_1, x_2, z, t), u_2(x_1, x_2, z, t), u_3(x_1, x_2, z, t)$.

The components of strain tensor ε_{ij} are determined from Cauchy relation as:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1},\ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2},\ \varepsilon_{33} = \frac{\partial u_3}{\partial z}, \tag{1.a}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),\ \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x_1}\right),\ \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial x_2}\right), \tag{1.b}$$

For orthotropic plate, the Hooke' law, in a general case, is written as:

$$\sigma_{11} = E_{11}\varepsilon_{11} + E_{12}\varepsilon_{22} + E_{13}\varepsilon_{33}, \tag{2.a}$$

$$\sigma_{22} = E_{21}\varepsilon_{11} + E_{22}\varepsilon_{22} + E_{23}\varepsilon_{33}, \tag{2.b}$$

$$\sigma_{33} = E_{31}\varepsilon_{11} + E_{32}\varepsilon_{22} + E_{33}\varepsilon_{33}, \tag{2.c}$$

$$\sigma_{12} = 2G_{12}\varepsilon_{12},\quad \sigma_{13} = 2G_{13}\varepsilon_{13},\quad \sigma_{23} = 2G_{23}\varepsilon_{23} \tag{2.d}$$

where $E_{11}, E_{12}, \cdots, E_{33}$ are the elastic constants, determined through Poisson ratio and the moduli of elasticity in the form [14] [15].

As an equation of motion of a plate we would use three-dimensional equations of dynamic theory of elasticity:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial z} = \rho \ddot{u}_2 \tag{3.a}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial z} = \rho \ddot{u}_2 \tag{3.b}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial z} = \rho \ddot{u}_3 \tag{3.c}$$

where ρ is a density of plate material.

Boundary conditions on lower and upper face surfaces of the plate $z = h$ and $z = -h$ are:

$$\sigma_{33} = q_3^{(+)}, \quad \sigma_{31} = q_1^{(+)}, \ \sigma_{32} = q_2^{(+)}, \quad \text{at } z = h; \tag{4.a}$$

$$\sigma_{33} = q_3^{(-)}, \quad \sigma_{31} = q_1^{(-)}, \ \sigma_{32} = q_2^{(-)}, \quad \text{at } z = -h. \tag{4.b}$$

Here $q_1^{(-)}, q_2^{(-)}, q_3^{(-)}$ and $q_1^{(+)}, q_2^{(+)}, q_3^{(+)}$ are distributed external loads, applied to upper and lower face surfaces of the plate $z = h$ and $z = -h$ along the direction of ox_1, ox_2, oz coordinates axes.

3. Method of Solution

The methods of building the bimoment theory of plates are based on Cauchy relation (1), generalized Hooke's law (2), three-dimensional equations of the theory of elasticity (3), boundary conditions on face surfaces (4). A proposed bimoment theory of plates is also described by two non-connected problems, each of which is formulated on the basis of six two-dimensional equations of motion with corresponding boundary conditions.

The components of displacement vector are expanded into Maclaurin infinite series in the form:

$$u_k = B_0^{(k)} + B_1^{(k)} \frac{z}{h} + B_2^{(k)} \left(\frac{z}{h}\right)^2 + B_3^{(k)} \left(\frac{z}{h}\right)^3 + \cdots + B_i^{(k)} \left(\frac{z}{h}\right)^i, \quad (k = 1, 2) \tag{5.a}$$

$$u_3 = A_0 + A_1 \frac{z}{h} + A_2 \left(\frac{z}{h}\right)^2 + A_3 \left(\frac{z}{h}\right)^3 + \cdots + A_i \left(\frac{z}{h}\right)^i \tag{5.b}$$

Here $B_i^{(k)}, A_i$ are unknown functions of two spatial coordinates and time $B_i^{(k)} = B_i^{(k)}(x_1, x_2, t)$, $A_i = A_i(x_1, x_2, t)$. In a general case, these functions are determined according to the formulae:

$$B_i^{(k)} = \frac{1}{i!} h^i \left(\frac{\partial^i u_k}{\partial z^i}\right)_{z=0}, \quad (k = 1, 2), \quad A_i = \frac{1}{i!} h^i \left(\frac{\partial^i u_3}{\partial z^i}\right)_{z=0}$$

The displacements in stresses in upper $z = -h$ and lower points $z = h$ in plate fibers we would designate as $u_i^{(-)}, u_i^{(+)}, (i = 1, 3)$ and $\sigma_{ij}^{(-)}, \sigma_{ij}^{(+)}, (i = 1, 3; j = 1, 3)$.

The first problem of bimoment theory describes tension-compression and transverse reduction of the plate, and the second one—the bending and transverse shear of the plate. Determinant relationships and corresponding equations of motion of the plate in the first and second problems are briefly described below.

The first problem is described by the forces and bimoments with six generalized functions $\overline{\psi}_1, \overline{\psi}_2, \overline{u}_1, \overline{u}_2, \overline{r}, \overline{W}$, which are determined by relationships:

$$\overline{u}_k = \frac{u_k^{(+)} - u_k^{(-)}}{2}, \quad \overline{\psi}_k = \frac{1}{2h} \int_{-h}^{h} u_k \mathrm{d}z, \quad (k = 1, 2), \tag{6}$$

$$\overline{W} = \frac{u_3^{(+)} - u_3^{(-)}}{2}, \quad \overline{r} = \frac{1}{2h^2} \int_{-h}^{h} u_3 z \mathrm{d}z \tag{7}$$

Introduce the external loads for the first problem

$$\bar{q}_1 = \frac{q_1^{(+)} - q_1^{(-)}}{2}, \quad \bar{q}_2 = \frac{q_2^{(+)} - q_2^{(-)}}{2}, \quad \bar{q}_3 = \frac{q_3^{(+)} + q_3^{(-)}}{2} \tag{8}$$

The expressions of longitudinal and tangential forces are written as [12]-[15]:

$$N_{11} = E_{11} H \frac{\partial \bar{\psi}_1}{\partial x_1} + E_{12} H \frac{\partial \bar{\psi}_2}{\partial x_2} + 2E_{13}\bar{W}, \quad N_{22} = E_{12} H \frac{\partial \bar{\psi}_1}{\partial x_1} + E_{22} H \frac{\partial \bar{\psi}_2}{\partial x_2} + 2E_{23}\bar{W} \tag{9.a}$$

$$N_{12} = N_{21} = G_{12}\left(H \frac{\partial \bar{\psi}_1}{\partial x_2} + H \frac{\partial \bar{\psi}_2}{\partial x_1} \right) \tag{9.b}$$

The intensities of the bimoments $\bar{p}_{13}, \bar{p}_{23}$ from tangential stresses σ_{13}, σ_{23} have the expressions

$$\bar{p}_{13} = G_{13}\left(\frac{2(\bar{u}_1 - \bar{\psi}_1)}{H} + \frac{\partial \bar{r}}{\partial x_1} \right), \quad \bar{p}_{23} = G_{23}\left(\frac{2(\bar{u}_2 - \bar{\psi}_2)}{H} + \frac{\partial \bar{r}}{\partial x_2} \right) \tag{10.a}$$

The intensity of the bimoment $\bar{p}_{33}$ from normal stress σ_{33} is written in the form:

$$\bar{p}_{33} = E_{31} \frac{\partial \bar{\psi}_1}{\partial x_1} + E_{32} \frac{\partial \bar{\psi}_2}{\partial x_2} + E_{33} \frac{2\bar{W}}{H} \tag{10.b}$$

The equations of motion relative to longitudinal and tangential forces and bimoments from tangential and normal stresses have the form [12]-[15]:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + 2\bar{q}_1 = \rho H \ddot{\bar{\psi}}_1, \quad \frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + 2\bar{q}_2 = \rho H \ddot{\bar{\psi}}_2 \tag{11}$$

$$\frac{\partial \bar{p}_{13}}{\partial x_1} + \frac{\partial \bar{p}_{23}}{\partial x_2} - \frac{2\bar{p}_{33}}{H} + \frac{2\bar{q}_3}{H} = \rho \ddot{\bar{r}} \tag{12}$$

Note, that the expressions of force factors (9), (10), and hence, the equations of motion of the system (11), (12) is rigorously built. This system consists of three equations relative to six unknown functions $\bar{\psi}_1, \bar{\psi}_2, \bar{u}_1, \bar{u}_2, \bar{r}, \bar{W}$. As could be seen, three equations are missed. If in expressions (9.a) the terms $2E_{13}\bar{W}, 2E_{13}\bar{W}$ are omitted, then we would obtain two equations of motion of classic theory of plates in the form (11), since the equation of motion (12) becomes isolated and fail.

The second problem of bimoment theory consists of the equations for bending moments, torsional moments, shear forces relative to six kinematic functions $\tilde{\psi}_1, \tilde{\psi}_2, \tilde{u}_1, \tilde{u}_2, \tilde{r}, \tilde{W}$, determined by formulae:

$$\tilde{u}_k = \frac{u_k^{(+)} - u_k^{(-)}}{2}, \quad \tilde{\psi}_k = \frac{1}{2h^2}\int_{-h}^{h} u_k z \mathrm{d}z, \quad (k = 1, 2), \tag{13}$$

$$\tilde{W} = \frac{u_3^{(+)} + u_3^{(-)}}{2}, \quad \tilde{r} = \frac{1}{2h}\int_{-h}^{h} u_3 \mathrm{d}z \tag{14}$$

Introduce the generalized external loads for the second problem

$$\tilde{q}_1 = \frac{q_1^{(+)} + q_1^{(-)}}{2}, \quad \tilde{q}_2 = \frac{q_2^{(+)} + q_2^{(-)}}{2}, \quad \tilde{q}_3 = \frac{q_3^{(+)} - q_3^{(-)}}{2} \tag{15}$$

Bending, torsional moments and shear forces, which are rigorously built, have the form [12]-[15]:

$$M_{11} = \frac{H^2}{2}\left(E_{11} \frac{\partial \tilde{\psi}_1}{\partial x_1} + E_{12} \frac{\partial \tilde{\psi}_2}{\partial x_2} - E_{13} \frac{2(\tilde{r} - \tilde{W})}{H} \right) \tag{16.a}$$

$$M_{22}=\frac{H^2}{2}\left(E_{12}\frac{\partial\tilde{\psi}_1}{\partial x_1}+E_{22}\frac{\partial\tilde{\psi}_2}{\partial x_2}-E_{23}\frac{2\left(\tilde{r}-\tilde{W}\right)}{H}\right) \tag{16.b}$$

$$M_{12}=\frac{H^2}{2}G_{12}\left(\frac{\partial\tilde{\psi}_1}{\partial x_2}+\frac{\partial\tilde{\psi}_2}{\partial x_1}\right) \tag{16.c}$$

$$Q_{13}=G_{13}\left(2\tilde{u}_1+H\frac{\partial\tilde{r}}{\partial x_1}\right),\quad Q_{23}=G_{23}\left(\tilde{u}_2+H\frac{\partial\tilde{r}}{\partial x_2}\right) \tag{16.d}$$

The system of equations of motion of the second problem consists of two equations relative to bending, torsional moments and one equation relative to shear force and it is written in the form [12]-[15]:

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-Q_{13}=\frac{H^2}{2}\rho\ddot{\tilde{\psi}}_1-H\tilde{q}_1,\quad \frac{\partial M_{21}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-Q_{23}=\frac{H^2}{2}\rho\ddot{\tilde{\psi}}_2-H\tilde{q}_2 \tag{17}$$

$$\frac{\partial Q_{13}}{\partial x_1}+\frac{\partial Q_{23}}{\partial x_2}=H\rho\ddot{\tilde{r}}-2\tilde{q}_3 \tag{18}$$

Note, that the expressions of forces and moments (16), hence, the equations of motion of the system (17), (18) are rigorously built. Similar to the first problem, here three equations are missed. The system of equations of motion (17), (18) consists of three equations relative to six unknown functions $\tilde{\psi}_1,\tilde{\psi}_2,\tilde{u}_1,\tilde{u}_2,\tilde{r},\tilde{W}$. If in expressions of forces and moments, into the equations of motion (17) and (18) conventionally introduce $\tilde{u}_1=3\tilde{\psi}_1,\tilde{u}_2=3\tilde{\psi}_2$ and $E_{13}=E_{23}=0$, and the shear modulus G_{13},G_{23} substitute for k_zG_{13},k_zG_{23}, (where k_z is a shear coefficient), then an equation of motion of plates could be obtained according to Timoshenko's theory.

To complete the systems (11), (12) and (17) and (18) it is necessary to build two more systems, with three equations in each. Write down three equations of motion of the theory of elasticity (3) on face surfaces of the plate $z=-h$ and $z=+h$. Adding and subtracting the equations of the theory of elasticity (3) on face surfaces of the plate $z=-h$ and $z=h$, and taking into account the Hooke's law (2), surface conditions (4) and designations (6), (7) and (13), (14), two independent systems with three equations in each could be obtained. The first of these systems describes the first problem and has the form:

$$\frac{\partial\bar{\sigma}_{11}}{\partial x_1}+\frac{\partial\bar{\sigma}_{12}}{\partial x_2}+\frac{\bar{\sigma}^*_{13}}{H}=\rho\ddot{\bar{u}}_1,\qquad \frac{\partial\bar{\sigma}_{21}}{\partial x_1}+\frac{\partial\bar{\sigma}_{22}}{\partial x_2}+\frac{\bar{\sigma}^*_{23}}{H}=\rho\ddot{\bar{u}}_2 \tag{19}$$

$$\frac{\partial\bar{q}_1}{\partial x_1}+\frac{\partial\bar{q}_2}{\partial x_2}+\frac{\bar{\sigma}^*_{33}}{H}=\rho\ddot{\bar{W}} \tag{20}$$

Here the intensities of the bimoments $\bar{\sigma}_{11},\bar{\sigma}_{22},\bar{\sigma}_{12}$—under transverse reduction and tension-compression of the plate, generated due to $\sigma_{11},\sigma_{22},\sigma_{12}$ are:

$$\bar{\sigma}_{ij}=\frac{\sigma_{ij}^{(+)}+\sigma_{ij}^{(-)}}{2},\quad (i=1,2;\ j=1,2) \tag{21}$$

$\bar{\sigma}^*_{13},\bar{\sigma}^*_{23},\bar{\sigma}^*_{33}$ are the intensities of the bimoments generated due to transverse stresses $\sigma_{13},\sigma_{23},\sigma_{33}$:

$$\frac{\bar{\sigma}^*_{3k}}{H}=\frac{1}{2}\left(\frac{\partial\sigma_{k3}^{(+)}}{\partial z}+\frac{\partial\sigma_{k3}^{(-)}}{\partial z}\right),\quad (k=1,2),\quad \frac{\bar{\sigma}^*_{33}}{H}=\frac{1}{2}\left(\frac{\sigma_{33}^{(+)}}{\partial z}-\frac{\partial\sigma_{33}^{(-)}}{\partial z}\right) \tag{22}$$

The second system of equations obtained from the equations of the theory of elasticity (3) is written in the form:

$$\frac{\partial\tilde{\sigma}_{11}}{\partial x_1}+\frac{\partial\tilde{\sigma}_{12}}{\partial x_2}+\frac{\tilde{\sigma}^*_{13}}{H}=\rho\ddot{\tilde{u}}_1,\quad \frac{\partial\tilde{\sigma}_{21}}{\partial x_1}+\frac{\partial\tilde{\sigma}_{22}}{\partial x_2}+\frac{\tilde{\sigma}^*_{23}}{H}=\rho\ddot{\tilde{u}}_2 \tag{23}$$

$$\frac{\partial \tilde{q}_1}{\partial x_1}+\frac{\partial \tilde{q}_2}{\partial x_2}+\frac{\tilde{\sigma}_{33}^*}{H}=\rho \ddot{\tilde{W}} \tag{24}$$

Here $\tilde{\sigma}_{11}, \tilde{\sigma}_{22}, \tilde{\sigma}_{12}$ are the intensities of the bimoments under transverse bending and shear for the second problem generated due to the stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$:

$$\tilde{\sigma}_{ij}=\frac{\sigma_{ij}^{(+)}-\sigma_{ij}^{(-)}}{2}, \quad (i=1,2;\ j=1,2) \tag{25}$$

The intensities of the bimoments $\tilde{\sigma}_{13}^*, \tilde{\sigma}_{23}^*, \tilde{\sigma}_{33}^*$, generated due to the stresses $\sigma_{33}, \sigma_{13}, \sigma_{23}$, under transverse shear and bending are written in the form:

$$\frac{\tilde{\sigma}_{3k}^*}{H}=\frac{1}{2}\left(\frac{\partial \sigma_{k3}^{(+)}}{\partial z}-\frac{\partial \sigma_{k3}^{(-)}}{\partial z}\right), \ (k=1,2), \quad \frac{\tilde{\sigma}_{33}^*}{H}=\frac{1}{2}\left(\frac{\sigma_{33}^{(+)}}{\partial z}+\frac{\partial \sigma_{33}^{(-)}}{\partial z}\right) \tag{26}$$

The intensities of the bimoments $\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12}, \tilde{\sigma}_{11}, \tilde{\sigma}_{12}, \tilde{\sigma}_{22}$ are determined from Hooke's law (2) with consideration of the conditions on face surfaces $z=-h$ and $z=h$ (4) as:

$$\bar{\sigma}_{11}=E_{11}^*\frac{\partial \bar{u}_1}{\partial x_1}+E_{12}^*\frac{\partial \bar{u}_2}{\partial x_2}+\frac{E_{13}}{E_{33}}\bar{q}_3, \quad \bar{\sigma}_{22}=E_{12}^*\frac{\partial \bar{u}_1}{\partial x_1}+E_{22}^*\frac{\partial \bar{u}_2}{\partial x_2}+\frac{E_{23}}{E_{33}}\bar{q}_3, \quad \bar{\sigma}_{12}=G_{12}\left(\frac{\partial \bar{u}_1}{\partial x_2}+\frac{\partial \bar{u}_2}{\partial x_1}\right) \tag{27}$$

$$\tilde{\sigma}_{11}=E_{11}^*\frac{\partial \tilde{u}_1}{\partial x_1}+E_{12}^*\frac{\partial \tilde{u}_2}{\partial x_2}+\frac{E_{13}}{E_{33}}\tilde{q}_3, \quad \tilde{\sigma}_{22}=E_{12}^*\frac{\partial \tilde{u}_1}{\partial x_1}+E_{22}^*\frac{\partial \tilde{u}_2}{\partial x_2}+\frac{E_{23}}{E_{33}}\tilde{q}_3, \quad \tilde{\sigma}_{12}=G_{12}\left(\frac{\partial \tilde{u}_1}{\partial x_2}+\frac{\partial \tilde{u}_2}{\partial x_1}\right) \tag{28}$$

Here $E_{11}^*=E_{11}-\frac{E_{13}}{E_{33}}E_{31}, E_{22}^*=E_{22}-\frac{E_{23}}{E_{33}}E_{32}, E_{12}^*=E_{21}-\frac{E_{23}}{E_{33}}E_{31}$.

The expressions of the intensities of the bimoments $\bar{\sigma}_{13}^*, \bar{\sigma}_{23}^*, \bar{\sigma}_{33}^*$ are determined by the solution of the system of linear algebraic equations relative to coefficients of Maclaurin series $B_{2i}^{(1)}, B_{2i}^{(2)}, A_{2i+1}, (i=0,1,2,\cdots)$, which are obtained by the substitution of the series (5) into the conditions on face surfaces at $z=-h$ and $z=h$ (4) and designations (6), (7).

$$\bar{\sigma}_{13}^*=G_{13}\left(60\frac{\bar{\psi}_1-\bar{u}_1}{H}-12\frac{\partial \bar{W}}{\partial x_1}\right)-\frac{G_{13}}{E_{33}}H\frac{\partial}{\partial x_1}\left(E_{31}\frac{\partial \bar{u}_1}{\partial x_1}+E_{32}\frac{\partial \bar{u}_2}{\partial x_2}-\bar{q}_3\right)+12\bar{q}_1 \tag{29.a}$$

$$\bar{\sigma}_{23}^*=G_{23}\left(60\frac{\bar{\psi}_2-\bar{u}_2}{H}-12\frac{\partial \bar{W}}{\partial x_2}\right)-\frac{G_{23}}{E_{33}}H\frac{\partial}{\partial x_2}\left(E_{31}\frac{\partial \bar{u}_1}{\partial x_1}+E_{32}\frac{\partial \bar{u}_2}{\partial x_2}-\bar{q}_3\right)+12\bar{q}_2 \tag{29.b}$$

$$\begin{aligned}\bar{\sigma}_{33}^*=E_{33}\left(420\frac{\bar{r}}{H}-180\frac{\bar{W}}{H}\right)-HE_{31}\frac{\partial}{\partial x_1}\left(\frac{\partial \bar{W}}{\partial x_1}-\frac{\bar{q}_1}{G_{13}}\right)-HE_{32}\frac{\partial}{\partial x_2}\left(\frac{\partial \bar{W}}{\partial x_2}-\frac{\bar{q}_2}{G_{23}}\right)\\ -20\left(E_{31}\frac{\partial \bar{u}_1}{\partial x_1}+E_{32}\frac{\partial \bar{u}_2}{\partial x_2}-\bar{q}_3\right)\end{aligned} \tag{30}$$

The expressions of the intensities of the bimoments $\tilde{\sigma}_{13}^*, \tilde{\sigma}_{23}^*, \tilde{\sigma}_{33}^*$ are determined by the solution of the system of linear algebraic equations relative to coefficients of Maclaurin series $B_{2i+1}^{(1)}, B_{2i+1}^{(2)}, A_{2i}, (i=0,1,2,\cdots)$, which are obtained by the substitution of the series (5) into the conditions on the face surfaces at $z=-h$ and $z=h$ (4) and designations (13), (14).

$$\tilde{\sigma}_{13}^*=G_{13}\left(420\frac{\tilde{\psi}_1}{H}-180\frac{\tilde{u}_1}{H}-20\frac{\partial \tilde{W}}{\partial x_1}\right)-\frac{G_{13}}{E_{33}}H\frac{\partial}{\partial x_1}\left(E_{31}\frac{\partial \tilde{u}_1}{\partial x_1}+E_{32}\frac{\partial \tilde{u}_2}{\partial x_2}-\tilde{q}_3\right)+20\tilde{q}_1, \tag{31.a}$$

$$\tilde{\sigma}_{23}^*=G_{23}\left(420\frac{\tilde{\psi}_2}{H}-180\frac{\tilde{u}_2}{H}-20\frac{\partial \tilde{W}}{\partial x_2}\right)-\frac{G_{23}}{E_{33}}H\frac{\partial}{\partial x_2}\left(E_{31}\frac{\partial \tilde{u}_1}{\partial x_1}+E_{32}\frac{\partial \tilde{u}_2}{\partial x_2}-\tilde{q}_3\right)+20\tilde{q}_2, \tag{31.b}$$

$$\tilde{\sigma}_{33}^{*} = 60E_{33}\frac{\tilde{r}-\tilde{W}}{H} - E_{31}H\frac{\partial}{\partial x_1}\left(\frac{\partial \tilde{W}}{\partial x_1} - \frac{\tilde{q}_1}{G_{31}}\right) - E_{32}H\frac{\partial}{\partial x_2}\left(\frac{\partial \tilde{W}}{\partial x_2} - \frac{\tilde{q}_2}{G_{32}}\right)$$
$$-12\left(E_{31}\frac{\partial \tilde{u}_1}{\partial x_1} + E_{32}\frac{\partial \tilde{u}_2}{\partial x_2} - \tilde{q}_3\right). \tag{32}$$

Write down the formulae to determine the displacements on the face surfaces of the plate $z=-h$ and $z=+h$:

$$u_i^{(-)} = \bar{u}_i - \tilde{u}_i,\quad u_i^{(+)} = \bar{u}_i + \tilde{u}_i,\ (i=1,2),\quad u_3^{(-)} = \tilde{W} - \bar{W},\quad u_3^{(+)} = \tilde{W} + \bar{W}. \tag{33}$$

Formulae for stresses on the face surfaces of the plate $z=-h$ and $z=h$ have the form:

$$\sigma_{ij}^{(-)} = \bar{\sigma}_{ij} - \tilde{\sigma}_{ij},\quad \sigma_{ij}^{(+)} = \bar{\sigma}_{ij} + \tilde{\sigma}_{ij},\ (i=1,2;\ j=1,2) \tag{34}$$

Maximum values of displacements and stresses of the plate are reached on the face surfaces of the plate and are determined by the solutions of the first and second problems by the formulae (33) and (34).

Note, that the expressions of intensities of the bimoments (10), (27), (28), (29), (30), (31) and (32) are built for the first time and are new in the theory of plates.

Consider the boundary conditions of a discussed problem for the thick plates.

1) On the border of the plate the displacements are zero. On the edges of the plate $x_1 = \text{const}$ and $x_2 = \text{const}$ the conditions should be as follows:

$$\bar{\psi}_1 = 0,\quad \bar{\psi}_2,\quad \bar{r} = 0,\quad \bar{u}_1 = 0;\quad \bar{u}_2 = 0,\quad \bar{W} = 0 \tag{35}$$

$$\tilde{\psi}_1 = 0,\quad \tilde{\psi}_2 = 0,\quad \tilde{r} = 0,\quad \tilde{u}_1 = 0;\quad \tilde{u}_2 = 0,\quad \tilde{W} = 0 \tag{36}$$

2) On the border $x_1 = \text{const}$ the plate is supported. The following conditions should be satisfied:

$$N_{11} = 0,\quad N_{12} = 0,\quad \bar{r} = 0,\quad \bar{\sigma}_{11} = 0,\quad \bar{\sigma}_{12} = 0,\quad \bar{W} = 0 \tag{37}$$

$$M_{11} = 0,\quad M_{12} = 0,\quad \tilde{r} = 0,\quad \tilde{\sigma}_{11} = 0,\quad \tilde{\sigma}_{12} = 0\quad \tilde{W} = 0 \tag{38}$$

3) On the border $x_1 = \text{const}$ the plate is free of supports. The following conditions should be satisfied

$$N_{11} = 0,\quad N_{12} = 0,\quad \bar{p}_{13} = 0,\quad \bar{\sigma}_{11} = 0,\quad \bar{\sigma}_{12} = 0,\quad \bar{\sigma}_{13}^{*} = 0 \tag{39}$$

$$M_{11} = 0,\quad M_{12} = 0,\quad Q_{13} = 0,\quad \tilde{\sigma}_{11} = 0,\quad \tilde{\sigma}_{12} = 0\quad \tilde{\sigma}_{13}^{*} = 0 \tag{40}$$

Boundary conditions on the border $x_2 = \text{const}$ are similarly written.

When studying the problem of transverse bending and shear it is enough to consider only the second problem with the equations of motion (17), (18), (23), (24) and boundary conditions (35)-(40).

4. Solution of Tests Problem

As an example, consider the forced harmonic vibrations of a cantilever rectangular plate fixed on both ends under the effect of harmonic periodic external load:

$$q_1^{(-)} = 0,\ q_2^{(-)} = 0,\ q_1^{(+)} = 0,\ q_2^{(+)} = 0,\ q_3^{(+)} = 0,\ q_3^{(-)} = -q_0 \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}\sin\left(\omega_0 t + \beta_0\right) \tag{41}$$

where q_0, ω_0, β_0 is an amplitude, frequency and the mode of vibration of an external load, respectively. Note, that if $\omega_0 = 0$, we obtain the problem of static bending of the plate.

Substituting (41) into (8) and (15) determine the load terms of the equation of motion. For a plate fixed on both ends the boundary conditions are written in the form (35) and (36).

5. Numeric Results

First determine eigenfrequencies of the plate. After dividing the variables by spatial coordinates and time, the problem is solved by Finite Difference Method. The step in spatial coordinates is $\Delta x_1 = \Delta x_2 = \frac{a}{30}$. In calculations, for isotropic plates $\nu_{21} = \nu_{13} = \nu_{32} = 0.3$ are given as an initial data.

For square plates with dimensions $a = b = 3H$ the value of eigenfrequency is $p_1 = 0.7469$. With increasing dimensions of the plate up to $a = b = 5H$ the value of eigenfrequency is $p_1 = 0.3906$. For square plates with dimensions $a = b = 8H$ the value of eigenfrequency is $p_1 = 0.1983$.

Table 1 shows the results obtained for the displacements, moments and forces in fixed square plates $a = b = 3H$ under different values of dimensionless frequency $\bar{\omega}_0 = \frac{\rho H^2 \omega_0}{E_1}$. When the value of the frequency of external effect ω_0 approaches the eigenfrequency $p_1 = 0.7469$ the values of the displacements, forces and moments dramatically increase; this testifies of gradual transition of the motion of plate points into resonant mode. As seen, an abrupt increase in the values of displacements, forces and moments could be observed.

Table 2 and **Table 3** show numeric values of displacements, moments and forces, calculated for the fixed square plates with dimensions $a = b = 5H$ and $a = b = 8H$, respectively, for different values of dimensionless frequency $\bar{\omega}_0$.

Calculations show that when the value of the frequency of external effect ω_0 approaches eigenfrequency, an increase in the values of displacements, forces and moments is observed; this testifies of gradual transition of the motion of plate points into resonant mode.

Table 1. Displacements, forces and moments at $a = b = 3H$.

$\bar{\omega}_0$	$\frac{\tilde{\psi}_1 E_1}{H q_0}$	$\frac{\tilde{r} E_1}{H q_0}$	$\frac{\tilde{W} E_1}{H q_0}$	M_{11}	Q_{13}
0.0000	−0.0158	0.8560	0.9402	0.0113	0.4591
0.3000	−0.0190	0.9882	1.0769	−0.0108	0.5277
0.4000	−0.0226	1.1302	1.2240	−0.0356	0.6010
0.5000	−0.0294	1.4039	1.5074	−0.0852	0.7419
0.6000	−0.0463	2.0663	2.1937	−0.2098	1.0816
0.7000	−0.1365	5.5665	5.8213	−0.8909	2.8704

Table 2. Displacements, forces and moments at $a = b = 5H$.

$\bar{\omega}_0$	$\frac{\tilde{\psi}_1 E_1}{H q_0}$	$\frac{\tilde{r} E_1}{H q_0}$	$\frac{\tilde{W} E_1}{H q_0}$	M_{11}	Q_{13}
0.0000	−0.0751	2.6428	2.7386	−0.1047	0.7774
0.1000	−0.0802	2.7835	2.8825	−0.1263	0.8155
0.2000	−0.1007	3.3478	3.4601	−0.2158	0.9676
0.3000	−0.1783	5.4420	5.6053	−0.5650	1.5275

Table 3. Displacements, forces and moments at $a = b = 8H$.

$\bar{\omega}_0$	$\frac{\tilde{\psi}_1 E_1}{H q_0}$	$\frac{\tilde{r} E_1}{H q_0}$	$\frac{\tilde{W} E_1}{H q_0}$	M_{11}	Q_{13}
0.0000	−0.3067	8.3941	8.5060	−0.3487	1.2598
0.1000	−0.3993	10.4739	10.6163	−0.6033	1.5244
0.1300	−0.5105	12.9496	13.1296	−0.9163	1.8364
0.1600	−0.8082	19.5290	19.8115	−1.7691	2.6596
0.1700	−1.0499	24.8541	25.2206	−2.4678	3.3234
0.1800	−1.5560	35.9866	36.5297	−3.9369	4.7087

Calculations show that the equations of motion of a plate (23) may be substituted by kinematic conditions relative to tangential stresses:

$$\tilde{\sigma}_{13} \equiv G_{31}\left(12\frac{\tilde{u}_1}{H} - 30\frac{\tilde{\psi}_1}{H} + \frac{\partial \tilde{W}}{\partial x_1}\right) = \tilde{q}_1, \quad \tilde{\sigma}_{23} \equiv G_{32}\left(12\frac{\tilde{u}_2}{H} - 30\frac{\tilde{\psi}_2}{H} + \frac{\partial \tilde{W}}{\partial x_2}\right) = \tilde{q}_2, \tag{26}$$

Kinematic equations serve to determine the generalized displacements $\tilde{u}_1, \tilde{u}_2$.

The equations (26) are determined by the solution of the system of linear algebraic equations relative to coefficients of the series (5) $B^{(1)}_{2i+1}, B^{(2)}_{2i+1}, A_{2i}, (i = 0, 1, 2, \cdots)$, which are obtained by the substitution of the series (5) into the conditions on the face surfaces at $z = -h$ and $z = h$ (4) and designations (13), (14).

6. Conclusion

Based on these studies, we would note that using the method of expansion in a series as part of three-dimensional dynamic theory of elasticity, a two-dimensional bimoment theory of orthotropic thick plates was developed and the equations of motion of the plate relative tot forces, moments and bimoments were built. It is shown that the problem in the general case is reduced to the definition of twelve unknown functions of two spatial coordinates and time. New expressions to determine the forces, moments and bimoments of the plates were built, as well as the methods for solving the problems of free and forced vibrations of plates based on Finite Difference Method.

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Modeling and Simulation of Real Gas Flow in a Pipeline

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Abstract

In this paper, a mathematical model that describes the flow of gas in a pipe is formulated. The model is simplified by making some assumptions. It is considered that the natural gas flowing in a long horizontal pipe, no heat source occurs inside the volume, transfer of heat due to heat conduction is dominated by heat exchange with the surrounding. The flow equations are coupled with equation of state. Different types of equations of state, ranging from the simple Ideal gas law to the more complex equation of state Benedict Webb Rubin Starling (BWRS), are considered. The flow equations are solved numerically using the Godunov scheme with Roe solver. Some numerical results are also presented.

Keywords

Gas Flow, Equation of State, Godunov Scheme, Roe Solver, Pipe

1. Introduction

The purpose of this paper is to describe the flow of natural gas in a pipeline by employing the full set of differential equations along with different types of equations of states(EOS), ranging from the simple Ideal gas law to the more complex equation of state, Benedict Webb Rubin Starling (BWRS). The flow equations are derived from the physical principles of conservation of mass, momentum, and energy. More detailed discussion of conservation laws can be found in [1]-[4]. The natural gas is inviscid and compressible. The gas flows in along a horizontal pipe, and then can be considered as one-dimensional flow. It is assumed no heat source occurs inside the pipe and transfer of heat due to the heat conduction is much less than the heat exchange with the surrounding.

In this paper, the results obtained by solving the flow equations along with different types of EOS are compared

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[5]. The ideal gas equation works reasonably well over limited temperature and pressure ranges for many substances. However, pipelines commonly operate outside these ranges and may move substances that are not ideal under any conditions. The more complicated EOS will approximate the real gas behavior for a wide range of pressure and temperature conditions.

The Godunov scheme with Roe solver [3] is used to solve the Euler equations numerically. The Godunov scheme for conservation laws is known for its shock-capturing capability.

The rest of the article is organized as follows. In Section (2) we review the set of partial differential equations which describe the flow of gas in a pipe. Several equations of states are discussed in this section. In Section (3) a thermodynamical relationships among the physical quantities are presented. One can refer [6] for more thermodynamical relationships. Section (4) contains the discussion of the numerical method used to solve the flow equations together with different types equation of states. Some numerical results are given in this section. Conclusions are deferred to Section (5).

2. Governing Equations of Real Gas Flow in a Pipe

Let us consider a gas occupying a sub domain Ω_0 at time $t=0$. Let $x=\Phi(\overline{x},t), \overline{x}\in\Omega_0$ describes the position of the particle $\overline{x}$ at time *t*. Then at time *t* the gas occupies the domain $\Omega_t=\{\Phi(\overline{x},t), \overline{x}\in\Omega_0\}$. The velocity of the gas at position *x* and time *t* is given by $u(x,t)=\frac{\partial}{\partial t}\Phi(\overline{x},t)$.

2.1. Transport Theorem

Let $f:\Omega_t\times[0,t]\to\mathbb{R}$ be some physical quantity transported by the fluid. The total amount $F(t)$ of the quantity *f* contained in Ω_t a time *t* is given by $F(t)=\int_{\Omega_t} f(x,t)\mathrm{d}x$.

Notation: $J(\overline{x},t)=\det\nabla\Phi(\overline{x},t)$

The rate of change of $F(t)$ is given by:

$$\begin{aligned}\frac{\mathrm{d}F(t)}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} f(x,t)\mathrm{d}x\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} f(\Phi(\overline{x},t))J(\overline{x},t)\mathrm{d}\overline{x}\\
&=\int_{\Omega_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}f(\Phi(\overline{x},t))J(\overline{x},t)+f(\Phi(\overline{x},t),t)\frac{\mathrm{d}}{\mathrm{d}t}J(\overline{x},t)\right)\mathrm{d}\overline{x}\quad\text{using product rule}\\
&=\int_{\Omega_0}\left(\left(\frac{\partial f}{\partial t}(\Phi(\overline{x},t),t)+(u\cdot\nabla)f(\Phi(\overline{x},t),t)\right)J(\overline{x},t)+f(\Phi(\overline{x},t),t)J\nabla\cdot u\right)\mathrm{d}\overline{x}\\
&=\int_{\Omega_t}\left(\frac{\partial f}{\partial t}(x,t)+(u\cdot\nabla)f(x,t)+f(x,t)\nabla\cdot u\right)\mathrm{d}x\\
&=\int_{\Omega_t}\left(\frac{\partial f}{\partial t}(x,t)+\nabla\cdot(fu)(x,t)\right)\mathrm{d}x\end{aligned}$$

Then we get the transport theorem: $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} f(x,t)\mathrm{d}x=\int_{\Omega_t}\left(\frac{\partial f}{\partial t}(x,t)+\nabla\cdot(fu)(x,t)\right)\mathrm{d}x$. The transport theorem is useful in the derivation of the governing equations.

2.2. Conservation of Mass (The Continuity Equation)

The total mass *m* in a volume Ω_t is given by $\int_{\Omega_t}\rho(x,t)\mathrm{d}x$. Mass is conserved during the deformation of $\Omega_0\to\Omega_t$ *i.e.* $\frac{\mathrm{d}m}{\mathrm{d}t}=0$, $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho(x,t)\mathrm{d}x=0$

$\int_{\Omega_t}\left(\frac{\partial\rho}{\mathrm{d}t}+\nabla\cdot(\rho u)\right)\mathrm{d}x=0$ (By transport theorem)

$$\frac{\partial\rho}{\mathrm{d}t}+\nabla\cdot(\rho u)=0 \tag{1}$$

Since the above integral holds true for arbitrary region Ω_t

2.3. Conservation of Momentum (Equation of the Motion)

The total momentum M of particles contained in Ω_t is given by $M = \int_{\Omega_t} \rho(x,t)u(x,t)dx$

According to Newton's second law: **The rate of change of momentum equals the action of all the forces *F* applied on** Ω_t

$$\frac{\mathrm{d}M}{\mathrm{d}t} = F$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho(x,t)u(x,t)\mathrm{d}x = F$$

$$\int_{\Omega_t}\left(\frac{\partial \rho u}{\partial t} + \nabla\cdot(\rho u \otimes u)\right)\mathrm{d}x = F$$

We have two types of forces acting on Ω_t:

1) Volume forces f_v, for example gravitation, which is given by

$$f_v = \int_{\Omega_t} \rho(x,t) g(x,t)\mathrm{d}x$$

where g is the gravitational acceleration.

2) Surface forces f_s acting on Ω_t through the boundary $\partial\Omega_t$ of Ω_t, such as pressure and inner friction forces.

Surface forces are given by

$$f_s = \int_{\partial\Omega_t} \sigma(x,t)\boldsymbol{n}\mathrm{d}s$$

where σ is the stress tensor defined as:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

and $\boldsymbol{n}$ is the outer normal. The total force $F = f_v + f_s$. Then, we have $\frac{\mathrm{d}M}{\mathrm{d}t} = f_v + f_s$

$$\int_{\Omega_t}\left(\frac{\partial \rho u}{\partial t} + \nabla\cdot(\rho u \otimes u)\right)\mathrm{d}x = \int_{\Omega_t} \rho(x,t) g(x,t)\mathrm{d}x + \int_{\partial\Omega_t} \sigma(x,t)\boldsymbol{n}\mathrm{d}s.$$

By applying divergence theorem, the second term on the right side of the above equation can be transformed to integral over the domain Ω_t and then we get:

$$\int_{\Omega_t}\left(\frac{\partial \rho u}{\partial t} + \nabla\cdot(\rho u \otimes u)\right)\mathrm{d}x = \int_{\Omega_t} \rho(x,t) g(x,t)\mathrm{d}x + \int_{\Omega_t} \nabla\cdot\sigma(x,t)\mathrm{d}x$$

$$\frac{\partial \rho u}{\partial t} + \nabla\cdot(\rho u \otimes u) - \rho g - \nabla\cdot\sigma = 0$$

or

$$\frac{\partial \rho u}{\partial t} + (u\cdot\nabla)(\rho u) + \rho u(\nabla\cdot u) - \rho g - \nabla\cdot\sigma = 0$$

For Newtonian fluid, the stress tensor depends linearly on the deformation velocity, ∇u, *i.e.*

$$\sigma = -pI + \tau = (-p + \lambda\nabla\cdot u)I + 2\mu D$$

where τ is the viscous part of σ, p is pressure, I is the identity matrix, λ and μ are friction coefficients, and D is the strain tensor given by

$$D = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right)$$

For inviscid fluid, friction is neglected and then $\sigma = -pI$

Therefore, the equation of motion for inviscid fluid becomes

$$\frac{\partial \rho u}{\partial t} + (u \cdot \nabla)\rho u + \rho u(\nabla \cdot u) - \rho g + \nabla p = 0 \tag{2}$$

2.4. Conservation of Energy

Conservation of energy accounts for effects of temperature variations on the flow or the transfer of heat with in the flow. The 1st Law of Thermodynamics states that: **The total energy of a system and its surroundings remains constant**.

Let ϵ be the total energy of the fluid in Ω_t and Q be the amount of heat transfered to Ω_t. The rate of change of the total energy of the fluid occupying Ω_t is the sum of powers of the volume force acting on the volume Ω_t, powers of the surface force acting on the surface $\partial\Omega_t$, and the amount of heat transmitted to Ω_t, *i.e.*

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \int_{\Omega_t} \rho(x,t) g(x,t) u(x,t) \mathrm{d}x + \int_{\partial\Omega_t} \sigma(x,t) u(x,t) \boldsymbol{n} \mathrm{d}s + Q$$

where $\epsilon = \int_{\Omega_t} \rho(x,t) E(x,t) \mathrm{d}x$ and $E = e + \frac{|u|^2}{2}$ is the density of energy (per unit mass), e is internal energy density, and $\frac{|u|^2}{2}$ is the density of kinetic energy.

$$Q = \int_{\Omega_t} \rho(x,t) q(x,t) \mathrm{d}x - \int_{\Omega_t} \bar{q}(x,t) \boldsymbol{n}(x) \mathrm{d}s + \int_{\Omega_t} \bar{\bar{q}}(x,t) \mathrm{d}x$$

where q is the density of heat sources (per unit mass), and

$\bar{q}$ is the heat flux (transfer of heat by conduction).

The transfer of heat by conduction is given by Fourier's law:

$\bar{q} = -\kappa \nabla T$ where T is the absolute temperature and $\kappa \geq 0$ is the coefficient of thermal conductivity of the fluid.

$\bar{\bar{q}}$ is the density of heat transfered from the surrounding and is given by:

$\bar{\bar{q}} = k_L \times \text{surface area} \times (T_{out} - T)$ where k_L is the total heat transfer coefficient and T_{out} is the temperature of the surrounding.

Then the energy equation for inviscid gas flow becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho(x,t) E(x,t) \mathrm{d}x &= \int_{\Omega_t} \rho(x,t) g(x,t) u(x,t) \mathrm{d}x + \int_{\partial\Omega_t} (-pI) u(x,t) \boldsymbol{n} \mathrm{d}s \\ &\quad + \int_{\Omega_t} \rho(x,t) q(x,t) \mathrm{d}x - \int_{\partial\Omega_t} \bar{q}(x,t) \boldsymbol{n} \mathrm{d}s + \int_{\Omega_t} \bar{\bar{q}}(x,t) \mathrm{d}x\end{aligned}$$

By applying the transport and divergence theorems to the above equation we obtain the following equation:

$$\begin{aligned}\int_{\Omega_t} \frac{\partial(\rho E)}{\partial t}(x,t) + \nabla \cdot (\rho E u)(x,t) \mathrm{d}x &= \int_{\Omega_t} \rho(x,t) g(x,t) u(x,t) \mathrm{d}x + \int_{\Omega_t} \nabla \cdot (pu) \mathrm{d}x \\ &\int_{\Omega_t} \rho(x,t) q(x,t) \mathrm{d}x - \int_{\Omega_t} \nabla \cdot \bar{q} \mathrm{d}x + \int_{\Omega_t} \bar{\bar{q}}(x,t) \mathrm{d}x\end{aligned}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E u) = \rho g u - \nabla \cdot (pu) + \rho q - \nabla \cdot \bar{q} + k_L \frac{\text{surface area}}{\text{volume}} (T_{out} - T).$$

$$\frac{\partial \rho E}{dt} + \nabla \cdot (\rho E u) = \rho g u - \nabla \cdot (pu) + \rho q + \nabla \cdot (\kappa \nabla T) + k_L \frac{\text{surface area}}{\text{volume}} (T_{out} - T) \tag{3}$$

There fore, from the equations (1), (2), (3) we get the following system of equations.

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \\ \frac{\partial \rho u}{\partial t} + (u \cdot \nabla)\rho u + \rho u(\nabla \cdot u) - \rho g + \nabla p = 0 \\ \frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E u) - \rho g u + \nabla \cdot (pu) - \rho q - \nabla \cdot (\kappa \nabla T) = k_L \frac{\text{surface area}}{\text{volume}} (T_{out} - T) \end{cases} \tag{4}$$

2.5. Simplifications

In practice the form of mathematical model varies with the assumptions made as regards of operation conditions of the pipeline. Simplified models are obtained by neglecting some terms in the basic equations. In our case, we consider natural gas (Methane) flowing in a long horizontal pipeline. Hence we can consider the flow as a one dimensional flow. By assuming the pipe is horizontal, we can neglect the contribution of the gravitational force. Assume also no heat source occurs inside the volume. For a cylindrical pipe, $\frac{\text{surface area}}{\text{volume}} = \frac{4}{D}$ where D is the diameter of the pipe. By applying the assumptions we made, (4) is reduced to

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x} = 0 \\ \dfrac{\partial \rho u}{\partial t} + \dfrac{\partial \rho u^2 + p}{\partial x} = 0 \\ \dfrac{\partial \rho E}{\partial t} + \dfrac{\partial (\rho E + p) u}{\partial x} - \dfrac{\partial^2 T}{\partial x^2} = \dfrac{4k_L}{D}(T_{out} - T) \end{cases} \tag{5}$$

Furthermore, Methane gas has the following properties. The specific heat capacity $c_p = 2165 [\text{J/kg} \cdot \text{K}]$, thermal conductivity $\kappa = 0.030 [\text{W/m} \cdot \text{K}]$, dynamic viscosity $\mu = 1.02E-5 [\text{kg/m} \cdot \text{s}]$. Typical values for the overall heat transfer coefficient k_L are $0.6 \left[\text{W/m}^2 \cdot \text{K}\right]$ for 0.5 m diameter insulated and buried in soil. If the pipe is exposed on the air k_L is $19 \left[\text{W/m}^2 \cdot \text{K}\right]$.

Prandtl number (Pr), defined as $Pr = \frac{c_p \mu}{\kappa}$, describes the relative strength of viscosity (the diffusion of momentum) to that of heat. It is entirely a property of the fluid not the flow. In our case the value of Pr is about 0.7, this enables us to regard the flow as inviscid flow. For gas flow typical values of Pr are between 0.7 and 1. Another dimensionless constant we can use to simplify our system of equations is the Nusselt number (Nu). The Nusselt number is defined as $Nu = \frac{k_L D}{\kappa}$, where D is a characteristic width of a flow, for example the diameter of the pipe. The Nusselt number compares convection heat transfer to fluid conduction heat transfer.

For Methane gas flowing through an insulated pipe of diameter 0.5 m buried underground, the value of Nu is approximately 10. If the pipe is exposed to air, it will be around 300. Therefore, the term included in the energy equation due to heat conduction $\left(\frac{\partial^2 T}{\partial x^2}\right)$ can be neglected in favor of the term due to heat exchange with the surrounding $\left(\frac{4k_L}{D}(T_{out} - T)\right)$. Incorporating these assumptions to Equation (5) yields:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x} = 0 \\ \dfrac{\partial \rho u}{\partial t} + \dfrac{\partial \rho u^2 + p}{\partial x} = 0 \\ \dfrac{\partial \rho E}{\partial t} + \dfrac{\partial (\rho E + p) u}{\partial x} = \dfrac{4k_L}{D}(T_{out} - T) \\ p = p(\rho, T) \end{cases} \tag{6}$$

where $p = p(\rho, T)$ is an equation of state used to complete the system of conservation laws. In the next chapter we will solve Equation (6) with different equation of state numerically.

2.6. Equation of State (EOS)

An equation of state is a relationship between state variables, such that specification of two state variables permits the calculation of the other state variables. For an ideal gas, the equation of state is the ideal gas law. More complicated EOS have been formulated by several workers to try to model the behavior of real gases over a range of pressures and temperatures. This includes Van der Waals (VDW), Sovae Redlich Kwong (SRK), Peng Robinson (PR), and Benedict Webb Rubin Starling (BWRS).

2.6.1. Ideal Gas law

The ideal gas law is given by

$$p = \rho RT \tag{7}$$

where p is the pressure, ρ is the density, R is the gas constant, and T is the absolute temperature.

The ideal gas law is derived based on two assumptions:

- The gas molecules occupy a negligible fraction of the total volume of the gas.
- The force of attraction between gas molecules is zero.

The ideal gas equation works reasonably well over limited temperature and pressure ranges for many substances. However, pipelines commonly operate outside these ranges and may move substances that are not ideal under any conditions. Hence, we need to look for equation of state with wider validity.

2.6.2. Van der Waals (VDW) EOS

It was observed that the ideal gas law didn't quite work for higher pressures and temperatures. The first assumption works at low pressures. But this assumption is not valid as the gas is compressed. Imagine for the moment that the molecules in a gas were all clustered in one corner of a cylinder, as shown in the figure below. At normal pressures, the volume occupied by these particles is a negligibly small fraction of the total volume of the gas. But at high pressures (when the gas is compressed), this is no longer true. As a result, real gases are not as compressible at high pressures as an ideal gas. The volume of real gas is therefore larger than expected from the ideal gas equation at high pressures. Van der Waals proposed that we correct for the fact that the volume of real gas is too large at high pressures by subtracting a term from the volume of the real gas before we substitute it in to the ideal gas equation. He therefore introduced a constant b in to the ideal gas equation that was equal to the volume actually occupied by the gas particles. When the pressure is small, and the volume is reasonably large, the subtracted term is too small to make any difference in the calculation. But at high pressures, when the volume of the gas is small, the subtracted term corrects for the fact that the volume of a real gas is larger than expected from the ideal gas equation.

The assumption that there is no force of attraction between the gas particles cannot be true. If it was, gases would never condense to form liquids. In reality, there is a small force of attraction between gas molecules that tends to hold the molecules together. This force of attraction has two consequences: (1) gases condense to form liquids at low temperatures and (2) the pressure of a real gas is sometimes smaller than expected for an ideal gas. To correct for the fact that the pressure of a real gas is smaller than expected from the ideal gas equation, Van der Waals added a term to the pressure in the ideal gas equation. This term contains a second constant a. The complete Van der Waals equation is written as follows:

$$p = \frac{\rho RT}{1 - b\rho} - a\rho^2 \tag{8}$$

Or in terms of molar volume

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT$$

where

$$a = \frac{27R^2T_c^2}{64P_c}$$

$$b = \frac{RT_c}{8P_c}$$

R is gas constant, P_c critical pressure, and T_c critical temperature Note that the values of the constants a and b differ from gas to gas. Even though, VDW EOS is better than Ideal gas law, still it is inadequate to describe real gas behavior.

We will consider three widely used equations of state that do work reasonably well near the dew point: Sovae-Redlich-Kwong (SRK), Peng-Robinson (PR), and Benedict-Webb-Rubin-Starling (BWRS). In addition to covering a wide range of conditions, these equations also can be expressed in generalized forms with mixing rules that permit the calculation of the coefficients for different compositions.

SRK and PR, along with VDW are called cubic equation of state, because expansion of the equations into a polynomial results in the highest order terms in density (or specific volume) being cubic. BWRS adds fifth and sixth power and exponential density terms. The cubic equation are all of the form

$$p = \frac{\rho RT}{1-\rho b} - \frac{a\rho^2}{1 + A\rho + B\rho^2} \tag{9}$$

2.6.3. The Sovae-Redlich-Kwong (SRK) EOS

The SRK EOS of state is given by

$$p = \frac{\rho RT}{1-b\rho} - \frac{a\rho^2}{1+b\rho} \tag{10}$$

where

$$a = a_1\left(1 + f_w\left(1-\sqrt{T_r}\right)\right)^2$$

$$a_1 = \frac{0.42748R^2T_c^2}{P_c}$$

$$f_w = 0.48 + 1.5746w - 0.176w^2$$

$$b = \frac{0.078664RT_c}{P_c}$$

w is the accentric factor which is a measure of the gas molecules deviation from the spherical symmetry, R is gas constant, P_c critical pressure, T_c critical temperature, and $T_r = \frac{T}{T_c}$ is the reduced temperature.

2.6.4. The Peng-Robinson (PR) EOS

The PR EOS is defined as

$$p = \frac{\rho RT}{1-b\rho} - \frac{a\rho^2}{1 + 2b\rho - b^2\rho^2} \tag{11}$$

where

$$a = a_1\left(1 + f_w\left(1-\sqrt{T_r}\right)\right)^2$$

$$a_1 = \frac{0.45724R^2T_c^2}{P_c}$$

$$f_w = 0.37464 + 1.54226w - 0.26992w^2$$

$$b = \frac{0.07780RT_c}{P_c}$$

2.6.5. Benedict-Webb-Rubin-Starling (BWRS) EOS

Probably because of its ability to cover both liquids and gases and the availability of coefficients and mixing rules for many hydrocarbons in one place, BWRS is the most widely used equation of state for simulation of pipelines with high density hydrocarbons, or with condensation.

Simplicity is not among the good qualities of the BWRS equation of state. The form of the equation is:

$$\begin{aligned} p = \rho RT &+ \left(BRT - A - \frac{C}{T^2} + \frac{D}{T^3} - \frac{E}{T^4} \right)\rho^2 + \left(bRT - a - \frac{d}{T} \right)\rho^3 \\ &+ \alpha\left(a + \frac{d}{T} \right)\rho^6 + \frac{c\rho^3}{T^2}\left(1 + \gamma\rho^2\right)\exp\left(-\gamma\rho^2\right) \end{aligned} \tag{12}$$

where the eleven coefficients $A, B, C, D, E, a, b, c, d, \alpha$, and γ are determined from ρ_c, T_c, P_c, and ω of the gas of interest and the universal constants A_i and B_i as follows.

$$B_0 = \frac{A_1 + B_1\omega}{\rho_c} \quad A_0 = \frac{A_2 + B_2\omega}{\rho_c} RTc$$

$$C_0 = \frac{A_3 + B_3\omega}{\rho_c} RTc^3 \quad \gamma = \frac{A_4 + B_4\omega}{\rho_c^2}$$

$$b = \frac{A_5 + B_5\omega}{\rho_c^2} \quad a = \frac{A_6 + B_6\omega}{\rho_c^2} RTc$$

$$\alpha = \frac{A_7 + B_7\omega}{\rho_c^3} \quad c = \frac{A_8 + B_8\omega}{\rho_c^2} RTc^3$$

$$D_0 = \frac{A_9 + B_9\omega}{\rho_c} RTc^4 \quad d = \frac{A_{10} + B_{10}\omega}{\rho_c^2} RTc^2$$

$$E_0 = \frac{A_{11} + B_{11}\omega\exp\left(-3.8\omega\right)}{\rho_c} RTc^5$$

where

$A_1 = 0.443690 \quad B_1 = 0.115449$

$A_2 = 1.28438 \quad B_2 = -0.920731$

$A_3 = 0.356306 \quad B_3 = 1.70871$

$A_4 = 0.544979 \quad B_4 = -0.270896$

$A_5 = 0.528629 \quad B_5 = 0.349261$

$A_6 = 0.484011 \quad B_6 = 0.754130$

$A_7 = 0.0705233 \quad B_7 = -0.04448$

$A_8 = 0.504087 \quad B_8 = 1.32245$

$A_9 = 0.0307452 \quad B_9 = 0.179433$

$A_{10} = 0.0732828 \quad B_{10} = 0.463492$

$A_{11} = 0.006450 \quad B_{11} = -0.022143$

BWRS can be adapted for mixtures by the rules:

$$B_0 = \sum x_i B_{0i} \quad A_0 = \sum\sum x_i x_j \sqrt{A_{0i} A_{0j}} \left(1 - k_{ij}\right)$$

$$C_0 = \sum\sum x_i x_j \sqrt{C_{0i} C_{0j}} \left(1 - k_{ij}\right)^{1/3} \quad \gamma = \left(\sum x_i \gamma_i^{1/3}\right)^3$$

$$b = \left(\sum x_i b_i^{1/3}\right)^3 \quad a = \left(\sum x_i a_i^{1/3}\right)^3$$

$$\alpha = \left(\sum x_i \alpha_i^{1/3}\right)^3 \quad c = \left(\sum x_i c_i^{1/3}\right)^3$$

$$D_0 = \sum\sum x_i x_j \sqrt{D_{0i} D_{0j}} \left(1 - k_{ij}\right)^{1/4} \quad d = \left(\sum x_i d_i^{1/3}\right)^3$$

$$E_0 = \sum\sum x_i x_j \sqrt{E_{0i} E_{0j}} \left(1 - k_{ij}\right)^{1/5}$$

where x_i is the mole fraction of the pure component *i* of the mixture, and k_{ij} are the binary interaction coefficients.

2.6.6. The Universal Gas Law

The universal gas law is $p = Z\rho RT$ where *Z* is called the compressibility factor (Real gas factor). It is a measure of how far the gas is from ideality. At atmospheric conditions, the value of *Z* is typically around 0.99. Under pipeline conditions, the value is typically around 0.9. A good equation of state can be selected by its ability to approximate the compressibility factor at critical conditions Z_c.

For example the experimental value of Z_c for Methane is 0.288. But its approximate value by VDW is 0.3025, by SRK is 0.2904, by *PR* it is 0.2894, and by BWRS it is 0.2890.

3. Thermodynamical Relations

In this section we will briefly discuss thermodynamical relations that exist among different physical quantities. First law of thermodynamics states that

$$\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}v \tag{13}$$

The specific total enthalpy is defined as $h = e + pv$ which implies

$$\mathrm{d}h = T\mathrm{d}s + v\mathrm{d}p \tag{14}$$

Derivative relationships:

Assume $e = e(s,v)$, then $\mathrm{d}e = \left(\frac{\partial e}{\partial s}\right)_v \mathrm{d}s + \left(\frac{\partial e}{\partial v}\right)_s \mathrm{d}v$. Comparing the coefficients of this equation to that of Equation (13) we get

$$\left(\frac{\partial e}{\partial s}\right)_v = T, \ \left(\frac{\partial e}{\partial v}\right)_s = -p \tag{15}$$

Similarly, assuming $h = h(s, p)$ we get

$$\mathrm{d}h = \left(\frac{\partial h}{\partial s}\right)_p \mathrm{d}s + \left(\frac{\partial h}{\partial p}\right)_s \mathrm{d}p$$

And comparing the coefficient of this equation with that of Equation (14) we get

$$\left(\frac{\partial h}{\partial s}\right)_p = T, \ \left(\frac{\partial h}{\partial p}\right)_s = v \tag{16}$$

Reciprocal relations involving internal energy *e* and entropy *s*:

Consider the internal energy and entropy to be a function of temperature and specific volume, *i.e*, $e = e(v,T)$, $s = s(v,T)$.

Then

$$\mathrm{d}e = \left(\frac{\partial e}{\partial v}\right)_T \mathrm{d}v + \left(\frac{\partial e}{\partial T}\right)_v \mathrm{d}T, \ \mathrm{d}s = \left(\frac{\partial s}{\partial v}\right)_T \mathrm{d}v + \left(\frac{\partial s}{\partial T}\right)_v \mathrm{d}T \tag{17}$$

The coefficient of d*T*, in the first equation, is by definition the heat capacity at constant volume, c_v. Substitute these two equations in (13) to get

$$\left(\frac{\partial e}{\partial v}\right)_T = T\left(\frac{\partial s}{\partial v}\right)_T - p, \ \left(\frac{\partial e}{\partial T}\right)_v = T\left(\frac{\partial s}{\partial T}\right)_v \tag{18}$$

Differentiating the first equation of (18) with respect to *T* and the second with respect to *v* gives us

$$\frac{\partial^2 e}{\partial v \partial T} = T\frac{\partial^2 s}{\partial v \partial T} + \left(\frac{\partial s}{\partial v}\right)_T - \left(\frac{\partial p}{\partial T}\right)_v$$

and

$$\frac{\partial^2 e}{\partial v \partial T} = T \frac{\partial^2 s}{\partial v \partial T}$$

$$\Rightarrow \left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v \tag{19}$$

Substituting (19) in the first equation of (18) yields

$$\left(\frac{\partial e}{\partial v}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_v - p \tag{20}$$

One useful form involving internal energy is obtained by substituting c_v for the coefficient of dT in (20) for the coefficient of dv in the first equation of (17).

$$\mathrm{d}e = c_v \mathrm{d}T + \left[T\left(\frac{\partial p}{\partial T}\right)_v - p\right]\mathrm{d}v \tag{21}$$

Reciprocal relations involving enthalpy h
Assume $h = h(p,T)$, $s = s(p,T)$
Then

$$\Rightarrow \mathrm{d}h = \left(\frac{\partial h}{\partial p}\right)_T \mathrm{d}p + \left(\frac{\partial h}{\partial T}\right)_p \mathrm{d}T \tag{22}$$

The coefficient of dT is by definition the heat capacity at constant pressure, c_p. In a similar procedure as in the internal energy and entropy case, above we get the following relationships.

$$\left(\frac{\partial h}{\partial p}\right)_T = T\left(\frac{\partial s}{\partial p}\right)_T + v,\ \left(\frac{\partial h}{\partial T}\right)_p = T\left(\frac{\partial s}{\partial T}\right)_p \tag{23}$$

By double differentiating we do get

$$\left(\frac{\partial h}{\partial p}\right)_T = v - T\left(\frac{\partial v}{\partial T}\right)_p \tag{24}$$

$$\mathrm{d}h = c_p \mathrm{d}T + \left[v - T\left(\frac{\partial v}{\partial T}\right)_p\right]\mathrm{d}p \tag{25}$$

Heat capacities

By equating the difference of (13) and (14) to the difference of (21) and (25) we get

$$\left(c_p - c_v\right)\mathrm{d}T = T\left(\frac{\partial v}{\partial T}\right)_p \mathrm{d}p + T\left(\frac{\partial p}{\partial T}\right)_v \mathrm{d}v \tag{26}$$

Dividing by $\mathrm{d}T$ and holding p constant gives

$$\left(c_p - c_v\right) = T\left(\frac{\partial v}{\partial T}\right)_p \left(\frac{\partial p}{\partial T}\right)_v \tag{27}$$

4. Numerical Methods: Godunov Scheme with Roe Solver

In this section we will consider a numerical scheme to solve homogeneous Euler equation with initial condition by employing different *EOS*. The Euler equation in vector form:

$$\begin{cases} U_t + F(U)_x = 0 \\ U(x,0) = U_0(x) \end{cases} \tag{28}$$

where

$$U=\begin{pmatrix}\rho\\ \rho u\\ \epsilon\end{pmatrix}\ \text{and}\ F(U)=\begin{pmatrix}\rho u\\ \rho u^2+p\\ (\epsilon+p)u\end{pmatrix}$$

And $p=p(\rho,T)$

One of the methods to solve a 1D nonlinear hyperbolic systems is the Godunov scheme

4.1. Godunov Scheme

Suppose we have subdivided our domain $[a,b]$ in to N subintervals with $x_1=a$ and $x_{N+1}=b$, so that $\Delta x=\frac{b-a}{N}$.

Let us define $U_i^0:=\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}U_0(x)\,\mathrm{d}x$. Assume U_i^n at time t^n is known and that u_i^n is piecewise constant on $\left[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right]$. Then we solve exactly the local Riemann problem for $U_t+F(U)_x=0$ on $[x_i,x_{i+1}]\times[t^n,t^{n+1}]$ with initial condition

$$U(x,t^n)=\begin{cases}U_i^n & \text{for } x<x_{i+\frac{1}{2}}\\ U_{i+1}^n & \text{for } x\ge x_{i+\frac{1}{2}}\end{cases}$$

Let us denote the solution by $w_i^n(x,t)$. Then the solution $w_i^n(x,t)$ of the local Riemann problems are used to define the global solution v as

$$v(x,t)=\begin{cases}w_i^n(x,t) & \text{if } x_i\le x\le x_{i+\frac{1}{2}} \text{ and } t^n\le t\le t^{n+1}\\ w_{i-1}^n(x,t) & \text{if } x_{i-\frac{1}{2}}\le x\le x_i \text{ and } t^n\le t\le t^{n+1}\end{cases}$$

Then the solution U_i^{n+1} is defined by

$$U_i^{n+1}=\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}v(x,t^{n+1})\,\mathrm{d}x$$

Conservation form:

Since v is an exact solution on $\left[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}\right]$, we have

$$\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}v_t(x,t)\,\mathrm{d}x\mathrm{d}t+\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}F_x(v(x,t))\,\mathrm{d}x\mathrm{d}t=0$$

$$\Rightarrow\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(v(x,t^{n+1})-v(x,t^n)\right)\mathrm{d}x+\int_{t^n}^{t^{n+1}}\left(F\left(v\left(x_{i+\frac{1}{2}},t\right)\right)-F\left(v\left(x_{i-\frac{1}{2}},t\right)\right)\right)\mathrm{d}t=0$$

$$\Rightarrow\Delta x\left(U_i^{n+1}-U_i^n\right)+\Delta t\left(F\left(U_{i+\frac{1}{2}}\right)-F\left(U_{i-\frac{1}{2}}\right)\right)=0$$

where $U_{i+\frac{1}{2}}=v\left(x_{i+\frac{1}{2}},t\right)$ is constant for $t^n\le t\le t^{n+1}$

$$\Rightarrow U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F\left(U_{i+\frac{1}{2}} \right) - F\left(U_{i-\frac{1}{2}} \right) \right)$$

With the numerical flux

$$g\left(U_{i-1}, U_i\right) = F\left(U_{i-\frac{1}{2}} \right)$$

This scheme is called Godunov scheme.

Solving a Riemann problem exactly is not always an easy task. Then we may need to consider an approximate solution of the Riemann problem.

4.2. Riemann Problem for a Linear System

Suppose we have a linear system $U_t + AU_x = 0$ with initial condition

$$U(x,0) = \begin{cases} U_l & \text{for } x < 0 \\ U_r & \text{for } x \ge 0 \end{cases}$$

Let $\lambda_1 < \lambda_2 < \lambda_3$ are the eigenvalues and r_1, r_2, r_3 are the corresponding eigenvectors. Define $\alpha_i, i = 1,2,3$ such that

$$U_r - U_l = \sum_{i=1}^{3} \alpha_i r_i .$$

Then the solution of the Riemann problem is given by

$$U(x,t) = \begin{cases} U_l & \text{for } \frac{x}{t} < \lambda_1 \\ U_k & \text{for } \lambda_k \le \frac{x}{t} < \lambda_{k+1},\ k = 1,2 \\ U_r & \text{for } \frac{x}{t} \ge \lambda_3 \end{cases}$$

where

$$U_k = U_l + \sum_{i=1}^{k} \alpha_i r_i$$

A variety of approximate Riemann solvers have been proposed that can be applied more easily than the exact Riemann solver. One of the most popular Riemann solvers currently in use is due to Roe.

Godunov scheme with Roe approximation.

The idea is to replace the non-linear Riemann problem solved at each interface by an approximate one.

$$U_t + A\left(U_l, U_r\right) U_x = 0$$

where U_l and U_r are the left and right values and $A\left(U_l, U_r\right)$ satisfies

$$F\left(U_r\right) - F\left(U_l\right) = A\left(U_l, U_r\right)\left(U_r - U_l\right)$$

$A\left(U_l, U_r\right)$ is diagonalizable with real eigenvectors.

$A\left(U_l, U_r\right) \to F'(U)$ as $U_l, U_r \to U$

Conservation form of the Roe scheme.

The Roe scheme can be written in conservation form as

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[g\left(U_i^n, U_{i+1}^n\right) - g\left(U_{i-1}^n, U_i^n\right) \right]$$

where

$$g(u,w)=\frac{1}{2}\left(F(u)+F(w)-\sum_{i=1}^{3}|\lambda_i|\alpha_i r_i\right)$$

where λ_i and r_i are the eigenvalues and eigenvectors of $A(u,w)$ and $w-u=\sum_{i=1}^{3}\alpha_i r_i$.

The main task in the Roe scheme is the determination of the matrix of linearization A.

Now let us consider our equation (28) together with an equation of state of the form $p=p(\rho,T)$.

Then we approximate this non-linear system with an approximate linear system as follows:

Define $A(U_l,U_r)=DF(\bar{U})$ where

$$\bar{U}=\begin{pmatrix}\bar{\rho}\\ \overline{\rho u}\\ \bar{\epsilon}\end{pmatrix}$$

And

$$\bar{\rho}=\sqrt{\rho_l\rho_r}$$

$$\bar{u}=\frac{\sqrt{\rho_l}u_l+\sqrt{\rho_r}u_r}{\sqrt{\rho_l}+\sqrt{\rho_r}}$$

$$\bar{h}=\frac{\sqrt{\rho_l}h_l+\sqrt{\rho_r}h_r}{\sqrt{\rho_l}+\sqrt{\rho_r}}$$

$h=\frac{e+p}{\rho}$ is the specific enthalpy. These averages are called the Roe mean values. $A(U_l,U_r)$ satisfies the Roe conditions.

To solve our problem with the Roe scheme, we need to calculate the eigenvalues and their eigenvectors of the Jacobian matrix $DF(\bar{U})$ which are needed to compute the Roe flux. But for complex *EOS* the determination of these eigenvectors may not be simple. One way of determining the eigenvectors of this Jacobian is by expressing the Euler equation in terms of primitive variables $V=(\rho,u,T)^t$. We choose the temperature T as one of primitive variables than the pressure p, because in most equation of state p is expressed in terms of T.

Let $V_t+BV_x=0$ be the Euler equation in terms of the primitive variables V and $U_t+F(U)_x=0$ be in conservative variables. The approximate linear system is $U_t+DF(\bar{U})U_x=0$

$$\Rightarrow\frac{\partial U}{\partial V}V_t+DF(\bar{U})\frac{\partial U}{\partial V}V_x=0$$

$$\Rightarrow V_t+M^{-1}DF(\bar{U})MV_x=0$$

where $M=\frac{\partial U}{\partial V}$

$$B=M^{-1}DF(\bar{U})M$$

$\Rightarrow$ the matrices B and $DF(\bar{U})$ have identical eigenvectors.

Further more, if $B=P\Lambda P^{-1}$ then $DF(\bar{U})=MP\Lambda P^{-1}M^{-1}$. Then $R=MP$ is the right eigenvectors of $DF(\bar{U})$

4.3. Solving Euler Equation Using the Ideal Gas Law

In this section we solve one dimensional Euler equation with Ideal gas EOS. Consider the Euler equation (28) with the ideal gas law $p=\rho RT$.

Using (21), the change of internal energy is given by $\mathrm{d}e=c_v\mathrm{d}T$ which implies $e=c_vT$, and the total energy

ϵ is given by: $\epsilon = \rho c_v T + \frac{\rho u^2}{2}$.

Now let us express (28) in terms of the primitive variables $V = (\rho, u, T)^t$, so that we can apply the Roe scheme easily.

Continuity equation:

$$\rho_t + (\rho u)_x = 0$$

$$\Rightarrow \rho_t + u\rho_x + \rho u_x$$

Momentum equation:

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$\Rightarrow u_t + uu_x + \frac{p_x}{\rho} = 0$$

Now using $p_x = \frac{\partial p}{\partial \rho}\rho_x + \frac{\partial p}{\partial T}T_x$, $\frac{\partial p}{\partial \rho} = RT$, and $\frac{\partial p}{\partial T} = \rho R$, the momentum equation in terms of the primitive variables is

$$u_t + \frac{RT}{\rho}\rho_x + uu_x + RT_x = 0$$

Energy Equation:

$$\epsilon_t + ((\epsilon + p)u)_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}\rho_t + \frac{\partial \epsilon}{\partial u}u_t + \frac{\partial \epsilon}{\partial T}T_t + u\left[\frac{\partial \epsilon}{\partial \rho}\rho_x + \frac{\partial \epsilon}{\partial u}u_x + \frac{\partial \epsilon}{\partial T}T_x + p_x\right] + (\epsilon + p)u_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(\rho_t + u\rho_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}(u_t + uu_x) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(-\rho u_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}\left(\frac{-p_x}{\rho}\right) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \left(-\rho\frac{\partial \epsilon}{\partial \rho} + \epsilon + p\right)u_x + \left(-\frac{1}{\rho}\frac{\partial \epsilon}{\partial u} + u\right)p_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0 \tag{29}$$

Now, using $\frac{\partial \epsilon}{\partial \rho} = c_v T + \frac{u^2}{2}$, $\frac{\partial \epsilon}{\partial u} = \rho u$, and $\frac{\partial \epsilon}{\partial T} = \rho c_v$, the coefficient of u_x in Equation (29) becomes ρRT

and the coefficient of p_x is 0.

Then equation (29) reduces to

$$T_t + uT_x + \frac{RT}{c_v}u_x = 0 \tag{30}$$

Then the Euler equation in primitive variables is written as

$$\begin{pmatrix} \rho \\ u \\ T \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ \frac{RT}{\rho} & u & R \\ 0 & \frac{RT}{c_v} & u \end{pmatrix}\begin{pmatrix} \rho \\ u \\ T \end{pmatrix}_x = 0 \tag{31}$$

Or in vector form

$$V_t + BV_x = 0$$

Eigenvalues and eigenvectors of the coefficient matrix *B* of (31) are computed as follows.

$$|\lambda I - B| = \begin{vmatrix} \lambda - u & -\rho & 0 \\ -\dfrac{RT}{\rho} & \lambda - u & -R \\ 0 & -\dfrac{RT}{c_v} & \lambda - u \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - u)\left[(\lambda - u)^2 - \frac{R^2T}{c_v}\right] + \rho\left[(\lambda - u)\left(-\frac{RT}{\rho}\right)\right] = 0$$

$$\Rightarrow \lambda = u \text{ or } \left[(\lambda - u)^2 - \frac{R^2T}{c_v} - RT\right] = 0$$

$$\lambda_1 = u - c, \lambda_2 = u, \text{ and } \lambda_3 = u + c$$

where the local speed of sound c is given by

$$c^2 = \frac{R + c_v}{c_v} RT = \gamma RT$$

The matrix of the corresponding eigenvectors is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -\dfrac{c}{\rho} & 0 & \dfrac{c}{\rho} \\ \dfrac{RT}{c_v \rho} & -RT & \dfrac{RT}{c_v \rho} \end{pmatrix}$$

To compute the eigenvectors of the Jacobian $DF(U)$ we need to compute the matrix $M = \dfrac{\partial U}{\partial V}$ where $U = (\rho, \rho u, \epsilon)^t$ and $V = (\rho, u, T)^t$

$$\begin{pmatrix} \rho \\ \rho u \\ \epsilon \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho c_v T + \dfrac{\rho u^2}{2} \end{pmatrix}$$

Hence

$$M = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ c_v T + \dfrac{u^2}{2} & \rho u & c_v \rho \end{pmatrix}$$

The matrix *R* of eigenvectors of $DF(U)$ is given by:

$$R = MP = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ c_vT + \frac{u^2}{2} - uc + RT & c_v\rho(1-T) + \frac{u^2}{2} & c_vT + \frac{u^2}{2} + uc + RT \end{pmatrix}$$

Since the total specific enthalpy h is given by $h = c_vT + \frac{u^2}{2} + RT$ we can write the eigenvectors in terms of h as

$$R = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ h-uc & c_v\rho(1-T) + \frac{u^2}{2} & h+uc \end{pmatrix}$$

4.4. Solving Euler Equation Using the Van der Waals (VDW) EOS

Here we solve one dimensional Euler equation with VDW EOS. Consider again the euler equation (28) with VDW EOS $p = \frac{\rho RT}{1-b\rho} - a\rho^2$ where $a = \frac{27R^2T_c^2}{64P_c}$ and $b = \frac{RT_c}{8P_c}$, R is gas constant, P_c critical pressure, T_c critical temperature, and $T_r = \frac{T}{T_c}$ is the reduced temperature.

Again using (21), the change of internal energy is given by:

$$\mathrm{d}e = c_v\mathrm{d}T - \frac{1}{\rho^2}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho - p\right]\mathrm{d}\rho$$

Here, $\left(\frac{\partial p}{\partial T}\right)_\rho = \frac{\rho R}{1-b\rho}$, $T\left(\frac{\partial p}{\partial T}\right)_\rho = \frac{\rho RT}{1-b\rho}$, and $T\left(\frac{\partial p}{\partial T}\right)_\rho - p = a\rho^2$.

Integrating the above differential equation gives the internal energy $e = c_vT - a\rho$.

The total energy ϵ is given by:

$$\epsilon = \rho c_vT - a\rho^2 + \frac{\rho u^2}{2}$$

Now let us express (28) in terms of the primitive variables $V = (\rho, u, T)^t$

Continuity equation:

$$\rho_t + (\rho u)_x = 0$$

$$\Rightarrow \rho_t + u\rho_x + \rho u_x$$

Momentum equation:

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$\Rightarrow u_t + uu_x + \frac{p_x}{\rho} = 0$$

Here, $p_x = \frac{\partial p}{\partial \rho}\rho_x + \frac{\partial p}{\partial T}T_x$, $\frac{\partial p}{\partial \rho} = \frac{RT}{(1-b\rho)^2} - 2a\rho$, and $\frac{\partial p}{\partial T} = \frac{\rho R}{1-b\rho}$.

Hence the momentum equation is reduced to

$$u_t + uu_x + \left(\frac{RT}{\rho(1-b\rho)^2} - 2a\right)\rho_x + \frac{R}{1-b\rho}T_x$$

Energy Equation:

$$\epsilon_t + \left((\epsilon + p)u\right)_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}\rho_t + \frac{\partial \epsilon}{\partial u}u_t + \frac{\partial \epsilon}{\partial T}T_t + u\left[\frac{\partial \epsilon}{\partial \rho}\rho_x + \frac{\partial \epsilon}{\partial u}u_x + \frac{\partial \epsilon}{\partial T}T_x + p_x\right] + (\epsilon + p)u_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(\rho_t + u\rho_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}(u_t + uu_x) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(-\rho u_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}\left(\frac{-p_x}{\rho}\right) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \left(-\rho\frac{\partial \epsilon}{\partial \rho} + \epsilon + p\right)u_x + \left(-\frac{1}{\rho}\frac{\partial \epsilon}{\partial u} + u\right)p_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0 \tag{32}$$

Using $\frac{\partial \epsilon}{\partial \rho} = c_v T - 2a\rho + \frac{u^2}{2}$, $\frac{\partial \epsilon}{\partial u} = \rho u$, and $\frac{\partial \epsilon}{\partial T} = \rho c_v$.

The coefficient of u_x in (32) becomes

$$\frac{\rho RT}{1-b\rho}$$

and the coefficient of p_x is 0.

Then (32) reduces to

$$T_t + uT_x + \frac{RT}{c_v(1-b\rho)}u_x = 0 \tag{33}$$

The Euler equation is written as

$$\begin{pmatrix}\rho \\ u \\ T\end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ a_{21} & u & a_{23} \\ 0 & a_{32} & u\end{pmatrix}\begin{pmatrix}\rho \\ u \\ T\end{pmatrix}_x = 0 \tag{34}$$

where $a_{21} = \frac{RT}{\rho(1-b\rho)^2} - 2a$, $a_{23} = \frac{R}{1-b\rho}$, and $a_{32} = \frac{RT}{c_v(1-b\rho)}$.

Eigenvalues and eigenvectors of the coefficient matrix B of (34) are computed as follows.

$$|\lambda I - B| = \begin{vmatrix} \lambda - u & -\rho & 0 \\ -a_{21} & \lambda - u & -a_{23} \\ 0 & -a_{32} & \lambda - u\end{vmatrix} = 0$$

$$\Rightarrow (\lambda - u)\left[(\lambda - u)^2 - a_{23}a_{32}\right] + \rho\left[(\lambda - u)(-a_{21})\right] = 0$$

$$\Rightarrow \lambda = u \text{ or } \left[(\lambda - u)^2 - a_{23}a_{32} - \rho a_{21}\right] = 0$$

$$\lambda_1 = u - c, \lambda_2 = u \text{ and } \lambda_3 = u + c$$

where the local speed of sound c is defined as

$$c^2 = a_{23}a_{32} + \rho a_{21}$$

The matrix of the corresponding eigenvectors is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -\dfrac{c}{\rho} & 0 & \dfrac{c}{\rho} \\ \dfrac{a_{32}}{\rho} & -\dfrac{a_{21}}{a_{23}} & \dfrac{a_{32}}{\rho} \end{pmatrix}$$

To compute the eigenvectors of the Jacobian $DF(U)$ we need to compute the matrix $M = \dfrac{\partial U}{\partial V}$ where $U = (\rho, \rho u, e)^t$ and $V = (\rho, u, T)^t$

$$\begin{pmatrix} \rho \\ \rho u \\ \epsilon \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho c_v T - a\rho^2 + \dfrac{\rho u^2}{2} \end{pmatrix}$$

Hence

$$M = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ c_v T - 2a\rho + \dfrac{u^2}{2} & \rho u & c_v \rho \end{pmatrix}$$

The matrix R of eigenvectors of $DF(U)$ is given by:

$$R = MP = \begin{pmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ m_{31} - uc + m_{33}\dfrac{a_{32}}{\rho} & m_{31} - m_{33}\dfrac{a_{21}}{a_{23}} & m_{31}uc + m_{33}\dfrac{a_{32}}{\rho} \end{pmatrix}$$

where $m_{31} = c_v T - 2a\rho + \dfrac{u^2}{2}$ and $m_{33} = \rho c_v$

Since the total specific enthalpy h is given by $h = m_{31} + m_{33}\dfrac{a_{32}}{\rho}$ we can write the eigenvectors in terms of h as

$$R = \begin{pmatrix} 1 & 1 & \\ u - c & u & u + c \\ h - uc & r_{23} & h + uc \end{pmatrix}$$

where $r_{23} = m_{31} - m_{33}\dfrac{a_{21}}{a_{23}}$.

4.5. Solving Euler Equation Using the Soave-Redlich-Kwong (SRK) EOS

Let us consider (28) with SRK EOS

$$p = \frac{\rho RT}{1 - b\rho} - \frac{a\rho^2}{1 + b\rho}$$

where $a = a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)^2$, $a_1 = \dfrac{0.42748R^2T_c^2}{P_c}$, $F_w = 0.48+1.5746w-0.176w^2$, $b = \dfrac{0.078664RT_c}{P_c}$, w is the accentric factor R is gas constant, P_c critical pressure, T_c critical temperature, and T_r is the reduced temperature.

The internal energy is given by:

$$\mathrm{d}e = c_v\mathrm{d}T - \frac{1}{\rho^2}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho - p\right]\mathrm{d}\rho$$

$$\left(\frac{\partial p}{\partial T}\right)_\rho = \frac{\rho R}{1-b\rho} + \frac{\rho^2}{1+b\rho}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)\left(\frac{f_w}{\sqrt{TT_c}}\right)$$

$$\Rightarrow T\left(\frac{\partial p}{\partial T}\right)_\rho = \frac{\rho RT}{1-b\rho} + \frac{\rho^2}{1+b\rho}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)f_w\sqrt{T_r}$$

$$\Rightarrow T\left(\frac{\partial p}{\partial T}\right)_\rho - p = \frac{\rho^2}{1+b\rho}a_1(1+f_w\left(1-\sqrt{T_r}\right)f_w\sqrt{T_r} + \frac{\rho^2}{1+b\rho}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)^2$$

$$= \frac{\rho^2}{1+b\rho}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)(1+f_w) = \frac{\rho^2}{1+b\rho}af_{wl}$$

where $f_{wl} = \dfrac{1+f_w}{1+f_w\left(1-\sqrt{T_r}\right)}$.

After integrating the differential equation of the internal energy, we get

$$e = c_vT + \frac{af_{wl}}{b}\log(1+b\rho)$$

The total energy ϵ is given by:

$$\epsilon = \rho c_vT + \frac{a\rho f_{wl}}{b}\log(1+b\rho) + \frac{\rho u^2}{2}$$

Continuity equation:

$$\rho_t + (\rho u)_x = 0$$

$$\Rightarrow \rho_t + u\rho_x + \rho u_x$$

Momentum equation:

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0$$

$$\Rightarrow u_t + uu_x + \frac{p_x}{\rho} = 0$$

using $p_x = \dfrac{\partial p}{\partial \rho}\rho_x + \dfrac{\partial p}{\partial T}T_x$, $\dfrac{\partial p}{\partial \rho} = \dfrac{RT}{(1-b\rho)^2} - \dfrac{a\rho(2+b\rho)}{(1+b\rho)^2}$, and $\dfrac{\partial p}{\partial T} = \dfrac{\rho R}{1-b\rho} + \dfrac{\rho^2 af_w f_{wl}}{(1+b\rho)\sqrt{TT_c}(1+f_w)}$ the momentum equation is written as

$$u_t + uu_x + \left(\frac{RT}{\rho(1-b\rho)^2} - \frac{a(2+b\rho)}{(1+b\rho)^2}\right)\rho_x + \left(\frac{R}{1-b\rho} + \frac{\rho af_w f_{wl}}{(1+b\rho)\sqrt{TT_c}(1+f_w)}\right)T_x$$

Energy Equation:

$$\epsilon_t + ((\epsilon + p)u)_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}\rho_t + \frac{\partial \epsilon}{\partial u}u_t + \frac{\partial \epsilon}{\partial T}T_t + u\left[\frac{\partial \epsilon}{\partial \rho}\rho_x + \frac{\partial \epsilon}{\partial u}u_x + \frac{\partial \epsilon}{\partial T}T_x + p_x\right] + (e+p)u_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(\rho_t + u\rho_x) + (\epsilon + p)u_x + \frac{\partial \varepsilon}{\partial u}(u_t + uu_x) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(-\rho u_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}\left(\frac{-p_x}{\rho}\right) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \left(-\rho\frac{\partial \epsilon}{\partial \rho} + \epsilon + p\right)u_x + \left(-\frac{1}{\rho}\frac{\partial \epsilon}{\partial u} + u\right)p_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0 \tag{35}$$

$$\frac{\partial \epsilon}{\partial \rho} = c_v T - \frac{af_{wl}}{b}\log(1+b\rho) - \frac{\rho af_{wl}}{1+b\rho} + \frac{u^2}{2}$$

$$\frac{\partial \epsilon}{\partial u} = \rho u$$

$$\frac{\partial \epsilon}{\partial T} = \rho c_v + \frac{\rho af_w f_{wl}^2}{b\sqrt{TT_c}(1+f_w)}\log(1+b\rho)$$

The coefficient of u_x in Equation (35) becomes

$$\frac{\rho RT}{1-b\rho} - \frac{a\rho^2}{1+b\rho} + \frac{\rho^2 af_{wl}}{1+b\rho}$$

And the coefficient of p_x is 0.
Notations: Let a_{321} denote the coefficient of u_x and a_{322} denote the coefficient of T_t *i.e.*

$$a_{322} = c_v\rho + \frac{\rho af_w f_{wl}^2}{b\sqrt{TT_c}(1+f_w)}\log(1+b\rho)$$

Then Equation (35) reduces to

$$T_t + uT_x + \frac{a_{321}}{a_{322}}u_x = 0 \tag{36}$$

The Euler equation is written as

$$\begin{pmatrix}\rho\\ u\\ T\end{pmatrix}_t + \begin{pmatrix} u & \rho & 0\\ a_{21} & u & a_{23}\\ 0 & a_{32} & u\end{pmatrix}\begin{pmatrix}\rho\\ u\\ T\end{pmatrix}_x = 0 \tag{37}$$

where,

$$a_{21} = \frac{RT}{\rho(1-b\rho)^2} - \frac{a(2+b\rho)}{(1+b\rho)^2}$$

$$a_{23} = \frac{R}{1-b\rho} + \frac{\rho af_w f_{wl}}{(1+b\rho)\sqrt{TT_c}(1+f_w)}$$

$$a_{32} = \frac{a_{321}}{a_{322}}$$

Eigenvalues and eigenvectors of the coefficient matrix A of Equation (37) are given as follows.

$$|\lambda I - B| = \begin{vmatrix} \lambda - u & -\rho & 0 \\ -a_{21} & \lambda - u & -a_{23} \\ 0 & -a_{32} & \lambda - u \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - u)\left[(\lambda - u)^2 - a_{23}a_{32}\right] + \rho\left[(\lambda - u)(-a_{21})\right] = 0$$

$$\Rightarrow \lambda = u \text{ or } \left[(\lambda - u)^2 - a_{23}a_{32} - \rho a_{21}\right] = 0$$

$$\lambda_1 = u - c, \lambda_2 = u \text{ and } \lambda_3 = u + c$$

where

$$c^2 = a_{23}a_{32} + \rho a_{21}$$

The matrix of the corresponding eigenvectors is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{c}{\rho} & 0 & \frac{c}{\rho} \\ \frac{a_{32}}{\rho} & -\frac{a_{21}}{a_{23}} & \frac{a_{32}}{\rho} \end{pmatrix}$$

To compute the eigenvectors of the Jacobian $DF(U)$ we need to compute the matrix $M = \frac{\partial U}{\partial V}$ where $U = (\rho, \rho u, e)^t$ and $V = (\rho, u, T)^t$

$$\begin{pmatrix} \rho \\ \rho u \\ \epsilon \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho c_v T - \frac{af_{wl}}{b}\log(1 + b\rho)\rho + \frac{\rho u^2}{2} \end{pmatrix}$$

Hence

$$M = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ m_{31} & \rho u & m_{33} \end{pmatrix}$$

where $m_{31} = c_v T - \frac{af_{wl}}{b}\log(1 + b\rho) - \frac{\rho af_{wl}}{1 + b\rho} + \frac{u^2}{2}$

And $m_{33} = c_v \rho + \frac{\rho af_w f_{wl}^2}{b\sqrt{TT_c}\,(1 + f_w)}\log(1 + b\rho)$

The matrix R of eigenvectors of $DF(U)$ is given by:

$$R = MP = \begin{pmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ m_{31} - uc + m_{33}\frac{a_{32}}{\rho} & m_{31} - m_{33}\frac{a_{21}}{a_{23}} & m_{31}uc + m_{33}\frac{a_{32}}{\rho} \end{pmatrix}$$

Since the specific enthalpy h is given by $h = m_{31} + m_{33}\frac{a_{32}}{\rho}$ we can write the eigenvectors in terms of h as

$$R=\begin{pmatrix} 1 & 1 & \\ u-c & u & u+c \\ h-uc & r_{23} & h+uc \end{pmatrix}$$

where $r_{23}=m_{31}-m_{33}\dfrac{a_{21}}{a_{23}}$.

4.6. Solving Euler Equation Using the Peng-Robinson (PR) EOS

Let us consider (28) with PR EOS.

$$p=\frac{\rho RT}{1-b\rho}-\frac{a\rho^2}{1+2b\rho-b^2\rho^2}$$

where $a=a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)^2$, $a_1=\dfrac{0.45724R^2T_c^2}{P_c}$, $F_w=0.37464+1.54226w-0.26992w^2$, $b=\dfrac{0.07780RT_c}{P_c}$, w is the accentric factor R is gas constant, P_c critical pressure, T_c critical temperature, and T_r is reduced temperature.

The internal energy is given by:

$$\mathrm{d}e=c_v\mathrm{d}T-\frac{1}{\rho^2}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho-p\right]\mathrm{d}\rho$$

Here,

$$\left(\frac{\partial p}{\partial T}\right)_\rho=\frac{\rho R}{1-b\rho}+\frac{\rho^2}{1+2b\rho-b^2\rho^2}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)\left(\frac{f_w}{\sqrt{TT_c}}\right)$$

$$\Rightarrow T\left(\frac{\partial p}{\partial T}\right)_\rho=\frac{\rho RT}{1-b\rho}+\frac{\rho^2}{1+2b\rho-b^2\rho^2}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)f_w\sqrt{T_r}$$

$$\begin{aligned}\Rightarrow T\left(\frac{\partial p}{\partial T}\right)_\rho-p&=\frac{\rho^2}{1+2b\rho-b^2\rho^2}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)f_w\sqrt{T_r}\\&\quad+\frac{\rho^2}{1+2b\rho-b^2\rho^2}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)^2\\&=\frac{\rho^2}{1+2b\rho-b^2\rho^2}a_1\left(1+f_w\left(1-\sqrt{T_r}\right)\right)\left(1+f_w\right)\\&=\frac{\rho^2}{1+2b\rho-b^2\rho^2}af_{wl}\end{aligned}$$

where $f_{wl}=\dfrac{1+f_w}{1+f_w\left(1-\sqrt{T_r}\right)}$.

Integrating the above differential equation for internal energy we get

$$e=c_vT+\frac{af_{wl}}{\sqrt{8}b}\log\frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}$$

The total energy ϵ is given by:

$$\epsilon=\rho c_vT+\frac{af_{wl}}{\sqrt{8}b}\log\frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}\rho+\frac{\rho u^2}{2}$$

Continuity equation:

$$\rho_t + (\rho u)_x = 0$$

$$\Rightarrow \rho_t + u\rho_x + \rho u_x = 0$$

Momentum equation:

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Using the continuity equation, it is reduced to

$$\Rightarrow u_t + uu_x + \frac{p_x}{\rho} = 0$$

Here, $p_x = \frac{\partial p}{\partial \rho}\rho_x + \frac{\partial p}{\partial T}T_x$, $\frac{\partial p}{\partial \rho} = \frac{RT}{(1-b\rho)^2} - \frac{2a\rho(1+b\rho)}{(1+2b\rho-b^2\rho^2)^2}$, and

$\frac{\partial p}{\partial T} = \frac{\rho R}{1-b\rho} + \frac{\rho^2 af_w f_{wl}}{(1+2b\rho-b^2\rho^2)\sqrt{TT_c}(1+f_w)}$.

The momentum equation is written as

$$u_t + uu_x + \left(\frac{RT}{\rho(1-b\rho)^2} - \frac{2a(1+b\rho)}{(1+2b\rho-b^2\rho^2)^2}\right)\rho_x + \left(\frac{R}{1-b\rho} + \frac{\rho af_w f_{wl}}{(1+2b\rho-b^2\rho^2)\sqrt{TT_c}(1+f_w)}\right)T_x$$

Energy Equation:

$$\epsilon_t + ((\epsilon + p)u)_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}\rho_t + \frac{\partial \epsilon}{\partial u}u_t + \frac{\partial \epsilon}{\partial T}T_t + u\left[\frac{\partial \epsilon}{\partial \rho}\rho_x + \frac{\partial \epsilon}{\partial u}u_x + \frac{\partial \epsilon}{\partial T}T_x + p_x\right] + (\epsilon + p)u_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(\rho_t + u\rho_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}(u_t + uu_x) + up_x + \frac{\partial e}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(-\rho u_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}\left(\frac{-p_x}{\rho}\right) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \left(-\rho\frac{\partial \epsilon}{\partial \rho} + \epsilon + p\right)u_x + \left(-\frac{1}{\rho}\frac{\partial \epsilon}{\partial u} + u\right)p_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0 \tag{38}$$

Using

$$\frac{\partial \epsilon}{\partial \rho} = c_v T + \frac{af_{wl}}{\sqrt{8}b}\log\frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}\rho - \frac{\rho}{1+2b\rho-b^2\rho^2}af_{wl} + \frac{u^2}{2}$$

$$\frac{\partial \epsilon}{\partial u} = \rho u$$

And

$$\frac{\partial \epsilon}{\partial T} = \rho c_v - \frac{\rho af_w f_{wl}^2}{\sqrt{8}b\sqrt{TT_c}(1+f_w)}\log\frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho},$$

The coefficient of u_x in (38) becomes

$$\frac{\rho RT}{1-b\rho} - \frac{a\rho^2}{1+2b\rho-b^2\rho^2} + \frac{\rho^2 af_{wl}}{1+2b\rho-b^2\rho^2}$$

And the coefficient of p_x is 0.

Notations: Let a_{321} denote the coefficient of u_x and a_{322} denote the coefficient of T_t

$$a_{322} = c_v\rho - \frac{\rho a f_w f_{wl}^2}{\sqrt{8}b\sqrt{TT_c}\left(1+f_w\right)} \log \frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}$$

Then (38) reduces to

$$T_t + uT_x + \frac{a_{321}}{a_{322}} u_x = 0 \tag{39}$$

The Euler equation is written as

$$\begin{pmatrix} \rho \\ u \\ T \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ a_{21} & u & a_{23} \\ 0 & a_{32} & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ T \end{pmatrix}_x = 0 \tag{40}$$

where,

$$a_{21} = \frac{RT}{\rho\left(1-b\rho\right)^2} - \frac{2a\left(1+b\rho\right)}{\left(1+2b\rho-b^2\rho^2\right)^2}$$

$$a_{23} = \frac{R}{1-b\rho} + \frac{\rho a f_w f_{wl}}{\left(1+2b\rho-b^2\rho^2\right)\sqrt{TT_c}\left(1+f_w\right)}$$

$$a_{32} = \frac{a_{321}}{a_{322}}$$

$$\left|\lambda I - B\right| = \begin{vmatrix} \lambda-u & -\rho & 0 \\ -a_{21} & \lambda-u & -a_{23} \\ 0 & -a_{32} & \lambda-u \end{vmatrix} = 0$$

$$\Rightarrow \left(\lambda-u\right)\left[\left(\lambda-u\right)^2 - a_{23}a_{32}\right] + \rho\left[\left(\lambda-u\right)\left(-a_{21}\right)\right] = 0$$

$$\Rightarrow \lambda = u \text{ or } \left[\left(\lambda-u\right)^2 - a_{23}a_{32} - \rho a_{21}\right] = 0$$

$$\lambda_1 = u-c, \lambda_2 = u \text{ and } \lambda_3 = u+c$$

where

$$c^2 = a_{23}a_{32} + \rho a_{21}$$

The matrix of the corresponding eigenvectors is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{c}{\rho} & 0 & \frac{c}{\rho} \\ \frac{a_{32}}{\rho} & -\frac{a_{21}}{a_{23}} & \frac{a_{32}}{\rho} \end{pmatrix}$$

To compute the eigenvectors of the Jacobian $DF\left(U\right)$ we need to compute the matrix $M = \frac{\partial U}{\partial V}$ where $U = \left(\rho, \rho u, e\right)^t$ and $V = \left(\rho, u, T\right)^t$

$$\begin{pmatrix} \rho \\ \rho u \\ \epsilon \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho c_v T + \frac{af_{wl}}{\sqrt{8b}} \log \frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}\rho + \frac{\rho u^2}{2} \end{pmatrix}$$

Hence

$$M = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ m_{31} & \rho u & m_{33} \end{pmatrix}$$

where $m_{31} = c_v T + \frac{af_{wl}}{\sqrt{8b}} \log \frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}\rho - \frac{\rho}{1+2b\rho - b^2\rho^2} af_{wl} + \frac{u^2}{2}$

and $m_{33} = \rho c_v - \frac{\rho a f_w f_{wl}^2}{\sqrt{8b}\sqrt{TT_c}\left(1+f_w\right)} \log \frac{\sqrt{2}+1-b\rho}{\sqrt{2}-1+b\rho}$.

The matrix R of eigenvectors of $DF(U)$ is given by:

$$R = MP = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ m_{31} - uc + m_{33}\frac{a_{32}}{\rho} & m_{31} - m_{33}\frac{a_{21}}{a_{23}} & m_{31}uc + m_{33}\frac{a_{32}}{\rho} \end{pmatrix}$$

Since the specific enthalpy h is given by $h = m_{31} + m_{33}\frac{a_{32}}{\rho}$ we can write the eigenvectors in terms of h as

$$R = \begin{pmatrix} 1 & 1 & \\ u-c & u & u+c \\ h-uc & r_{23} & h+uc \end{pmatrix}$$

where $r_{23} = m_{31} - m_{33}\frac{a_{21}}{a_{23}}$.

4.7. Solving Euler Equation Using the Benedict-Webb-Rubin-Starling (BWRS) EOS

Let us consider (28) with BWRS EOS.

$$p = \rho RT + \left(BRT - A - \frac{C}{T^2} + \frac{D}{T^3} - \frac{E}{T^4}\right)\rho^2 + \left(bRT - a - \frac{d}{T}\right)\rho^3 + \alpha\left(a + \frac{d}{T}\right)\rho^6 + \frac{c\rho^3}{T^2}\left(1+\gamma\rho^2\right)\exp\left(-\gamma\rho^2\right)$$

The internal energy is given by:

$$\mathrm{d}e = c_v \mathrm{d}T - \frac{1}{\rho^2}\left[T\left(\frac{\partial p}{\partial T}\right)_\rho - p\right]\mathrm{d}\rho$$

$$\left(\frac{\partial p}{\partial T}\right)_\rho = \rho R + (BR + \frac{2C}{T^2} - \frac{3D}{T^4} + \frac{4E}{T^5})\rho^2 + (bR + \frac{d}{T^2})\rho^3 - \frac{\alpha d}{T^2}\rho^6 - \frac{2c\rho^3}{T^3}(1+\gamma\rho^2)\exp(-\gamma\rho^2)$$

$$\Rightarrow T\left(\frac{\partial p}{\partial T}\right)_\rho - p = \left(A + \frac{3C}{T^2} - \frac{4D}{T^3} + \frac{5E}{T^4}\right)\rho^2 + \left(a + \frac{2d}{T}\right)\rho^3 - \alpha\left(a + \frac{2d}{T}\right)\rho^6 - \frac{3c}{T^2}\left(\rho^3 + \gamma\rho^5\right)\exp\left(-\gamma\rho^2\right)$$

$$\Rightarrow e = c_v T - \left(A + \frac{3C}{T^2} - \frac{4D}{T^3} + \frac{5E}{T^4}\right)\rho - \left(a + \frac{2d}{T}\right)\frac{\rho^2}{2} + \alpha\left(a + \frac{2d}{T}\right)\frac{\rho^5}{5} - \frac{3c}{T^2}\left(\frac{1}{\gamma} + \frac{\rho^2}{2}\right)\exp\left(-\gamma\rho^2\right)$$

The total energy ϵ is given by:

$$\epsilon = \rho c_v T - \left(A + \frac{3C}{T^2} - \frac{4D}{T^3} + \frac{5E}{T^4}\right)\rho^2 - \left(a + \frac{2d}{T}\right)\frac{\rho^3}{2} + \alpha\left(a + \frac{2d}{T}\right)\frac{\rho^6}{5} - \frac{3c}{T^2}\left(\frac{\rho}{\gamma} + \frac{\rho^3}{2}\right)\exp\left(-\gamma\rho^2\right) + \rho\frac{u^2}{2}$$

Continuity equation:

$$\rho_t + \left(\rho u\right)_x = 0$$

$$\Rightarrow \rho_t + u\rho_x + \rho u_x$$

Momentum equation:

$$\left(\rho u\right)_t + \left(\rho u^2 + p\right)_x = 0$$

$$\Rightarrow u_t + u u_x + \frac{p_x}{\rho} = 0$$

$$p_x = \frac{\partial p}{\partial \rho}\rho_x + \frac{\partial p}{\partial T}T_x$$

$$\frac{\partial p}{\partial \rho} = RT + 2\left(BRT - A - \frac{C}{T^2} + \frac{D}{T^3} - \frac{E}{T^4}\right)\rho + 3\left(bRT - a - \frac{d}{T}\right)\rho^2 + 6\alpha\left(a + \frac{d}{T}\right)\rho^5 + \frac{c}{T^2}\left(3\rho^2 + 3\gamma\rho^4 - 2\gamma^2\rho^6\right)\exp\left(-\gamma\rho^2\right)$$

$$\frac{\partial p}{\partial T} = \rho R + \left(BR + \frac{2C}{T^3} - \frac{3D}{T^4} + \frac{4E}{T^5}\right)\rho^2 + \left(bR + \frac{d}{T^2}\right)\rho^3 - \alpha\left(\frac{d}{T}\right)\rho^6 - \frac{2c\rho^3}{T^2}\left(1 + \gamma\rho^2\right)\exp\left(-\gamma\rho^2\right)$$

Let

$$a_{21} = \frac{\frac{\partial p}{\partial \rho}}{\rho} = \frac{RT}{\rho} + 2\left(BRT - A - \frac{C}{T^2} + \frac{D}{T^3} - \frac{E}{T^4}\right) + 3\left(bRT - a - \frac{d}{T}\right)\rho + 6\alpha\left(a + \frac{d}{T}\right)\rho^4 + \frac{c\rho}{T^2}\left(3 + 3\gamma\rho^2 - 2\gamma^2\rho^4\right)\exp\left(-\gamma\rho^2\right)$$

$$a_{23} = \frac{\frac{\partial p}{\partial T}}{\rho} = R + \left(BR + \frac{2C}{T^3} - \frac{3D}{T^4} + \frac{4E}{T^5}\right)\rho + \left(bR + \frac{d}{T^2}\right)\rho^2 - \left(\frac{\alpha d}{T^2}\right)\rho^5 - \frac{2c\rho^2}{T^2}\left(1 + \gamma\rho^2\right)\exp\left(-\gamma\rho^2\right)$$

The momentum equation is written as

$$u_t + u u_x + a_{21}\rho_x + a_{23}T_x = 0$$

Energy Equation:

$$\epsilon_t + \left(\left(\epsilon + p\right)u\right)_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}\rho_t + \frac{\partial \epsilon}{\partial u}u_t + \frac{\partial \epsilon}{\partial T}T_t + u\left[\frac{\partial \epsilon}{\partial \rho}\rho_x + \frac{\partial \epsilon}{\partial u}u_x + \frac{\partial \epsilon}{\partial T}T_x + p_x\right] + \left(\epsilon + p\right)u_x = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(\rho_t + u\rho_x) + (\epsilon + p)u_x + \frac{\partial \varepsilon}{\partial u}(u_t + uu_x) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \frac{\partial \epsilon}{\partial \rho}(-\rho u_x) + (\epsilon + p)u_x + \frac{\partial \epsilon}{\partial u}\left(\frac{-p_x}{\rho}\right) + up_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0$$

$$\Rightarrow \left(-\rho\frac{\partial \epsilon}{\partial \rho} + \epsilon + p\right)u_x + \left(-\frac{1}{\rho}\frac{\partial \epsilon}{\partial u} + u\right)p_x + \frac{\partial \epsilon}{\partial T}(T_t + uT_x) = 0 \quad (41)$$

$$\frac{\partial \epsilon}{\partial \rho} = c_v T - 2\left(A + \frac{3C}{T^2} - \frac{4D}{T^3} + \frac{5E}{T^4}\right)\rho - \frac{3}{2}\left(a + \frac{2d}{T}\right)\rho^2$$
$$+\frac{6}{5}\alpha\left(a + \frac{2d}{T}\right)\rho^5 - \frac{3c}{T^2}\left(\frac{1}{\gamma} - \frac{\rho^2}{2} - \gamma\rho^4\right)\exp\left(-\gamma\rho^2\right) + \frac{u^2}{2}$$

$$\frac{\partial \epsilon}{\partial u} = \rho u$$

$$\frac{\partial \epsilon}{\partial T} = \rho c_v + \left(\frac{6C}{T^3} - \frac{12D}{T^4} + \frac{20E}{T^5}\right)\rho^2 + \frac{d}{T^2}\rho^3 - \frac{2\alpha d}{5T^2}\rho^6 + \frac{6c}{T^3}\left(\frac{\rho}{\gamma} + \frac{\rho^3}{2}\right)\exp\left(-\gamma\rho^2\right)$$

The coefficient of u_x in Equation (41) becomes

$$\rho RT + \left(BRT + \frac{2C}{T^2} - \frac{3D}{T^3} + \frac{4E}{T^4}\right)\rho^2 + \left(bRT + \frac{d}{T}\right)\rho^3 - \alpha\frac{d}{T}\rho^6 - \frac{2c}{T^2}\left(\rho^3 + \gamma\rho^5\right)\exp\left(-\gamma^2\right)$$

and the coefficient of p_x is 0.

Notations: Let a_{321} denote the coefficient of u_x and a_{322} denote the coefficient of T_t *i.e*, $a_{322} = \frac{\partial \epsilon}{\partial T}$
Then (41) reduces to

$$T_t + uT_x + \frac{a_{321}}{a_{322}}u_x = 0 \quad (42)$$

The Euler equation is written as

$$\begin{pmatrix} \rho \\ u \\ T \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ a_{21} & u & a_{23} \\ 0 & a_{32} & u \end{pmatrix}\begin{pmatrix} \rho \\ u \\ T \end{pmatrix}_x = 0 \quad (43)$$

where,

$$a_{32} = \frac{a_{321}}{a_{322}}$$

Eigenvalues and eigenvectors of the coefficient matrix B of Equation (43) are computed as follows.

$$|\lambda I - B| = \begin{vmatrix} \lambda - u & -\rho & 0 \\ -a_{21} & \lambda - u & -a_{23} \\ 0 & -a_{32} & \lambda - u \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - u)\left[(\lambda - u)^2 - a_{23}a_{32}\right] + \rho\left[(\lambda - u)(-a_{21})\right] = 0$$

$$\Rightarrow \lambda = u \text{ or } \left[(\lambda - u)^2 - a_{23}a_{32} - \rho a_{21}\right] = 0$$

$$\lambda_1 = u - c, \lambda_2 = u \text{ and } \lambda_3 = u + c$$

where

$$c^2 = a_{23}a_{32} + \rho a_{21}$$

The matrix of the corresponding eigenvectors is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -\dfrac{c}{\rho} & 0 & \dfrac{c}{\rho} \\ \dfrac{a_{32}}{\rho} & -\dfrac{a_{21}}{a_{23}} & \dfrac{a_{32}}{\rho} \end{pmatrix}$$

To compute the eigenvectors of the Jacobian $DF(U)$ we need to compute the matrix $M = \dfrac{\partial U}{\partial V}$ where $U = (\rho, \rho u, e)^t$ and $V = (\rho, u, T)^t$

$$\begin{pmatrix} \rho \\ \rho u \\ \epsilon \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho c_v T - \left(A + \dfrac{3C}{T^2} - \dfrac{4D}{T^3} + \dfrac{5E}{T^4}\right)\rho^2 + \left(a + \dfrac{2d}{T}\right)\dfrac{\rho^3}{2} + \alpha\left(a + \dfrac{2d}{T}\right)\dfrac{\rho^6}{5} - \dfrac{3c}{T^2}\left(\dfrac{\rho}{\gamma} + \dfrac{\rho^3}{2}\right)\exp\left(-\gamma\rho^2\right) + \rho\dfrac{u^2}{2} \end{pmatrix}$$

Hence

$$M = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ m_{31} & \rho u & m_{33} \end{pmatrix}$$

where

$$m_{31} = c_v T - 2\left(A + \frac{3C}{T^2} - \frac{4D}{T^3} + \frac{5E}{T^4}\right)\rho - \frac{3}{2}\left(a + \frac{2d}{T}\right)\rho^2 + \frac{6}{5}\alpha\left(a + \frac{2d}{T}\right)\rho^5 - \frac{3c}{T^2}\left(\frac{1}{\gamma} - \frac{\rho^2}{2} - \gamma\rho^4\right)\exp\left(-\gamma\rho^2\right) + \frac{u^2}{2}$$

and $m_{33} = \rho c_v + \left(\dfrac{6c}{T^3} - \dfrac{12D}{T^4} + \dfrac{20E}{T^5}\right)\rho^2 + \dfrac{d}{T^2}\rho^3 - \dfrac{\alpha d}{5T^2}\rho^6 + \dfrac{6c}{T^3}\left(\dfrac{\rho}{\gamma} + \dfrac{\rho^3}{2}\right)\exp\left(-\gamma\rho^2\right)$

The matrix R of eigenvectors of $DF(U)$ is given by:

$$R = MP = \begin{pmatrix} 1 & 1 & 1 \\ u - c & u & u + c \\ m_{31} - uc + m_{33}\dfrac{a_{32}}{\rho} & m_{31} - m_{33}\dfrac{a_{21}}{a_{23}} & m_{31}uc + m_{33}\dfrac{a_{32}}{\rho} \end{pmatrix}$$

Since the specific enthalpy h is given by $h = m_{31} + m_{33}\dfrac{a_{32}}{\rho}$ we can write the eigenvectors in terms of h as

$$R = \begin{pmatrix} 1 & 1 & \\ u - c & u & u + c \\ h - uc & r_{23} & h + uc \end{pmatrix}$$

where $r_{32} = m_{31} - m_{33}\dfrac{a_{21}}{a_{23}}$.

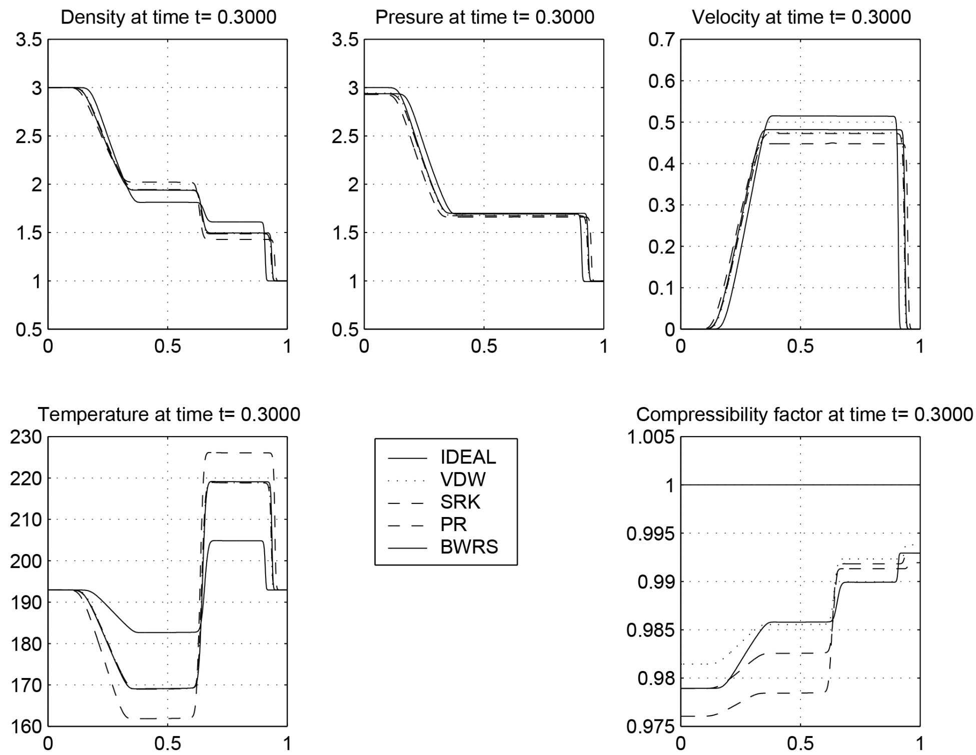

Figure 1. The results obtained by solving the homogeneous Euler equation by employing the ideal gas law and the other four equation of states.

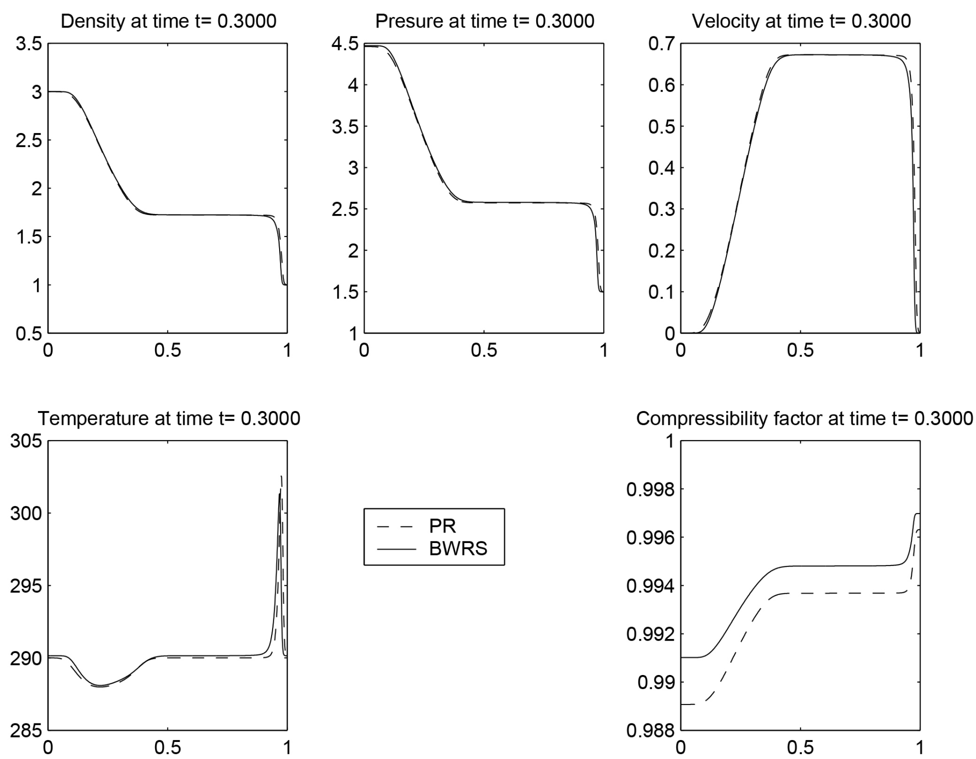

Figure 2. The results obtained by solving the Euler equation (including the source term) by employing the *PR* and *BWRS EOS*.

4.8. Application of the Roe solver

Now to apply the Roe scheme on (28), on each cell $\left[x_i, x_{i+1}\right]$, we approximate the system by

$$U_t + AU_x = 0$$

$$U\left(x,t^n\right) = \begin{cases} U_i^n & \text{for } x < x_{i+\frac{1}{2}} \\ U_{i+1}^n & \text{for } x > x_{i+\frac{1}{2}} \end{cases}$$

where $A = D\left(F\bar{U}\right)$ and $\bar{U}$ is determined from the Roe averages. The solution is determined as:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[g\left(U_i^n, U_{i+1}^n\right) - g\left(U_{i-1}^n, U_i^n\right)\right]$$

where

$$g\left(u,w\right) = \frac{1}{2}\left(F\left(u\right) + F\left(w\right) - \sum_{i=1}^{3}\left|\lambda_i\right|\alpha_i r_i\right)$$

where λ_i and r_i are the eigenvalues and eigenvectors of $A\left(u,w\right)$ and $w - u = \sum_{i=1}^{3}\alpha_i r_i$.

The last equation is a system of simultaneous algebraic equations for the variables α_i.

The conservative variables $\left(\rho, \rho u, \epsilon\right)$ are determined by the scheme. The velocity is obtained from ρ and ρu. But to determine the value of the temperature *T* we use an iteration method (especially for the cases of complex EOS). Then the pressure *P* is computed from the *EOS*

4.9. Numerical Results

In this section we present some numerical results. We consider a tube of length 1, filled by Methane gas, the initial discontinuity is located at $x_0 = 0.5$. In our simulation the following initial data is used.

$\rho_l = 3$, $p_l = 3$, $u_l = 0$ for $x \le 0.5$

$\rho_r = 1$, $p_r = 1$, $u_r = 0$ for $x > 0.5$.

In **Figure 1**, we have plotted the density, pressure, velocity, temperature, and the real gas compressibility factor computed by using each of EOS we discussed.

Figure 2 depicts results of (6), *i.e*, the Euler equation with the source term included, obtained by applying *PR*, and *BWRS EOS*.

5. Conclusion

The model that describes the flow of gas in a pipe is presented. Simplifications to the equations are made using appropriate assumptions. Several Equations of states that close the system of equations are examined and the results obtained for each equation of state are compared.

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A Maximum Principle Result for a General Fourth Order Semilinear Elliptic Equation

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Abstract

We obtain maximum principles for solutions of some general fourth order elliptic equations by modifying an auxiliary function introduced by L.E. Payne. We give a brief application of these maximum principles by deducing apriori bounds on a certain quantity of interest.

Keywords

Nonlinear, Fourth Order, Partial Differential Equation, Semilinear

1. Introduction

In [1], Payne obtains maximum principle results for the semilinear fourth order elliptic equation

$$\Delta^2 u = f(u) \tag{1}$$

by proving that certain functionals defined on the solution of (1) are subharmonic. In this work, functionals containing the terms $\left|\nabla^2 u\right| - u_{,i}\Delta u_{,i}$ are utilized and apriori bounds on the integral of the square of the second gradient and on the square of the gradient of the solution are deduced. Since then, many authors [2]-[11] and references therein have used this technique to obtain maximum principle results for other fourth order elliptic differential equations whose principal part is the biharmonic operator.

Other works deal with the more general fourth order elliptic operator $L^2 u$, where $Lu := a_{ij}u_{,ij}$ and $a_{ij} = a_{ji}$. In [12], Dunninger mentions that functionals containing the term $(Lu)^2$ can be used to obtain maximum principle results for such linear equations as

$$L^2 u + aLu + bu = 0.$$

A similar approach is taken in [13] for a class of nonlinear fourth order equations.

In this paper, we modify the results in [1] and a matrix result from [14] to deduce maximum principles defined on the solutions to semilinear fourth order elliptic equations of the form:

$$L^2 u = f(u). \tag{2}$$

Then we briefly indicate how these maximum principles can be used to obtain apriori bounds on a certain quantity of interest.

2. Results

Throughout this paper, the summation convention on repeated indices is used; commas denote partial differentiation. Let $a_{ij}(x)$ be a symmetric matrix. Moreover let $Lu := a_{ij}u_{,ij}$, be a uniformly elliptic operator, *i.e*, the symmetric matrix $a_{ij}(x)$ is positive definite and satisfies the uniform ellipticity condition:
$a_{ij}(x)v_i v_j \ge |v|^2, x \in \Omega, v \in R^n$, where Ω is a bounded domain in R^n and $n \ge 2$.

Let u be a C^5 solution to the equation

$$L^2 u = f(u) \text{ in } \Omega. \tag{3}$$

where f is say, a C^1 function. Now we define the functional

$$P = c_1(Lu)^2 - (a_{mn}Lu)_{,n} u_{,m} + c_2|\nabla u|^2 + 2(1-c_1)\int_0^u f(s)\mathrm{d}s + \beta(x).$$

We show that $L(P)$ is subharmonic and note that the constants c_1 and c_2 and any constraints on f are yet to be determined.

By a straight-forward calculation, we have

$$\begin{aligned} P_{,i} &= 2c_1 LuLu_{,i} - (a_{nm}Lu)_{,ni} u_{,m} - (a_{mn}Lu)_{,n} u_{,mi} \\ &\quad + 2c_2 u_{,m}u_{,mi} + 2(1-c_1) f(u)u_{,i} + \beta_{,i}. \end{aligned}$$

Now we write

$$\begin{aligned} L(P) &= a_{ij}P_{,ij} \\ &= 2c_1 a_{ij} LuLu_{,ij} + 2c_1 a_{ij} Lu_{,i} Lu_{,j} - a_{ij}(a_{mn}Lu)_{,nij} u_{,m} \\ &\quad - a_{ij}(a_{mn}Lu)_{,ni} u_{,mj} - a_{ij}(a_{nm}Lu)_{,nj} u_{,mi} - a_{ij}(a_{mn}Lu)_{,n} u_{,mij} \\ &\quad + 2c_2 a_{ij} u_{,mi} u_{,mj} + 2c_2 a_{ij} u_{,m} u_{,mij} + 2(1-c_1) f(u) a_{ij} u_{,ij} \\ &\quad + 2(1-c_1) f'(u) a_{ij} u_{,i} u_{,j} + L(\beta). \end{aligned} \tag{4}$$

By expanding out the derivative terms in parentheses, we see that $L(P)$ is

$$\begin{aligned} &= 2c_1 f(u)Lu + 2c_1 a_{ij} Lu_{,i} Lu_{,j} - a_{ij} u_{,m} \big[a_{mn,nij} Lu + a_{mn,ni} Lu_{,j} + a_{mn,nj} Lu_{,i} \\ &\quad + a_{mn,n} Lu_{,ij} + a_{mn,ij} Lu_{,n} + a_{mn,i} Lu_{,nj} + a_{mn,j} Lu_{,ni} + a_{mn} Lu_{,nij} \big] \\ &\quad - 2a_{ij} u_{,mj} \left(a_{mn,i} Lu_{,n} + a_{mn} Lu_{,ni} + a_{mn,ni} Lu + a_{mn,n} Lu_{,i} \right) \\ &\quad - a_{ij} u_{,mij} \left(a_{mn,n} Lu + a_{mn} Lu_{,n} \right) + 2c_2 a_{ij} u_{,mi} u_{,mj} + 2c_2 a_{ij} u_{,m} u_{,mij} \\ &\quad + 2(1-c_1) f(u) Lu + 2(1-c_1) f'(u) a_{ij} u_{,i} u_{,j} + L(\beta). \end{aligned} \tag{5}$$

The terms in lines 2 and 3 above containing two or more derivatives of Lu can be rewritten using (3) in the form $A^{ij} f(u) = Lu_{,ij}$, where A^{ij} denotes the matrix which is the inverse of the positive definite matrix (a_{ij}). Furthermore, we use the identity $a_{ij}u_{,mij} = Lu_{,m} - a_{ij,m}u_{,ij}$ to rewrite the last two terms in line 4. Hence,

$$\begin{aligned} L(P) &= 2f(u)Lu + 2c_1 a_{ij} Lu_{,i} Lu_{,j} - f(u) u_{,m} a_{mn,n} - a_{ij} Lu \left(u_{,m} a_{mn,nij} + 2a_{mn,ni} u_{,mj} \right) \\ &\quad - a_{mn,n} LuLu_{,m} + a_{mn,n} Lu a_{ij,m} u_{,ij} - a_{mn} Lu_{,n} Lu_{,m} + a_{mn} a_{ij,m} u_{,ij} Lu_{,n} \\ &\quad - a_{ij} a_{mn} u_{,m} \left(A^{ni}_{,j} f(u) + A^{ni} f'(u) u_{,j} \right) - 2a_{ij} a_{mn} u_{,mj} A^{ni} f(u) \\ &\quad - 2a_{ij} a_{mn,i} u_{,m} A^{nj} f(u) - 2a_{ij} u_{,m} a_{mn,ni} Lu_{,j} - 2a_{mn,n} a_{ij} u_{,mj} Lu_{,i} \\ &\quad - a_{ij} Lu_{,n} u_{,m} a_{mn,ij} - 2a_{ij} Lu_{,n} u_{,mj} a_{mn,i} + 2c_2 a_{ij} u_{,mi} u_{,mj} \\ &\quad + 2c_2 a_{ij} u_{,m} u_{,mij} + 2(1-c_1) f'(u) a_{ij} u_{,i} u_{,j} + L(\beta). \end{aligned} \tag{6}$$

Using the identity above for $a_{ij}u_{,mij}$ and the additional identity, $a_{ij}A^{ni}_{,j} = -a_{ij,j}A^{ni}$, which can be obtained by computing $\left(a_{ij}A^{ni}\right)_{,j}$, for the terms at the ends of lines 6 and 3 respectively, we obtain

$$\begin{aligned} L(P) = {} & L(\beta) + (1-2c_1)a_{ij}f'(u)u_{,i}u_{,j} - 2a_{mn,n}u_{,m}f(u) + 2c_2u_{,k}Lu_{,k} \\ & + 2c_2a_{ij}u_{,ik}u_{,jk} - 2c_2a_{ij,k}u_{,ij}u_{,k} - a_{mn,n}LuLu_{,m} + a_{mn,n}Lua_{ij,m}u_{,ij} \\ & - a_{rs}a_{ij}u_{,m}a_{mn,nrs}u_{,ij} - 2a_{mn,n}a_{ij}Lu_{,i}u_{,mj} - 2a_{ij}Lua_{mn,ni}u_{,mj} \\ & - 2a_{ij}a_{mn,i}Lu_{,n}u_{,mj} + a_{mn}a_{ij,m}Lu_{,n}u_{,ij} + (2c_1-1)a_{ij}Lu_{,i}Lu_{,j} \\ & - a_{ij}a_{mn,ij}u_{,m}Lu_{,n} - 2a_{ij}a_{mn,ni}u_{,m}Lu_{,j}. \end{aligned} \tag{7}$$

To show that $L(P)$ is nonnegative, we establish a series of inequalities based on the following one from [14]: Let (s_{pk}) be any $n\times n$ matrix. From the inequality

$$a_{ij}\left(u_{ik} + \frac{1}{2}A^{ip}s_{pk}\right)\left(u_{,jk} + \frac{1}{2}A^{jq}s_{qk}\right) \geq 0, \tag{8}$$

One can deduce

$$a_{ij}u_{,kj}u_{,ki} + s_{ki}u_{,ki} \geq -\frac{1}{4}A^{pq}s_{pk}s_{qk}. \tag{9}$$

Repeated use of (9) on terms in lines 2, 3, 4, 5 in (7) yields the following:

$$a_{ij}u_{,ik}u_{,jk} + a_{mn}a_{ij,m}Lu_{,n}u_{,ij} \geq -\frac{1}{4}A^{pq}\left(a_{mn}a_{pi,m}Lu_{,n}a_{rs}a_{qi,r}Lu_{,s}\right) \tag{10}$$

$$a_{ij}u_{,ik}u_{,jk} - 2a_{ij}a_{mn,i}Lu_{,n}u_{,mj} \geq -A^{pq}\left(a_{pm,r}a_{ri}Lu_{,n}a_{qn,s}a_{si}Lu_{,m}\right) \tag{11}$$

$$a_{ij}u_{,ik}u_{,jk} - 2a_{ij}a_{mn,ni}Luu_{,mj} \geq -A^{pq}\left(Lua_{ps,sr}a_{ri}Lua_{ql,lw}a_{wi}\right) \tag{12}$$

$$a_{ij}u_{,ik}u_{,jk} + a_{mn,n}a_{ij,m}Lu_{,n}u_{,ij} \geq -\frac{1}{4}A^{pq}\left(a_{mn,n}a_{pi,m}Lua_{rs,s}a_{qi,r}Lu\right) \tag{13}$$

$$a_{ij}u_{,ik}u_{,jk} - a_{ij}a_{rs}a_{mn,nrs}u_{,ij}u_{,m} \geq -\frac{1}{4}A^{pq}\left(a_{rs}a_{pi}u_{,m}a_{mn,nrs}a_{lw}a_{qi}u_{,z}a_{zt,tlw}\right) \tag{14}$$

$$a_{ij}Lu_{,i}Lu_{,j} - 2a_{mn,ni}a_{ij}u_{,m}Lu_{,j} \geq -A^{pq}\left(a_{sk}u_{,r}a_{rn,ns}a_{mq}u_{,p}a_{kw,wm}\right) \tag{15}$$

$$a_{ij}Lu_{,i}Lu_{j} - a_{ij}a_{mn,ij}u_{,m}Lu_{,n} \geq -\frac{1}{4}A^{pq}\left(a_{in}u_{,r}a_{rk,in}a_{lm}u_{,q}a_{kp,lm}\right) \tag{16}$$

$$a_{ij}u_{,ik}u_{jk} - 2c_2a_{ij,m}u_{,m}u_{,ij} \geq -c_2^2A^{pq}\left(a_{pi,m}u_{,m}a_{qi,l}u_{,l}\right) \tag{17}$$

$$a_{ij}u_{,ik}u_{jk} - 2a_{ij}a_{mn,n}Lu_{,i}u_{,mj} \geq -A^{pq}\left(a_{pr,r}a_{si}Lu_{,s}a_{ql,l}a_{mi}Lu_{,m}\right) \tag{18}$$

Furthermore, by completing the square, we obtain useful inequalities for the last two terms in line 1 and the third term in line 2 of (7):

$$2c_2u_{,k}Lu_{,k} \geq -c_2u_{,m}u_{,m} - c_2Lu_{,m}Lu_{,m} \tag{19}$$

$$-2a_{mn,n}u_{,m}f(u) \geq -a_{rp,p}u_{,r}a_{sq,q}u_{,s} - f^2 \tag{20}$$

$$-a_{mn,n}LuLu_{,m} \geq -a_{mp,p}Lua_{mq,q}Lu - Lu_{,m}Lu_{,m} \tag{21}$$

We add (10)-(21) and label the resulting inequality, for part of $L(P)$, as

$$
\begin{aligned}
\hat{L}(P) = {} & 7a_{ij}u_{ik}u_{jk} + 2a_{ij}Lu_{,i}Lu_{,j} + 2c_2u_{,k}Lu_{,k} - 2a_{mn,n}u_{,m}f(u) + a_{mn}a_{ij,m}Li_{,n}u_{,ij} \\
& - 2a_{ij}a_{mn,i}Lu_{,n}u_{,mj} - 2a_{ij}a_{mn,ni}Luu_{,mj} + a_{mn,n}a_{ij,m}Lu_{,n}u_{,ij} - a_{ij}a_{rs}a_{mn,nrs}u_{,ij}u_{,m} \\
& - 2a_{mn,ni}a_{ij}u_{,m}Lu_{,j} - 2c_2a_{ij,m}u_{,m}u_{,ij} - 2a_{ij}a_{mn,n}Lu_{,i}u_{,mj} + a_{mn,n}Lua_{ij,m}u_{,ij} \\
\geq {} & -A^{pq}\left(a_{sk}u_{,r}a_{rn,ns}a_{mq}u_{,p}a_{kw,wm}\right) - \frac{1}{4}A^{pq}\left(a_{in}u_{,r}a_{rk,in}a_{lm}u_{,q}a_{kp,lm}\right) \\
& - c_2^2A^{pq}\left(a_{pi,m}u_{,m}a_{qi,l}u_{,l}\right) - A^{pq}\left(a_{pr,r}a_{si}Lu_{,s}a_{ql,l}a_{mi}Lu_{,m}\right) \\
& - A^{pq}\left(a_{pm,r}a_{ri}Lu_{,n}a_{qn,s}a_{si}Lu_{,m}\right) - c_2u_{,m}u_{,m} - c_2Lu_{,m}Lu_{,m} \\
& - a_{rp,p}u_{,r}a_{sq,q}u_{,s} - a_{mp,p}Lua_{mq,q}Lu - Lu_{,m}Lu_{,m} - a_{mn,n}LuLu_{,m} - f^2 \\
& - \frac{1}{4}A^{pq}\left(a_{mn}a_{pi,m}Lu_{,n}a_{rs}a_{qi,r}Lu_{,s}\right) - \frac{1}{4}A^{pq}\left(a_{mn,n}a_{pi,m}Lua_{rs,s}a_{qi,r}Lu\right) \\
& - \frac{1}{4}A^{pq}\left(a_{rs}a_{pi}u_{,m}a_{mn,nrs}a_{lw}a_{qi}u_{,z}a_{zt,tlw}\right) - A^{pq}\left(Lua_{ps,sr}a_{ri}Lua_{ql,lw}a_{wi}\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
L(P) = {} & \hat{L}(P) + L(\beta) + (1-2c_1)a_{ij}f'u_{,i}u_{,j} + (2c_2-7)a_{ij}u_{,ik}u_{,jk} + (2c_1-3)a_{ij}Lu_{,i}Lu_{,j} \\
\geq {} & \left((2c_2-7)a_{ij} - a_{mp,p}a_{mq,q}a_{ik}a_{jk} - \frac{1}{4}A^{pq}a_{mn,n}a_{pi,m}a_{rs,s}a_{qi,r}a_{ik}a_{jk} - A^{pq}a_{ps,sr}a_{ri}a_{ql,lw}a_{wi}a_{ik}a_{jk}\right)u_{,ik}u_{,jk} \\
& + \left((2c_1-3)a_{ij} - (c_2+1)\delta_{ij} - A^{pq}a_{pj,r}a_{rl}a_{qi,s}a_{sl} - A^{pq}a_{pr,r}a_{is}a_{ql,l}a_{js} - \frac{1}{4}A^{pq}a_{mi}a_{pl,m}a_{rj}a_{ql,r}\right)Lu_{,i}Lu_{,j} \\
& + \left((1-2c_1)f'a_{ij} - c_2\delta_{ij} - A^{jq}a_{sk}a_{in,ns}a_{mq}a_{kw,wm} - c_2^2A^{pq}a_{pm,i}a_{qm,j} - \frac{1}{4}A^{pj}a_{rn}a_{ik,rn}a_{lm}a_{kp,lm}\right. \\
& \left. - a_{ip,p}a_{jq,q} - \frac{1}{4}A^{pq}a_{rs}a_{pm}a_{in,nrs}a_{lw}a_{qm}a_{jt,tlw}\right)u_{,i}u_{,j} + L(\beta) - f^2.
\end{aligned}
$$

Since $a_{ij}(x)$ is positive definite, for a sufficiently large value of c_2, where c_2 depends on the coefficients a_{ij} and their derivatives, and for a sufficiently large value of c_1, say (>1), where c_1 depends on the constants c_2, γ, a_{ij}, and various derivatives of a_{ij}, $L(P)$ can be made nonnegative as desired. Thus we have the following result.

Theorem 1. *Suppose that* $u \in C^5(\Omega) \cap C^3(\bar{\Omega})$ *is a solution of* (2) *and* $f \in C^1(R)$. *If* $f^2 \leq \gamma$, *where* $\gamma > 0$, $f'(u) \leq \alpha, \alpha < 0$, $\beta(x)$ *is a nonnegative function such that* $L(\beta) \geq \gamma$ *then there exists positive constants* c_2 *and* c_1 *sufficiently large* $(c_1 > 1)$ *such that P cannot attain its maximum value in* Ω *unless it is a constant.*

We note that the function $f(u) = -(u + u^3)$ satisfies the conditions stated in Theorem 1 for a solution that is bounded above.

3. Bounds

Here we give a brief application of Theorem 1.

Suppose that

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

By Theorem 1,

$$P \leq \max_{\partial\Omega}\left(c_1(Lu)^2 + \beta(x)\right).$$

Using integration by parts on the first two terms of P yields the identity

$$\int_\Omega a^{nm}u_{,km}a^{kp}u_{,pn} - \left(a^{mn}Lu\right)_{,n}u_{,m}\mathrm{d}x = 2\int_\Omega (Lu)^2\,\mathrm{d}x.$$

Upon integrating both sides of the previous inequality we deduce

$$2\int_{\Omega}(Lu)^2\,\mathrm{d}x+2(1-c_1)\int_{\Omega}\left(\int_0^u f(s)\,\mathrm{d}s\right)\mathrm{d}x \tag{22}$$

$$\leq\left[\max_{\partial\Omega}\left(c_1(Lu)^2+\beta(x)\right)\right]\operatorname{area}(\Omega). \tag{23}$$

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