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Explanation of Relation between Wave Function and Probability Density Based on Quantum Mechanics in Phase Space

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Abstract

The main problem of quantum mechanics is to elucidate why the probability density is the modulus square of wave function. For the purpose of solving this problem, we explored the possibility of deducing the fundamental equation of quantum mechanics by starting with the probability density. To do so, it is necessary to formulate a new theory of quantum mechanics distinguished from the previous ones. Our investigation shows that it is possible to construct quantum mechanics in phase space as an alternative autonomous formulation and such a possibility enables us to study quantum mechanics by starting with the probability density rather than the wave function. This direction of research is contrary to configuration-space formulation of quantum mechanics starting with the wave function. Our work leads to a full understanding of the wave function as the both mathematically and physically sufficient representation of quantum-mechanical state which supplements information on quantum state given solely by the probability density with phase information on quantum state. The final result of our work is that quantum mechanics in phase space satisfactorily elucidates the relation between the wave function and the probability density by using the consistent procedure starting with the probability density, thus corroborating the ontological interpretation of the wave function and withdrawing a main assumption of quantum mechanics.

Keywords

Quantum Ensemble Theory, Quantum Mechanics In Phase Space, Wave Function, Probability Density, Schrödinger Equation

1. Introduction

The most important task of the studies of quantum mechanics is to elucidate why the probability density is the modulus square of wave function, *i.e.*, Born's rule: $\rho = |\psi|^2$ [1]. This assumption called Born's probabilistic interpretation on the wave function was initially proposed by Born who believed that the ψ-function was only a mathematical device [2]. Born's proposition dictates that the wave function determines only the probability that a particle which brings with itself energy and momentum takes a path, but no energy and no momentum pertains to the wave.

This problem has been a fundamental subject of a long debate that had begun soon after quantum theory was formulated in the main in the 1920s, but the solution to the problem even now remains undetermined [3]. There are different interpretations of the wave function in quantum mechanics that are incompatible with one another, which involve the Copenhagen interpretation [2], the de Broglie's pilot wave interpretation [4], many-world interpretation [5], the realistic interpretation and so on [6] [7] [8] [9] [10]. The recent researches show that the wave function as a very peculiar object is interpreted still differently by two strategies: either something really existing [11]-[16] or an abstract entity for describing the behavior of particles [11] [17] [18]. Without correct interpretation of the wave function, it is impossible to completely solve the important foundation problems of quantum mechanics including the wave function collapse [19] and quantum entanglement.

As is well known, within the framework of the standard theory of quantum mechanics, the relation between the wave function and the probability density is assumed rather than derived. To the best of our knowledge, successful researches which aim to elucidate this assumption theoretically are not yet known.

We consider that this problem is related to in what space quantum mechanics should be constructed. In this connection, it is necessary to review several formulations of quantum mechanics distinguished by spaces where they are built. Different opinions about in what space the construction of quantum theory is possible gave rise to distinguished formulations of quantum mechanics.

Three self-standing formulations of quantum mechanics include the conventional Hilbert space, path integral and phase space formalism [20] [21] [22] [23] [24]. Until now, the standard formalism of quantum mechanics has maintained the dominant status amongst formulations of quantum mechanics by providing an excellent mathematical framework for explaining phenomena in microscopic world [25] [26]. This formalism makes use of Hermitian operators defined in configuration space instead of the phase-space functions of classical mechanics. The position variable and momentum operator defined in position space are the fundamental operators constituting Hermitian operators corresponding to all dynamical quantities of classical mechanics. Therefore, this formulation is establishes formally in configuration space.

It remains an important question whether quantum mechanics can be established as a consistent theory in phase space. If it would be possible, the study of quantum mechanics could be promoted from a new starting point and as a result, the quantum puzzle at issue might be easily solved. Since the advent of quantum mechanics, there have been steady studies to develop the phase-space formalism of quantum mechanics [20] [21] [27] [28] [29] [30]. The causal theory of quantum mechanics including quantum mechanics in phase space (abbreviated as QMPS) aims to formulate an alternative theory for the purpose of mathematical treatment and interpretation of quantum dynamical problems by adopting such concepts of classical mechanics as trajectory as they are [31] [32] [33] [34] [35]. The causal theory involves QMPS and Bohmian mechanics [21] [27] [29] [36] [37] [38] [39]. In this connection, it is important to note that the hydrodynamic equation of quantum mechanics has good potential to show in depth the dynamical characteristics of quantum mechanics [38] [40] [41] [42]. This equation sheds light on dynamical structure of quantum mechanics by use of the polar form of wave function. Quantum trajectory theory is distinguished from the standard theory of quantum mechanics by adopting the concept of trajectory even for microscopic particles [43] [44] [45] [46]. It is striking that Bohmian mechanics [47] as the most representative form of quantum trajectory theory is a deterministic but non-classical dynamics which provides statistical predictions in perfect accord with quantum theory [48].

The quantum trajectory at issue is fundamentally different from classical trajectory, since the former is assigned a definite probability unlike the latter [44] [46] [49].

The formulations of the causal theory of quantum mechanics are based on the confidence that quantum mechanics can be established consistently without contradicting the uncertainty principle, even if it is allowed to use both position and momentum together as fundamental variables of quantum state [30] [50]. The ground of this standpoint is that for the theory of the configuration-space formulation of quantum mechanics the introduction of the momentum operator is identified with the one-to-one correspondence between position and momentum according to the diagram shown in **Figure 1**.

This shows that the simultaneous determination of positions and momenta does not contradict even the standard theory of quantum mechanics.

The systematic developments of theory and remarkable contributions to applications in this field highlight the significance of QMPS [24] [51] [52] [53] [54] [55] [56]. As a formal theory of QMPS, the quantum tomography formulation [57] [58] [59] is developed based on a set of probability distributions as the dynamical variables, which possess classical-like characteristics being non-negative, normalized and measurable.

An important task of QMPS is to establish the formalism as a self-standing one [60]. The solution to this problem may lead to the formation of more general formalism of quantum mechanics. In this connection, it is noticeable that Moyal's method does not employ the Schrödinger equation and assumes an autonomous formalism of quantum mechanics [27] [61]. Moyal's equation, even though not airtight, shows an important view that there may be other ways

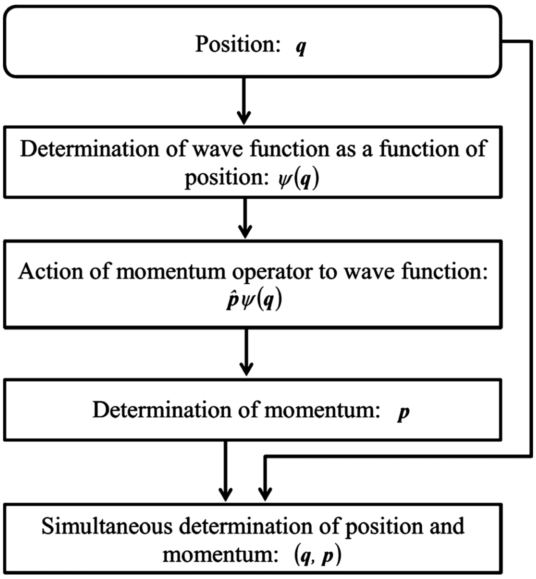

Figure 1. Simultaneous determination of position and momentum in terms of operator and wave function: For the configuration formulation of quantum mechanics the introduction of the momentum operator presupposes the one-to-one correspondence between position and momentum.

capable of describing quantal phenomena without recourse to the Schrödinger equation.

Meanwhile, it is necessary to emphasize that the idea of quantum mechanics with trajectories is identified with that of the path integral formulation [62], since this formulation employs the concept of the action indicating paths.

An increase of application domains of quantum theory needs to formulate an all-embracing quantum theory applicable even to both the largest scales such as the universe and the smallest where quantum gravity is studied. It is striking that despite such requirement, a compelling alternative theory has not so far emerged which has the potential to become complete as distinguished from the preceding formulations of quantum mechanics [63].

There are several directions of investigation which aim at an emergent theory of quantum mechanics. Steve Adler proposed a statistical mechanics of deterministic matrix models from which quantum mechanics is emergent [64]. On the other hand, Gerard ‘t Hooft developed a set of distinct ideas for quantum mechanics underlain by determinism [65]. An idea for applying the conception of the state vector reduction to gravity that Roger Penrose proposed makes us suggest experimental situations in which the decay of quantum superposition could be observed [66] [67] [68].

The present status of quantum mechanics indicates that quantum mechanics still has not been satisfactorily formulated in the aspect of formalism and for this reason, the theoretical development of quantum mechanics even now needs original self-standing formulations. In this regard, it is useful to introduce Takabayasi’ remarks [30].

The ordinary formulation of quantum mechanics, as established by the fusion *of Heisenberg's matrix mechanics and Schrödinger's wave mechanics, is certainly the most fundamental and powerful one, having its own 'picture' in a broad sense essentially non-classical. Nevertheless we may consider another consistent formulation of quantum mechanics with its associated picture, for instance, path integral formulation by Feynman". Generally such a new formulation and picture would reveal new aspects of physical and mathematical construction of quantum mechanics, and might serve to suggest new clues to future progress of quantum theory itself, apart from its usefulness for practical applications to specified class of problems.*

A newly established formalism should contribute to the solutions of open questions of quantum mechanics and lay a better foundation of the quantum theory. From this point of view, we aim to establish an alternative formalism in terms of statistical ensemble in phase space reflecting the probabilistic and dynamical structure of quantum mechanics in order to explain the relation between the wave function and the probability density.

To be a genuine autonomous formalism, it must have its fundamental equation independent of other formulations. We have the view that the statistical formalism of quantum mechanics based on the statistical ensemble in phase space is qualified for an autonomous formalism of quantum mechanics. Based on phase space, our methodology makes an intelligible and natural inference from the probability wave to obtain an alternative fundamental quantum-mechanical equation without recourse to the other formulations of quantization, thereby satisfactorily explaining the relation between the wave function and the probability density. Eventually, it can be concluded that to solve open questions of quantum mechanics including the relation between the wave function and the probability density, a new formulation distinguished from the previous ones is required and really there is such a possibility.

Hereafter, we describe the phase-space formalism of quantum mechanics to reach the goal of this paper which is to explain the relation between the wave function and the probability density.

The remaining paper is organized as follows. In Sect. 2, we deduce the fundamental equation of quantum mechanics in phase space and formulate the phase-space theory of quantum mechanics. In Sect. 3, we describe why the phase-space formalism of quantum mechanics is a generalized theory containing the configuration-space formulation and how this formalism explains the foundational questions of quantum mechanics to prove its validity. In Sect. 4, the results and discussion are given. The paper is concluded in Sect. 5.

2. An Alternative Phase-Space Formulation of Quantum Mechanics

Through this section, we shall show how an alternative formalism of quantum mechanics in phase space can be constructed independently and why this formalism plays a crucial role in elucidating the relation between the wave function and the probability density.

2.1. Consistent Procedure: From Probability Density to Wave Function

To explain the relation between the wave function and the probability density may depend on with which we start the derivation of the fundamental equation, i. e., either with the wave function or with the probability density. Probably, the solution of this problem essentially results in the construction of a new consistent formalism and as a result the solution of some open problems relevant to the foundations of quantum mechanics.

The Schrödinger equation, which was derived in configuration space, essentially should be assessed to be assumed. With the Schrödinger equation, we can infer the conception of operators corresponding to observables. While the Schrödinger equation is a nonrelativistic equation, it gives the clue as to how to find out the relativistic wave equation. It is with the help of the momentum operator inferred from the Schrödinger equation that the Klein-Gordon equation and Dirac's equation were obtained. Therefore, we should consider that the Schrödinger equation is the starting point for searching for relativistic wave equations and a source providing the important concepts of quantum mechanics including quantum operator. This shows that quantum operators such as momentum operator which the Schrödinger equation gives have general meaning.

The solutions of the Schrödinger equation in good agreement with a wide range of experiments confirm its validity. However, to the best of our knowledge, whether the equation is exact or approximate still remains undetermined. This is because in essence, the Schrödinger equation was assumed. What should be stressed is the fact that the Schrödinger equation itself derived with the help of the assumed wave function which is not originally ontological cannot explain the relation between the wave function and probability density in a logical way. So it is necessary to note that there is no need to exclude the possibilities of investigating any other formalism independent of the Schrödinger equation, since it could contribute to elucidating incomplete aspects of quantum mechanics and resolving some quantum puzzles. For this reason, we explore for a novel fundamental equation of quantum mechanics in phase space inferred from the probability wave. If it were possible to use phase space to formulate a new version of quantum mechanics, we could begin to deduce the fundamental equation of quantum mechanics starting with statistical ensemble in phase space.

The conception of statistical ensemble in phase space due to the wave field is the starting point obtaining an alternative fundamental equation of QMPS. Our formalism is based on the view that quantum mechanics has the statistical structure. Without using the Schrödinger equation, we obtain a new fundamental equation in phase space from the picture of statistical ensemble representing the wave field. Such a procedure leads our exploration from the probability density to the wave function without the help of assumption. However, the reverse procedure developing from the wave function to the probability density inevitably requires the assumption about the relation between the wave function and the probability density. Our purpose is to demonstrate that the formulation starting with the probability density in phase space can provide exact wave equation free of assumption and logically elucidate the relation between wave function and probability density.

2.2. Basic Premises of Quantum Mechanics in Phase Space

With solid foundation, we employ the wave function and probability density in phase space to construct quantum mechanics with minor assumptions, although the conventional understanding of the uncertainty principle hinders us from using a pair of canonical conjugate variables together. Starting with statistical ensemble in phase space which gives the probability density, our formulation yields its autonomous fundamental equation in phase space.

In order to frame a new formulation of quantum mechanics in phase space, we start with the following premises.

(I) *Quantum state of microscopic particles is represented in phase space.*

From the beginning of development of quantum mechanics, a major problem is whether position and momentum variables can be utilized together to represent states of microscopic particles. Seen against the background of QMPS, such a situation is a matter of course. According to the standard theory of quantum mechanics, choosing positions and the corresponding canonical conjugate momentum variables at once is regarded as violating the uncertainty principle. On the contrary, our view is that taking both position and momentum variable as basic variables of a quantum-mechanical state does not involve any inconsistency.

The standard theory of quantum mechanics excludes the possibility of determining canonically conjugate position and momentum simultaneously. However, such a standpoint of the standard theory of quantum mechanics cannot explain the fact that the application of the momentum operator to a wave function definitely determines particles' momenta. In fact, a momentum operator enables the one-to-one correspondence between position and momentum via a wave function in configuration space. Therefore, there is no doubt about the fact that the existence of the momentum operator means the one-to-one correspondence between a position and a momentum. However, it should be considered that in phase space there is not the one to one correspondence between a position and a momentum but multi-correspondence relation. Nevertheless, simultaneous determination of position and momentum is not problematic. This is the first logic of the simultaneous determination of position and momentum.

Also, there is the second logic of adducing reason in support of the possibility of the simultaneous determination of position and momentum. It is necessary to give careful thought to the fact that the velocity of a particle is calculated by the time series of positions, and the momentum is determined as the product of mass and velocity. Meanwhile, according to the standard quantum theory, it is possible to exactly measure without limitation both time and position simultaneously, since they are not canonically conjugate. This possibility therefore makes us circumvent the uncertainty principle to determine the position and momentum simultaneously as exactly as we want.

In principle, there is the third logic that makes us avoid the disturbance which the simultaneous measurement of position and momentum causes. If we make not simultaneous but alternate measurement of position and momentum, according to the uncertainty principle the results of measurement prove to be free of disturbance. Then the data on position and momentum obtained by the alternate measurement become non-perturbed information reflecting the reality of a microscopic particle. In this case, it is obvious that the position function or the momentum function interpolated in terms of the time series of position or momentum observable should coincide with the other by differentiation or integration, since the two observables are free from the disturbance due to measurement of each other. It shows that position and momentum have correspondence relation and the alternate and simultaneous measurements are equivalent.

Heisenberg's interpretation of the uncertainty relation is based on the disturbance that the joint measurement of canonically conjugate observables brings about for each other. If the uncertainty of momentum is ascribed to the disturbance due to the measurement of position, then we should regard the sequential measurement of position itself as being disturbed by previous measurements of position. If so, it is impossible to study quantum phenomena with exactly measured values of observables. The wave function and operators as the starting point for investigating the uncertainty relation are not involved in the measurement by observer at all and purely represent the probabilistic possibilities inherent in quantum system. Therefore, the uncertainty principle has nothing to do with measurement.

This logic of the simultaneous determination of position and momentum can be represented by the diagram shown in **Figure 2**.

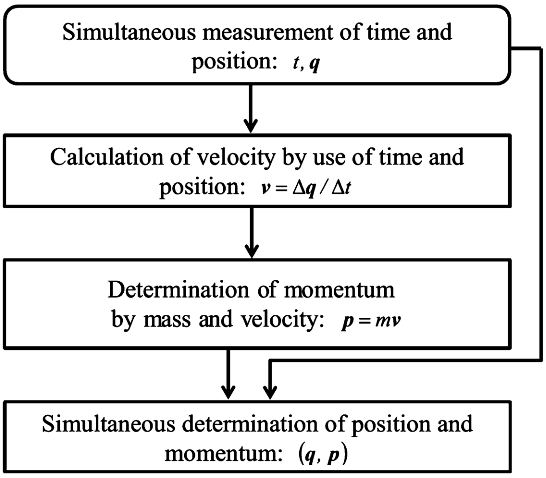

Figure 2. Simultaneous determination of position and momentum in terms of time and position: The possibility of simultaneously determining time and position ensures the unique determination of momentum corresponding to position.

It is necessary to note that until the sixties it was generally believed that it is impossible to measure two conjugate observables together, but since then the matter has not been the case: it is currently accepted that it is possible to measure two conjugate observables together [1] [69]. The emergence of the theory of the positive-operator valued measurement (POVM) is a significant example that shows the renovation in the understanding of the uncertainty principle [70] [71].

Judging from this ground, we can imagine a definite correspondence between position and momentum. This context yields an ensemble in position space and the corresponding ensemble in momentum space which give distributions of positions and momenta.

The main problem is to explain how these two distributions determined by ensembles are related. Our formalism will show that this relation can be rigorously proved rather than assumed by treating the statistical ensemble.

The quantum states can be described by ensemble in phase space which obeys the wave-like rule. The quantal ensemble displays the correlation between phase trajectories, *i.e.*, self-interference due to the wave field [72]. Then the phase points in the quantal ensemble behave in a wave-like manner with the correlation according to de Broglie's relation. Therefore, the quantal ensemble should obey both the dynamical laws for particles and the wave-like rule for the phase-space ensemble.

The unavoidable one-to-one correspondence between position and momentum demonstrated purely by the standard theory of quantum mechanics shows that it is necessary to renew our understanding of the uncertainty relation. For this reason, we are bound to use definitely phase space even for the investigation of microscopic world. So as in statistical mechanics, we can conceive the density of phase points given by a statistical ensemble in phase space which represents the probability of finding particles in the volume element centered on a phase point $(\mathbf{q},\mathbf{p})$ as

$$\rho=\rho(\mathbf{q},\mathbf{p},t). \tag{1}$$

ρ contains important information about the wave field. In fact, such a definition is natural in statistical mechanics, since ρ gives complete information involving particles' interaction and the constraints on motion.

To obtain ρ being always real-valued and positive, it is necessary to assume that as a necessary mathematical requirement, the probability density is given by the modulus square of the so-called wave function in phase space, which is generally a complex-valued function. Namely,

$$\rho=\psi^*(\mathbf{q},\mathbf{p},t)\psi(\mathbf{q},\mathbf{p},t). \tag{2}$$

This assumption is purely a mathematical requirement, but it is necessary also for introducing a certain state function including the phase information of quantum state. Thus, wave function ψ as a complex function includes phase to become the complete state function which contains not only information on probability distribution but also that on phase of quantum state. As a result, the wave function contains more information than the probability density. Eventually, the wave function substantiates the necessary and sufficient condition for quantum state.

The wave function normalized by

$$\psi(\mathbf{q},\mathbf{p},t) \leftarrow \frac{\psi(\mathbf{q},\mathbf{p},t)}{\left(\iint \psi^*(\mathbf{q},\mathbf{p},t)\psi(\mathbf{q},\mathbf{p},t)\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}\right)^{1/2}} \tag{3}$$

should satisfy the following normalization condition:

$$\iint \psi^*(\mathbf{q},\mathbf{p},t)\psi(\mathbf{q},\mathbf{p},t)\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p} = 1. \tag{4}$$

By definition, the probability density in configuration space is expressed as

$$\rho_q(\mathbf{q}) = \int \rho(\mathbf{q},\mathbf{p})\mathrm{d}\mathbf{p},$$

the probability density in momentum space, as

$$\rho_p(\mathbf{p}) = \int \rho(\mathbf{q},\mathbf{p})\mathrm{d}\mathbf{q}.$$

The mean value of a dynamical quantity is defined as

$$\bar{F} = \iint \psi^*(\mathbf{q},\mathbf{p})F(\mathbf{q},\mathbf{p})\psi(\mathbf{q},\mathbf{p})\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}. \tag{5}$$

Essentially, the statistical ensemble stands for the wave field as a physical reality, so it reflects the dynamical causality in microscopic world. Choosing both position and momentum as basic dynamical variables of quantum mechanics provides the possibility of introducing methodology of statistical mechanics, thereby extending the methodological realm of quantum mechanics.

(II) *de Broglie's relation represents the behavior of the wave field and statistical ensemble in phase space thereof.*

The de Broglie relation characterizes the wave field producing an ensemble of microscopic states. On the other hand, this field is represented by a statistical ensemble in phase space. In the end, the de Broglie relation stands for an ensemble in phase space regulated by the wave field.

(III) *The form of the wave function is represented in terms of the action.*

Specifically, for a many-particle system, we write the polar form of wave function as

$$\psi(\mathbf{q},\mathbf{p},t) = \varphi(\mathbf{q},\mathbf{p},t)\exp\left(\mathrm{i}\frac{S}{\hbar}\right), \tag{6}$$

where φ is a real-valued function and *S*, the action represented as

$$S(\mathbf{q},\mathbf{p},t) = \int_0^{\mathbf{q}} \mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}' - \int_0^t H(t')\mathrm{d}t'. \tag{7}$$

Here, $\mathbf{q}$ is the whole of coordinates and $\mathbf{p}$, that of momentum components of the system under consideration.

As it is, we can give a reasonable explanation for the form of the wave function, Equation (6) based on the de Broglie relation. If it is possible, it turns out that from premise (II) issues premise (III). In the end, the premises amount to the adoption of phase space and the de Broglie relation.

The action which reflects the ensemble of trajectories of a given system can be considered as a characteristic integral for a microscopic system. In Refs. [73] and as its extension [73], we explained with the help of mechanical picture the reason why the wave function should be represented as a complex number function. This assumption implies that the wave field has the phase determined by the action, and h is the quantum of the action. The fact that Equation (7) comes from the de Broglie relation naturally will lead to grasping sound meanings of the dualism.

Now, let us consider this matter. It is obvious that the de Broglie's relation defines the frequency and the wave vector of the de Broglie wave. Using this relation, we can determine the phase of the wave without loss of generality as

$$\Phi = \int_0^{\mathbf{q}} \mathbf{k}\left(\mathbf{q}'\right)\mathrm{d}\mathbf{q}' - \int_0^{t} \omega\left(t'\right)\mathrm{d}t'$$

Of course, the phase of a free particle is represented as

$$\Phi = \mathbf{kq} - \omega t.$$

For a period, the phase relation should satisfy

$$\int_0^{\mathbf{q}} \mathbf{k}\left(\mathbf{q}'\right)\mathrm{d}\mathbf{q}' - \int_0^{t} \omega\left(t'\right)\mathrm{d}t' = 2\pi. \tag{8}$$

The above relation indicates the minimal condition necessary for a wave. In order for a wave to exist, it should satisfy at least this physical condition. Multiplying the both sides of Equation (8) by h and taking into consideration the de Broglie relation, we get

$$\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\mathrm{d}\mathbf{q}' - \int_0^{t} E\left(t'\right)\mathrm{d}t' = 2\pi\hbar = h. \tag{9}$$

Here we supposed that even though the momentum of a particle changes, the de Broglie relation is applicable as in case of a free particle.

Hence, we can adopt the general condition of periodicity as

$$\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\mathrm{d}\mathbf{q}' - \int_0^{t} E\left(t'\right)\mathrm{d}t' = nh, \tag{10}$$

where n is an integer. We can split Equation (10) into

$$\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\mathrm{d}\mathbf{q}' = kh$$

and

$$\int_0^{t} E\left(t'\right)\mathrm{d}t' = lh,$$

where k and l are an integer. The condition of periodicity of a free particle is represented as

$$\mathbf{pq} - Et = nh.$$

Equation (10) is nothing but the Bohr-Sommerfeld quantization condition. Based on the above arguments, we in general can write the phase of the de Broglie wave as

$$\Phi = 2\pi \frac{\int_0^{\mathbf{q}} \mathbf{p}\left(\mathbf{q}'\right)\mathrm{d}\mathbf{q}' - \int_0^{t} E\left(t'\right)\mathrm{d}t'}{h}. \tag{11}$$

Here, the numerator evidently is the action. Hereafter, we denote the action by

$$S(\mathbf{q},\mathbf{p},t)=\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^t H(t')\mathrm{d}t'.$$

Obviously, the action of a free particle is written as

$$S(\mathbf{q},\mathbf{p},t)=\mathbf{pq}-Ht.$$

We can suppose from Equation (9) that the quantum in phase space should be represented as

$$\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'=h, \tag{12}$$

while the quantum in energy-time space should be represented as

$$\int_0^t H(t')\mathrm{d}t'=h. \tag{13}$$

These relations really shed light on the nature of space-time quantization.

It is natural to interpret the above relations as characterizing an ensemble that consists of pairs of position and momentum, and of time and energy. From Equation (11), the phase of the probability wave is written in terms of the action as

$$\Phi=\frac{S(\mathbf{q},\mathbf{p},t)}{\hbar}.$$

In the end, the de Broglie wave can be represented as

$$\psi(\mathbf{q},\mathbf{p},t)=\varphi(\mathbf{q},\mathbf{p},t)\exp\left(\mathrm{i}\frac{S(\mathbf{q},\mathbf{p},t)}{\hbar}\right),$$

where $\varphi(\mathbf{q},\mathbf{p},t)$ as the probability amplitude is a real-valued function. Thus, the third premise, Equation (6) has been explained and as a result, in the true sense, premise (II) and premise (III) have unified.

We therefore can conclude that the de Broglie relation is represented in phase space and enables us to determine the form of the wave function. Meanwhile, Equation (10) shows essential contents of the uncertainty relation reflecting ensemble in phase space and the broad context of classical mechanics and quantum mechanics. In fact, Equation (10) shows that a quantum state is determined by the distribution of pairs of position and momentum, and by the distribution of pairs of time and energy. Expression (12), (13) tell us that the quantum of the action is h. At the same time, it shows the necessity and validity of the simultaneous determination of position and momentum, and of time and energy. This is because if it were not to be possible, we could not imagine the phase of a wave.

Of course, this assumption is not regarded to be new, since such a form of wave function has already been used in the preceding formulations [74]. It is necessary to recall the fact that the Schrödinger equation was obtained, implicitly employing this assumption. In fact, for the Schrödinger equation the phase part of the wave function assumed for a free particle agrees with this assumption. Such an understanding of the wave field serves as the basis for establishing our formalism of quantum mechanics in phase space.

2.3. A Complete System of Quantum Operators Based on Phase-Space Formalism of Quantum Mechanics

We manipulate the wave function, Equation (6) to form the conception of operator. The assumed form of wave function enables us to get the idea for quantum operators. To infer the concept of quantum operator, let us determine the derivatives of the action in the extended phase space,

$$S(\mathbf{q},\mathbf{p},t)=\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^t H(t')\mathrm{d}t'$$

with respect to q, p, t. It is due to the introduction of ensemble of paths to regard momentum as an independent variable. The calculation runs as follows.

$$\begin{aligned}\frac{\partial S(\mathbf{q},\mathbf{p},t)}{\partial q_i}&=\frac{\partial}{\partial q_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^t H(t')\mathrm{d}t'\right)\\&=\frac{\partial}{\partial q_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'\right)\\&=\frac{\partial}{\partial q_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'\Big|_0^{\mathbf{q}}-\int_0^{\mathbf{q}}\mathbf{q}'\frac{\partial\mathbf{p}(\mathbf{q}')}{\partial\mathbf{q}'}\mathrm{d}\mathbf{q}'\right)\\&=\frac{\partial}{\partial q_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'\Big|_0^{\mathbf{q}}\right)=\frac{\partial}{\partial q_i}(\mathbf{pq})=p_i,\end{aligned}\tag{14}$$

$$\begin{aligned}\frac{\partial S(\mathbf{q},\mathbf{p},t)}{\partial p_i}&=\frac{\partial}{\partial p_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^t H(t')\mathrm{d}t'\right)\\&=\frac{\partial}{\partial p_i}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'\right)\\&=\frac{\partial}{\partial p_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'\Big|_0^{\mathbf{q}}-\int_0^{\mathbf{q}}\mathbf{q}'\frac{\partial\mathbf{p}(\mathbf{q}')}{\partial\mathbf{q}'}\mathrm{d}\mathbf{q}'\right)\\&=\frac{\partial}{\partial p_i}\left(\mathbf{p}(\mathbf{q}')\mathbf{q}'\Big|_0^{\mathbf{q}}\right)=\frac{\partial}{\partial p_i}(\mathbf{pq})=q_i,\end{aligned}\tag{15}$$

$$\frac{\partial S(\mathbf{q},\mathbf{p},t)}{\partial t}=\frac{\partial}{\partial t}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^t H(t')\mathrm{d}t'\right)=-H.\tag{16}$$

For the purpose of finding operators, in view of Equations (14)-(16), we conceive the following operators:

$$\mathrm{i}\hbar\frac{\partial}{\partial t},\quad -\mathrm{i}\hbar\frac{\partial}{\partial p_i},\quad -\mathrm{i}\hbar\frac{\partial}{\partial q_i}.$$

These operators become the tools for deriving dynamical quantities from the wave function. In fact, the application of a quantum operator to the wave function yields the corresponding dynamical quantity. Thus, we come to find the operator relations equal or analogous to ones in configuration-space formalism.

To confirm this fact, first let us review $\mathrm{i}\hbar\frac{\partial\psi}{\partial t}$.

The application of $\mathrm{i}\hbar\frac{\partial}{\partial t}$ to the wave function,

$$\psi(\mathbf{q},\mathbf{p},t)=\varphi(\mathbf{q},\mathbf{p},t)\exp\left\{\mathrm{i}\frac{S(\mathbf{q},\mathbf{p},t)}{\hbar}\right\}$$

yields

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t}=\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial t}\psi+\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial t}\psi=H\psi+\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial t}\psi. \tag{17}$$

Here, we took Equation (16) into consideration. From the above expression, we can interpret $\mathrm{i}\hbar\frac{\partial}{\partial t}$ as the operator relative to total energy, since it makes H be derived from the wave function.

Next, let us consider $-\mathrm{i}\hbar\frac{\partial\psi}{\partial p_i}$. The application of this operator to the wave function gives

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial p_i}=-\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial p_i}\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial p_i}\psi=q_i\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial p_i}\psi. \tag{18}$$

Since this operation gives q_i, we can regard $-\mathrm{i}\hbar\frac{\partial}{\partial p_i}$ as the position operator.

Similarly, we get

$$-\mathrm{i}\hbar\frac{\partial\psi}{\partial q_i}=-\mathrm{i}\hbar\frac{\mathrm{i}}{\hbar}\frac{\partial S}{\partial q_i}\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial q_i}\psi=p_i\psi-\mathrm{i}\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial q_i}\psi. \tag{19}$$

As a consequence, $-\mathrm{i}\hbar\frac{\partial}{\partial q_i}$ is adopted as the momentum operator. The obtained results naturally bring forth the idea for quantum operator. From Equations (17), (18) and (19), we can interpret the meaning of the relation between an observable, L and the corresponding operator, $\hat{L}$ as

$$L=\mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right). \tag{20}$$

In fact, this relation naturally results from the definition of mean value. By definition, the mean value with respect to $\hat{L}$ is written as

$$\bar{L}=\int\psi^*\hat{L}\psi\mathrm{d}\tau=\int\psi^*\psi\frac{\hat{L}\psi}{\psi}\mathrm{d}\tau=\int\rho\frac{\hat{L}\psi}{\psi}\mathrm{d}\tau. \tag{21}$$

Accordingly, with the real-valued property of $\bar{L}$ in mind, we reach the conclusion that $\mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right)$ should be regarded as the observable with respect to operator $\hat{L}$.

Such an interpretation of quantum observables naturally leads to adopting time as an ordinary quantum observable. Extending the phase space furthermore, we can take the action as

$$S(\mathbf{q},\mathbf{p},H,t)=\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^{t}H(t')\mathrm{d}t'. \tag{22}$$

Then we get

$$\begin{aligned}\frac{\partial S(\mathbf{q},\mathbf{p},H,t)}{\partial H}&=\frac{\partial}{\partial H}\left(\int_0^{\mathbf{q}}\mathbf{p}(\mathbf{q}')\mathrm{d}\mathbf{q}'-\int_0^t H(t')\mathrm{d}t'\right)\\&=-\frac{\partial}{\partial H}\left(\int_0^t H(t')\mathrm{d}t'\right)\\&=-\frac{\partial}{\partial H}\left(H(t')t'\Big|_0^t-\int_0^t t'\frac{\partial H(t')}{\partial t'}\mathrm{d}t'\right)\\&=-\frac{\partial}{\partial H}\left(H(t')t'\Big|_0^t\right)=-\frac{\partial}{\partial H}(Ht)=-t.\end{aligned}\tag{23}$$

Therefore, the time operator becomes

$$\hat{t}=\mathrm{i}\hbar\frac{\partial}{\partial H}.\tag{24}$$

Thus, it is clarified that the time observable also has the corresponding operator and the complete system of basic operators of quantum mechanics should be determined based on the action.

The introduction of these operators helps to elucidate the relations of this formalism with the others.

2.4. Fundamental Equation of Quantum Mechanics in Phase Space

Our purpose is to derive the equation for the wave equation starting with the probability density. If this goal would have been achieved, the relationship between the wave function and the probability density could be naturally explained. Of course, what the wave function is already was specified by Equation (2). The concept of wave function issued from the probability density on the basis of Equation (2) should be regarded as a mathematically and physically necessary requirement instead of an assumption. In order to obtain the fundamental equation of quantum mechanics in phase space, we start with the probability density identified with the statistical ensemble of phase points ($\mathbf{q},\mathbf{p}$).

Taking into account that $\frac{\mathrm{d}\rho}{\mathrm{d}t}$ remains constant on the wave front of ρ, we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=0.\tag{25}$$

Also, if we observe the change in the probability density in the reference system moving at the phase velocity, we have Equation (25) as the probability continuity equation, which we may refer to as Liouville's theorem for quantal ensemble. From Equation (25), we have the following equation of motion for the probability density:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\frac{\partial\rho}{\partial t}+\sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i\frac{\partial\rho}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial\rho}{\partial p_i}\right]=0,\tag{26}$$

where f is the number of degrees of freedom, and $\left(\dot{q}_{phase}\right)_i$ and $\left(\dot{p}_{phase}\right)_i$ are components of the phase velocities in phase space. Meanwhile, we can adopt $\left(\dot{p}_{phase}\right)_i$ as

$$\left(\dot{p}_{phase}\right)_i = m \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_{phase}\right)_i . \tag{27}$$

For convenience, Equation (26) is written as

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i \frac{\partial \rho}{\partial q_i} + \left(\dot{p}_{phase}\right)_i \frac{\partial \rho}{\partial p_i}\right]. \tag{28}$$

Now, we should express the phase velocities as the corresponding group velocities with the help of the relation between phase and group velocity. The de Broglie relation gives the following relation between the phase and group velocity:

$$v_{phase} = \frac{\omega}{k} = \frac{E}{p}. \tag{29}$$

For non-relativistic case, considering only the kinetic energy of a particle as the energy fulfilling the de Broglie relation leads to the relation between the phase and group velocity,

$$v_{phase} = \frac{v_{group}}{2}. \tag{30}$$

Next, the relation between the phase and group acceleration can be calculated to get

$$\frac{\mathrm{d}v_{phase}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K}{p}\right) = \left[\frac{K}{p}, H\right]_{\mathrm{PB}} + \frac{\partial}{\partial t}\left(\frac{K}{p}\right), \tag{31}$$

where subscript PB indicates the Poisson bracket and K the kinetic energy. For non-relativistic case, we have

$$\frac{\mathrm{d}v_{phase}}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}v_{group}}{\mathrm{d}t}. \tag{32}$$

Note that for a particle in a potential field, de Broglie's relation should be extended. For a particle in a potential field, the phase velocity should be defined with kinetic energy, *i.e.*, the remainder determined by subtracting potential energy from total energy as

$$v_{phase} = \frac{E-U}{\sqrt{2m\left(E-U\right)}} = \frac{K}{p}. \tag{33}$$

Without such a generalized definition, the de Broglie relation cannot ensure generality for quantum theory [75]. Obviously, it should be considered that only kinetic energy is related to the wave-like characteristics. We shall prove Equation (33) rigorously at the end of this section.

We adopt Equations (29) and (31) as ones valid for both relativistic and non-relativistic case. After inserting Equation (2) into Equation (28) to obtain the wave equation, we go through the following steps:

$$\frac{\partial\left(\psi^*\psi\right)}{\partial t} + \sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i \frac{\partial\left(\psi^*\psi\right)}{\partial q_i} + \left(\dot{p}_{phase}\right)_i \frac{\partial\left(\psi^*\psi\right)}{\partial p_i}\right] = 0,$$

$$\psi\frac{\partial\psi^*}{\partial t}+\sum_{i=1}^{f}\psi\left[\left(\dot{q}_{phase}\right)_i\frac{\partial\psi^*}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial\psi^*}{\partial p_i}\right]$$
$$+\psi^*\frac{\partial\psi}{\partial t}+\sum_{i=1}^{f}\psi^*\left[\left(\dot{q}_{phase}\right)_i\frac{\partial\psi}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial\psi}{\partial p_i}\right]=0, \tag{34}$$

$$\psi^*\frac{\partial\psi}{\partial t}+\sum_{i=1}^{f}\psi^*\left[\left(\dot{q}_{phase}\right)_i\frac{\partial\psi}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial\psi}{\partial p_i}\right]+complex\ conjugate=0. \tag{35}$$

Dividing both sides of the above equation by $\psi^*\psi$ gives

$$\frac{1}{\psi}\frac{\partial\psi}{\partial t}+\frac{1}{\psi}\sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i\frac{\partial\psi}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial\psi}{\partial p_i}\right]+complex\ conjugate=0. \tag{36}$$

On the other hand, we come to see that the real part of the expression,

$$\frac{1}{\psi}\frac{\partial\psi}{\partial t}+\frac{1}{\psi}\sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i\frac{\partial\psi}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial\psi}{\partial p_i}\right] \tag{37}$$

and that of its complex conjugate are the same, and are significant. Therefore, the real part of Equation (37) should be zero. Using the representation of operator

$$\Lambda=\sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i\frac{\partial}{\partial q_i}+\left(\dot{p}_{phase}\right)_i\frac{\partial}{\partial p_i}\right]+\frac{\partial}{\partial t}, \tag{38}$$

otherwise in the vectorial form

$$\Lambda=\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase}\frac{\partial}{\partial\mathbf{q}}+\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{phase}\frac{\partial}{\partial\mathbf{p}}+\frac{\partial}{\partial t},$$

we can write Equation (36) briefly as

$$\frac{1}{\psi}\Lambda\psi+\frac{1}{\psi^*}\Lambda\psi^*=0. \tag{39}$$

In view of $\psi=\varphi\exp\left(\mathrm{i}\frac{S}{\hbar}\right)=\varphi\exp\left(\mathrm{i}\Phi\right)$, we rewrite the above equation as

$$2\frac{1}{\varphi}\Lambda\varphi+\frac{1}{\exp\left(\mathrm{i}\Phi\right)}\Lambda\exp\left(\mathrm{i}\Phi\right)+\frac{1}{\exp\left(-\mathrm{i}\Phi\right)}\Lambda\exp\left(-\mathrm{i}\Phi\right)=0. \tag{40}$$

Since the real and imaginary parts should be zero, respectively, for the real part, we get

$$\Lambda\varphi=0. \tag{41}$$

Supposing

$$\Lambda\exp\left(\mathrm{i}\Phi\right)=0, \tag{42}$$

we get

$$\Lambda\psi=0. \tag{43}$$

Of course, we can fix the imaginary part of Equation (36) in an arbitrary way. But we should recall that we already fixed the phase part of wave function on the basis of the de Broglie relation and thus do not need to determine it. If the equation for ψ exists, then its real part should coincide with its imaginary part because operator Λ is linear differential operator.

To confirm that Equation (42) is valid, we calculate the following:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} = \Lambda S &= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \mathbf{q}}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} + \frac{\partial S}{\partial \mathbf{p}}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{phase} \\ &= \frac{\partial S}{\partial t} + \mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} + \mathbf{q}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{phase} \\ &= \frac{\partial S}{\partial t} + \frac{1}{2}\mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{group} + \frac{1}{2}\mathbf{q}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{group} \\ &= \frac{\partial S}{\partial t} + K + U \\ &= \frac{\partial S}{\partial t} + H.\end{aligned} \tag{44}$$

Since the variation in phase with respect to an observer moving at the phase velocity vanishes, we get

$$\frac{\partial S}{\partial t} + H = 0. \tag{45}$$

This is nothing but the Hamilton-Jacobi equation.

Since Equation (45) is equivalent to Equation (42), it follows that the case of Equation (42) contains dynamical relation correctly, so is physically possible. Of course, from the aspect of mathematics, it is possible the case that expression (37) is purely imaginary function as well. In this case, we should take

$$\Lambda \exp(\mathrm{i}\Phi) = \mathrm{i}\alpha \exp(\mathrm{i}\Phi), \tag{46}$$

which means the dynamical equation

$$\frac{\partial S}{\partial t} + H = \hbar\alpha. \tag{47}$$

Obviously, the above equation is physically meaningless.

From this, we get as the necessary and sufficient condition for satisfying physical and mathematical requirements the wave equation,

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i \frac{\partial \psi}{\partial q_i} + \left(\dot{p}_{phase}\right)_i \frac{\partial \psi}{\partial p_i}\right] = 0. \tag{48}$$

Although insignificant and even meaningless in the mathematical aspect, in order both to reveal the physical structure of the wave equation and to clarify the role of quantum operators, we multiply the both sides of Equation (48) by $\mathrm{i}\hbar$. Thus, Equation (48) becomes

$$\mathrm{i}\hbar\frac{\partial \psi}{\partial t} = -\mathrm{i}\hbar\sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i \frac{\partial \psi}{\partial q_i} + \left(\dot{p}_{phase}\right)_i \frac{\partial \psi}{\partial p_i}\right].$$

With the help of quantum operators, the above equation is written as

$$\hat{E}\psi = \sum_{i=1}^{f}\left[\left(\dot{q}_{phase}\right)_i \hat{p}_i\psi + \left(\dot{p}_{phase}\right)_i \hat{q}_i\psi\right]. \tag{49}$$

Equation (49) shows that the multiplication by $\mathrm{i}\hbar$ helps us to grasp quantum operators as the tools for determining physical quantities in microscopic world in terms of the wave function. It should be emphasized that the dynamical quantities obtained with the help of operators and wave function are not the same as classical ones, and get quantal. On the other hand, in view of Equations (30) and (31), we substitute particle-like quantities for wave-like quantities, $\left(\dot{q}_{phase}\right)_i$ and $\left(\dot{p}_{phase}\right)_i$ in the wave function. It is possible to introduce instead of the phase velocities in Equation (48) the group velocities represented by means of the Hamiltonian function,

$$\left(\dot{q}_{group}\right)_i = \frac{\partial H}{\partial p_i},$$

$$\left(\dot{p}_{group}\right)_i = -\frac{\partial H}{\partial q_i}.$$

In doing so, we obtain the wave equation for non-relativistic case written with the help of dynamical quantities,

$$\frac{\partial \psi}{\partial t} = -\frac{1}{2}\sum_{i=1}^{f}\left[\frac{\partial H}{\partial p_i}\frac{\partial \psi}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial \psi}{\partial p_i}\right].$$

With the help of the obtained operators, we can recast the above equation as

$$\hat{E}\psi = \frac{1}{2}\left\{\sum_{i=1}^{f}\left(\frac{\partial H}{\partial p_i}\hat{p}_i - \frac{\partial H}{\partial q_i}\hat{q}_i\right)\right\}\psi \tag{50}$$

or

$$\hat{E}\psi = \frac{1}{2}\left\{\sum_{i=1}^{f}\left[\left(\dot{q}_{group}\right)_i \hat{p}_i + \left(\dot{p}_{group}\right)_i \hat{q}_i\right]\right\}\psi. \tag{51}$$

In a more compact form, we write

$$\hat{E}\psi = \hat{H}\psi,$$

where the Hamiltonian operator takes the following form:

$$\hat{H} = \hat{T} + \hat{U}. \tag{52}$$

Generally, the solution of Equation (49), ψ is a complex-valued function. Equation (49) is the wave equation in phase space that is represented as a first-order partial differential equation. For this reason, ψ can be referred to as the wave function.

The fundamental equation of this formalism is distinguished from the Schrödinger equation because the wave function is defined not in configuration space but in phase space. As is already mentioned, for the configuration-space formalism there is the one-to-one correspondence relation between position and momentum. This fact is obvious from Equation (20), specifically,

$$\mathbf{p} = \mathrm{Re}\left\{\frac{\hat{\mathbf{p}}\,\psi(\mathbf{q})}{\psi(\mathbf{q})}\right\}.$$

Obviously, the above expression shows the one-to-one correspondence between position and momentum. It is important to note that on that account, the configuration-space formalism of quantum mechanics cannot give spectrum of momentum at a given position. Therefore, the momentum operator in configuration-space formulation of quantum mechanics should be regarded as the operator producing the mean value of momentum at a given position. This fact shows that the configuration-space formalism of quantum mechanics has limitation in describing the statistical characteristics of quantum mechanics.

On the contrary, the phase-space formalism of quantum mechanics takes advantage of describing statistical characteristics of quantum mechanics completely. For example, the momentum operator of the phase-space formalism enables us to determine momenta by means of the following relation:

$$\mathbf{p} = \mathrm{Re}\left\{\frac{\hat{\mathbf{p}}\,\psi(\mathbf{q},\mathbf{p})}{\psi(\mathbf{q},\mathbf{p})}\right\}. \tag{53}$$

This equation can be considered as the eigenvalue equation with respect to momentum operator. This relation indicates that when $\mathbf{q}$ is definitely given, $\mathbf{p}$ can be produced as a spectrum. For this reason, this formalism is expected to be useful to elucidate the statistical characteristics of quantum process. Similarly, for position operator we can conceive eigenvalue equation,

$$\mathbf{q} = \mathrm{Re}\left\{\frac{\hat{\mathbf{q}}\,\psi(\mathbf{q},\mathbf{p})}{\psi(\mathbf{q},\mathbf{p})}\right\}.$$

This means that a definite $\mathbf{p}$ has a spectrum of $\mathbf{q}$. The above discussion illustrates that quantum mechanics in phase space describes statistical features of quantum mechanics more properly than configuration-space formulation of quantum mechanics.

With the already defined phase of wave function, we can explain why the phase velocity is associated with kinetic energy rather than total energy. For the purpose of clarifying this matter, we calculate the velocity of propagation of an equal phase, *i.e.*, the time derivative of the phase with respect to the reference system moving at the phase velocity as

$$\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \Lambda\Phi &= \frac{1}{\hbar}\frac{\mathrm{d}S}{\mathrm{d}t} \\
&= \frac{1}{\hbar}\left[\frac{\partial S}{\partial \mathbf{q}}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} + \frac{\partial S}{\partial \mathbf{p}}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{phase} + \frac{\partial S}{\partial t}\right] \\
&= \frac{1}{\hbar}\left[\mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} + \mathbf{q}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{phase} + \frac{\partial S}{\partial t}\right] \\
&= \frac{1}{\hbar}\left[\mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} + \mathbf{q}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_{phase} - E\right] \\
&= \frac{1}{\hbar}\left[\mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} + U - E\right] \\
&= \frac{1}{\hbar}\left[\mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} - K\right],
\end{aligned} \tag{54}$$

where $\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase}$ is the phase velocity and K, the kinetic energy entering the total energy $E = K + U$. Since the phase velocity with respect to an observer moving at the phase velocity vanishes, we obtain

$$\mathbf{p}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{phase} - K = 0. \tag{55}$$

From Equation (55), we can define the phase velocity as

$$\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)_{phase} = \frac{K}{p}. \tag{56}$$

This is in perfect accord with our definition of the phase velocity, *i.e.*, Equation (33). Obviously, the action stands for the state of a statistical ensemble given by position, momentum, time and energy of particles just in the sense of phase. In the end, the phase velocity in quantum mechanics indicates the propagation rate of the state of a quantum system, which state is represented by a statistical ensemble in phase space carrying the meaning of phase as in a physical wave. We consider that the notion of the phase velocity can be naturally accepted from the point of view that an ensemble given as a set of position, momentum, time and energy behaves like a wave. It should be emphasized that we refer to the behavior of an ensemble as the matter wave. In this sense, the phase velocity is physically meaningful, so the velocity cannot exceed the light speed. Our investigation has demonstrated this fact in a reasonable way. It is the next task to explain the cause of the occurrence of the ensemble in microscopic world.

In doing so, we have obtained the wave equation for the wave function by beginning with the probability density. The abovementioned statement gives the obvious explanation of why the probability density is the modulus square of the wave function.

Altogether, the operators corresponding to fundamental observables are enumerated as

$$\hat{E} = \mathrm{i}\hbar\frac{\partial}{\partial t}, \tag{57}$$

$$\hat{p}_i = -\mathrm{i}\hbar\frac{\partial}{\partial q_i}, \tag{58}$$

$$\hat{q}_i = -\mathrm{i}\hbar\frac{\partial}{\partial p_i}, \tag{59}$$

$$\hat{U} = -\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{f} -\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} = \frac{1}{2}\sum_{i=1}^{f}\left(\dot{p}_{group}\right)_i \hat{q}_i, \tag{60}$$

$$\hat{T} = -\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{f}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} = \frac{1}{2}\sum_{i=1}^{f}\left(\dot{q}_{group}\right)_i \hat{p}_i, \tag{61}$$

where $\left(\dot{q}_{group}\right)_i$ and $\left(\dot{p}_{group}\right)_i$ denote the *i*th components of the group velocity with respect to position and momentum, respectively. The successive three operators correspond to energy, momentum and position, respectively, which become basic dynamical quantities. The fourth operator should be considered as the potential energy operator, since it corresponds to a potential energy function. This operator suggests nothing but the virial theorem of statistical mechanics. Thus, we can arrive at the important conclusion that for quantum mechanics, the potential energy should be represented as the virial of the system under consideration.

Meanwhile, the fifth operator should be considered as the kinetic energy operator, since it corresponds to kinetic energy.

There are important differences between operators of our formalism and ones of the configuration formalism because the wave functions applied by them are defined in different spaces. For the Schrödinger equation, the wave function is the state function defined in configuration space, whereas for the fundamental equation of QMPS, the wave function is the state function defined in phase space.

Thus, it has been explained why the probability density is the modulus square of the wave function, based on quantum mechanics in phase space. Evidently, such a explanation is impossible for the formulations starting with the wave function in configuration space: the standard theory of quantum mechanics and the previous phase-space formalism of quantum mechanics.

2.5. Relativistic Wave Equation of Phase-Space Formalism

The deduction of relativistic wave equation within this theory is useful for demonstrating the validity of this formalism. For this formalism, the phase velocity is considered to have real physical meaning. Since the phase velocity plays an important role in deducing the fundamental equation of this formalism, it should be exactly explained whether or not it has physical meaning. We already had explained this matter in Ref. [73]. See App. A.

Now, we can go over to the problem of obtaining a relativistic wave equation. To obtain the relativistic wave equation, we should replace $\left(\dot{q}_{phase}\right)_i$ and $\left(\dot{p}_{phase}\right)_i$ in Equation (48) with the relativistic relation. For the sake of simplicity, we take the Cartesian coordinate system. Following the preceding argument about the phase velocity, we adopt Equation (A.4) as the relativistic phase velocity.

From Equation (A.4)

$$v_{phase} = \frac{K}{p} = \frac{\sqrt{c^2 p^2 - m_0^2 c^4} - m_0 c^2}{p}, \tag{62}$$

we can represent the phase velocity in the vectorial form as

$$\mathbf{v}_{phase} = \frac{\sqrt{c^2 p^2 - m_0^2 c^4} - m c_0^2}{p^2}\mathbf{p}.$$

Accordingly, the ith component of the phase velocity is written as

$$\left(v_{phase}\right)_i = \frac{\sqrt{c^2 p^2 - m_0^2 c^4} - m_0 c^2}{p^2} p_i, \tag{63}$$

where p_i is the momentum component in the Cartesian coordinate system.

In the next place, we should determine $\left(\dot{p}_{phase}\right)_i$. We adopt the phase velocity in momentum space as

$$\begin{aligned}\left(\dot{p}_{phase}\right)_i &= m\cdot\frac{\mathrm{d}\left(v_{phase}\right)_i}{\mathrm{d}t} = m\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K}{p^2}p_i\right)\\ &= m\cdot\left\{\left[\left(\frac{K}{p^2}p_i\right),H\right]_{\mathrm{PB}} + \frac{\partial\left(\frac{K}{p^2}p_i\right)}{\partial t}\right\}\\ &= m\cdot\left[\left(\frac{K}{p^2}p_i\right),H\right]_{\mathrm{PB}}.\end{aligned} \tag{64}$$

Accordingly, we have

$$\left(\dot{p}_{phase}\right)_i = m\cdot\left[\left(\frac{K}{p^2}p_i\right),H\right]_{\mathrm{PB}}. \tag{65}$$

Inserting Equations (63) and (65) into (48) gives the relativistic wave equation. It is evident that for $c \gg v$, the relativistic wave equation turns into the non-relativistic wave Equation (51). Therefore, the obtained relativistic equation becomes the generalized wave equation of quantum mechanics in phase space. This implies an advance in combining the quantum theory with the relativity theory.

Thus, we have deduced the fundamental equation of quantum mechanics in phase space by making inference without jumps of logic. The ultimate outcome of our investigation is that we have obtained the fundamental equation of quantum mechanics in phase space starting with the probability density, thus having explaining the relation between the wave function and the probability density.

3. Generality and Validity of Quantum Mechanics in Phase Space

First, we shall demonstrate the generality of our formalism by showing that our formalism comprises the formulation of quantum mechanics in configuration space as its special case. Furthermore, we shall demonstrate how this formalism lays the foundation for the interpretation of important principles by taking the generalized proof of the uncertainty relations as an example. Thus, it is illustrated that this formalism provides a complete and consistent theory of quantum mechanics.

If the presented phase-space formalism would yield the Schrödinger equation as an approximation of its fundamental equation, it turns out that the theory becomes generalized. On the other hand, if the phase-space formalism of quantum mechanics would be possible, the fundamental equation of this formalism should necessarily contain the Schrödinger equation as its special case.

If this fact is confirmed, it turns out that the phase-space formalism possesses generality. We already had described this matter in Ref. [73]. We show in Apps. B and C that the Schrödinger equation can be derived from the fundamental equation of our formalism as an approximation. Obviously, introducing the conception of operators into the equation of QMPS enables us to easily interpret the relation between the fundamental equation of QMPS and the Schrödinger equation.

Thus, the generality of our formalism has been successfully demonstrated.

Next, we should demonstrate that our formalism gives reasonable explanations of the principles of quantum mechanics to show the validity of the formalism. For this purpose, we take the uncertainty principle as an example. The statistical formalism of quantum mechanics enables us to prove the uncertainty relations in a general way in the sense of the standard deviation by means of the commutation relation. The momentum operator assumed in this formalism, which applies to the wave function in phase space, is in perfect accord with that in the standard theory of quantum mechanics. Therefore, it is concluded that for the two cases the commutation relation between position and momentum operator is identical. In fact, the uncertainty relation can be similarly proved for the two cases.

With Apps. D and E as the citations of Ref. [73], we describe how our formalism offers the generalized proof of the uncertainty relations in terms of commutation relations of operators. Thus, the uncertainty relation not only for position and momentum but also for energy and time has been proved in a general way in terms of commutation relation of operators.

The aforementioned description of the generalized proof of the uncertainty relation shows that our phase-space formalism is consistent with the fundamentals of quantum mechanics and has significant potential capable of solving open questions relative to the foundations and interpretation of quantum mechanics.

Furthermore, it is useful to examine what the uncertainty relations, (D.6) and (D.10) mean based on our formalism. To tell the truth, the uncertainty relation dictates that the minimum of product of distributions of conjugate coordinate and momentum cannot get smaller than the Planck constant and that the distributions of two canonically conjugate observables are inversely proportional.

However, the information is not satisfactory as such. If the product of distributions of conjugate coordinate and momentum is larger than the Planck constant, how can we explain the uncertainty relation in a quantitative way? In fact, relation (D.6) cannot give a satisfactory answer to this question because it is represented as an inequality. As a question, is it possible that the product of two uncertainties becomes 2.1 h or 3.4 h or 20 h? If we understand the uncertainty relation as it is, these three values are all possible, since they are greater than the Planck constant. In fact, it is a task to clarify which case is true.

The solution to this problem can be given by considering on the basis of the de Broglie relation. It is significant to show the fact that the contents of the uncertainty relation are sufficiently explained on the basis of the de Broglie relation without using the commutation relation between operators and wave function. If it be possible, it will be the confirmation that the de Broglie relation is the foundation of the uncertainty relations and there is a different way to explain the relations more reasonably than the previous ones.

The analysis of the de Broglie relation has shown that the action gives the phase of material wave.

Let us consider that starting with relation (10), the uncertainty relation obtained in the sense of the standard deviation can be derived directly.

We believe that the nature of quantum lies in the quantization of the action. In other words, it is in every quantum-physical process that the action should be quantized. As an example, in three-dimensional case, the action as an integral along a path is a function in phase space to be written as

$$S=\int_{\mathbf{r}_0}^{\mathbf{r}}\mathbf{p}(\mathbf{r}')\mathrm{d}\mathbf{r}'=\int_{x_0}^{x}p(x')\mathrm{d}x'+\int_{y_0}^{y}p(y')\mathrm{d}y'+\int_{z_0}^{z}p(z')\mathrm{d}z'=kh,$$

where k is an integer. For convenience, we shall consider one-dimensional case. Then the action is represented as

$$S=\int_{x_0}^{x}p(x')\mathrm{d}x'=nh. \tag{66}$$

By the mean value theorem of the integral calculus, we have

$$S=\int_{x_0}^{x}p(x')\mathrm{d}x'=(x-x_0)p(\xi)=\Delta x\overline{p}, \tag{67}$$

where ξ is between x_0 and x, and $\overline{p}_n$ can be interpreted as the mean magnitude of momentum. Hence, we can write the quantization condition for momentum as

$$\Delta x\overline{p}_n=nh. \tag{68}$$

Therefore, a quantized momentum in a given interval Δx is written as

$$\overline{p}_n=\frac{nh}{\Delta x}.$$

It is obvious that for $n=1$, Δx is the wavelength of material wave. It means that for a quantum state to be realized, for a given momentum the magnitude of space necessary for a quantum state at least should be equal to the wavelength of material wave. If the magnitude of space is less than the wave length, a quantum state is impossible to be realized.

For a given n, magnitude of momentum and distribution of coordinate are in the inversely proportional relation. This shows that the greater momentum is, as a result of wavelength of material wave to get smaller, the weaker the quantization of space is and thus the character of the given system approaches that of classical system characterized by continuum. For a given Δx, the smallest variation in momentum due to quantum fluctuation is

$$\Delta p=\frac{h}{\Delta x}. \tag{69}$$

Therefore, if $\overline{p}_n\gg\Delta p$, then the quantum fluctuation in state of a system is negligibly small and thus the system becomes classical.

On the other hand, if $\overline{p}_n \approx \Delta p$, the quantum fluctuation proves to cause remarkable change in state of a particle and therefore its state becomes quantum. The influence of Δp due to quantum fluctuation on the velocity of a particle is the criterion for distinguishing between microscopic particle and macroscopic one. For a quantum fluctuation in momentum, Δp, on account of relative difference between masses of macroscopic particle and microscopic one the change in velocity of macroscopic particle is considerably small compared to that of microscopic particle and as a result the macroscopic particle is insensitive to quantum fluctuation.

On the other hand, we can consider the uncertainty relation in the sense of the standard deviations of canonically conjugate quantities.

Our view is that quantum fluctuation puts in action a family of virtual trajectories, *i.e.*, an ensemble of trajectories around a classical trajectory and thus it is possible to imagine the deviation of actions in reference to the classical trajectory.

The action integral for the classical path usually has a minimum according to the principle of least action. For the classical path, we have as a minimum

$$S_0 = \int_{x_0}^{x} p_0\left(x'\right)\mathrm{d}x' = \Delta x \overline{p}_0 = n_0 h, \tag{70}$$

where n_0 is an integer. For quantum processes taking place in the equal interval, actions are represented as

$$S_n = \int_{x_0}^{x} p_n\left(x'\right)\mathrm{d}x' = \Delta x \overline{p}_n = nh, \tag{71}$$

where n is an integer.

The subtraction of Equation (71) from Equation (70) determines the quantum fluctuation as

$$\left|S_n - S_0\right| = \Delta S_n = \Delta x\left|\overline{p}_n - \overline{p}_0\right| = \Delta x \Delta \overline{p}_n = \left|n - n_0\right| h. \tag{72}$$

From Equation (72), we have as the standard deviation of action

$$\Delta S = \sum_n W_n \left|S_n - S_0\right| = \Delta x \sum_n W_n \Delta \overline{p}_n,$$

where W_n refers to the probability of a fluctuating quantum path.

Identifying $\sum_n W_n \Delta \overline{p}_n$ with the mean value of momentum deviation, Δp, we get

$$\Delta x \sum_n W_n \Delta \overline{p}_n = \Delta x \cdot \Delta p = kh, \tag{73}$$

where k is an integer.

From Equation (73), it follows that the minimum condition of quantum fluctuation is

$$\Delta x \cdot \Delta p = h. \tag{74}$$

Since there is no quantum fluctuation in momentum in the case of a free particle of a definite momentum, Equation (73) reads

$$\Delta x \cdot \Delta p = 0, \tag{75}$$

which should be considered to represent a classical state free of quantum fluctuation. Equations (73)-(75) imply that the uncertainty can change exactly by an integer multiple of the definite value, h, which integer multiple begins from zero.

Thus, as the essential contents of the uncertainty relation, the facts are explained satisfactorily in terms of the action in phase space that there exists the quantum of phase space, h and the uncertainty relation is determined by an integer multiple of h. As an important understanding of classical limit of this principle, it is satisfactorily explained also the fact that for a particle of great momentum the uncertainty of position due to quantum fluctuation is small. This method explains more than that proving the uncertainty relation in terms of the commutation relation of operators in the sense of standard deviation. In fact, the relation of Equation (68) is not possible to be expounded correctly in the sense of the standard deviation.

Next, let us consider the uncertainty relation for time and energy. Quantum states in energy-time space are represented by action as

$$S = \int_{t_0}^{t} E(t')\mathrm{d}t' = nh. \tag{76}$$

According to the mean value theorem of the integral calculus, the above expression can be recast as

$$S = \int_{t_0}^{t} E(t')\mathrm{d}t' = (t - t_0)E(\tau) = \Delta t\bar{E} = nh, \tag{77}$$

where τ is between t_0 and t and Δt is a duration of quantum process. Therefore, the relation between the magnitude of energy and the duration of time is represented as

$$\bar{E} = \frac{nh}{\Delta t}, \tag{78}$$

otherwise,

$$\Delta t = \frac{nh}{\bar{E}}. \tag{79}$$

If n is given, the magnitude of energy and the duration of time are inversely proportional to each other. This indicates that the greater the energy is, the weaker the quantization of time, and as a result, the character of a system approaches that of classical system assuming continuous time.

As in the case of dealing with the uncertainty relation for position and momentum, for classical and quantum processes we determine the deviation of action as

$$|S_n - S_0| = \Delta S_n = \Delta t|\bar{E}_n - \bar{E}_0| = \Delta t \cdot \Delta\bar{E}_n = |n - n_0|h. \tag{80}$$

With the help of Equation (80), the standard deviation of action is written as

$$\Delta S = \sum_n W_n |S_n - S_0| = \Delta t \sum_n W_n \Delta\bar{E}_n.$$

Since $\sum_n W_n \Delta\bar{E}_n$ is the mean value of energy deviation, ΔE, it follows that

$$\Delta S = \Delta t \cdot \Delta E = kh, \tag{81}$$

where k is an integer.

Naturally, from Equation (81), as the minimum condition of quantum fluctuation, we get

$$\Delta t \cdot \Delta E = h. \tag{82}$$

For a definite energy, since quantum fluctuation vanishes, Equation (81) is written as

$$\Delta t \cdot \Delta E = 0. \tag{83}$$

It is essential that from Equations (12) and (13) issue Equations (D.6) and (E.10). Therefore, it would be correct to start with Equation (10) in order to establish the foundation of the uncertainty relation. Obviously, this interpretation is different from Heisenberg's interpretation of the uncertainty relation, since it is not related to measurement.

It is important to discuss the lowest limit of uncertainty. The matter is whether it is h or $\hbar$ or $\hbar/2$. It is reasonable to take the limit as h in that the foundation of the uncertainty relation is the de Broglie relation and the direct explanation of the uncertainty relation in terms of the action function is more straightforward and general than that in terms of standard deviation. In fact, we take the cell in phase space as h in statistical mechanics. Ultimately, we can conclude that the explanation of the uncertainty relation in terms of the action is more intuitive and general than the previous, and contains the complete contents of the uncertainty principle.

It should be emphasized that our formalism gives satisfactory answers to the correspondence principle as well. In fact, the complete system of operators of our formalism explains how to interpret the correspondence between dynamical quantities and quantum operators. In addition, it should be noted that the fundamental equation of our formalism is a linear equation, so our formalism agrees with the superposition principle.

In this manner, it is demonstrated that our formalism is consistent with all the principles of quantum mechanics.

4. Results and Discussion

We can easily grasp three striking features of quantum theory as its success, its rejection by some of our deepest thinkers, and the absence of compelling alternatives [76]. Of course, the standard theory of quantum mechanics must be acknowledged as the most successful of all physical theories. In fact, based on quantum mechanics, a vast range of phenomena can be understood and the validity of quantum theory has been confirmed by experiments of high precision.

However, to quantum mechanics remain the demanding tasks to solve open questions on the foundations and interpretation, and there exist several formulations irreconcilable with one another from the point of view of quantum foundations. Such a situation of a lack of consensus on theoretical perspective sets up the important task to open up new avenues for the successful solution of open questions and to explain the relations between the existing formulations. A key question here is to explain the relation between the wave function and the probability density as the cornerstone of the construction of quantum mechanics. In our view, the solution of this task is impossible without adopting a new formulation of quantum mechanics.

An alternative formulation should not only explain existing experiments exactly but also incorporate other seemingly secure fields of modern physics such as special relativity, field theory, and the theory of elementary particle.

To achieve the goal, we have established an alternative formalism in phase space which contains the present configuration-space formulation as its special case. Our work has shown how to obtain within the framework of its theory the fundamental equation of quantum mechanics in phase space without recourse to the other formulations of quantum mechanics, and how to get the idea for operators pertaining to dynamical quantities.

What is important here is that the wave equation for microscopic particles independent of the previous ones can be easily obtained without a jump of logic, provided that the research starts with statistical ensemble in phase space embodying the wave field. Interestingly, the derivation of the fundamental equation of our formalism does not need the Schrödinger equation, while the former contains the latter. In fact, this equation yields the Schrödinger equation by making a definite approximation. This fact shows that the phase-space formulation possesses generality as an extension of the configuration-space formulation.

On the other hand, with the help of the fundamental equation of our formalism, we have obtained reasonable results such as the reproduction of the Liouville theorem and the virial theorem for quantum mechanics.

Importantly, it has been demonstrated that our formalism can easily give the relativistic wave equation without treating the problem of linearizing the Hamiltonian operator in such a way that one takes into consideration the point that the fundamental equation is a first-order partial differential equation with respect to time, position and momentum variables, and uses the representation of the relativistic phase velocity.

Since the fundamental equation is a first-order partial differential equation with respect to position, momentum and time, from the point of view of the Lorentz symmetry it is superior to the Schrödinger equation. Moreover, this equation takes advantage over those of conventional formulations in that it uses generalized coordinates.

For our formalism, the idea for quantum operators is given naturally and logically from the structure of the wave function defined in phase space in terms of the action. Specifically, our formalism provides a reasonable explanation of why operators corresponding to canonically conjugate dynamical quantities should be introduced to quantum mechanics and, in particular, what the time operator is. Here, the action plays a key role. With the action constituting the phase part of the wave function, we can find a complete set of canonical conjugate variables and the corresponding operators. It is notable that the predetermined form of the wave function of our formalism inevitably forms the conception of the time operator, which makes the system of quantum operators complete.

Furthermore, it has been shown that this formalism has good potential of providing reasonable results of quantization by dealing with some important problems including the uncertainty relation, which confirm the validity of this formalism.

The final result of our investigation leads to the conclusion that important open questions of quantum mechanics can be studied successfully based on the presented formalism of quantum mechanics in phase space. What is best of all, this formalism, starting with ensemble in phase space indicating a really existing object, successfully has offered the interpretation of the relation between the wave function and the probability density. As shown in subsection 2.4, the deduction of the fundamental equation of our formalism involves the elucidation of the relation between the wave function and the probability density. Our formalism has given an ontological interpretation on Born's rule because it develops passing the milestones of concepts: wave field → ensemble in phase space → the deduction of wave equation → the elucidation of Born's rule thereof.

Thus, our formalism has a mathematical structure sufficient to unravel seemingly mysterious quantum phenomena based on statistical perspective. The outcomes of research show that our formalism is possessed of generality and thus the present configuration formalism is an approximation of the presented phase-space formalism.

In this regard, it is necessary to recall that even now, some of physicists indicate that since quantum mechanics remains incomplete, an alternative formulation with firmer foundations and more consistent structure is required. It is well known that an acute topic of the controversy over the foundations of quantum mechanics is whether quantum mechanics is complete.

James B. Hartle who was a rapporteur of the 23rd Solvay Conference on Physics wrote as follows [76].

Even while acknowledging its undoubted empirical success, many of our greatest minds have rejected quantum mechanics as a framework for fundamental theory. Among the pioneers, the names of Einstein, Schrödinger, De Broglie, and Bohm stand out in this regard. Among our distinguished contemporaries, Adler, Leggett, Penrose, and 't Hooft could probably be counted in this category. Much of this thought has in common the intuition that quantum mechanics is an effective approximation of a more fundamental theory built on a notion of reality closer to that classical physics. Remarkably, despite eighty years of unease with its basic premises, and despite having been tested only in a limited, largely microscopic, domain, no fully satisfactory alternative to quantum theory has emerged.

The gist of this statement is that not a few physicists who considered and consider the present quantum theory to be a good approximation of a more fundamental theory were in the past and also are even now, and next that there is not yet any other promising theory superior to the existing theories.

The problem of the foundations of quantum mechanics has the history of long and serious debates. As a result, we witness the diversity of views on the foundations of quantum mechanics.

Up to now, no prediction of quantum phenomena such as energy spectra, transition probabilities, cross sections etc. by virtue of quantum mechanics has been disproved experimentally. Nevertheless, there exist conceptual problems related to the foundations of quantum theory. Many physicists put aside these problems to the field of philosophy, but the consideration of these problems concerning the philosophy of science can initiate a new progress of theoretical revolution in physical science.

To understand the significance of an emergent theory of quantum mechanics, it is useful to recall what Roger Penrose wrote [33]:

Although his (Einstein's voice) *was not a lone voice-for Schrödinger and Dirac also regarded the quantum theory as being in an important way "provisional"-Einstein's criticisms were made more openly, and they attained a particular weight owing to his reputation……We find that matters of "interpretation" of quantum theory, that have for decades been regarded by physicists as "mere matters of philosophy", can lead to very significant physical effects, some even having important current commercial implications, such as quantum cryptography, and others having more remote potentialities, such as quantum computation.*

In this connection, it is necessary to again remember Takabayasi's remaks that a new formulation and picture of quantum mechanics may lead to elucidating new aspects of physical and mathematical structure of quantum mechanics, and to finding new clues to the future progress of quantum theory, irrespective of its usefulness of practical applications.

To satisfy such a requirement, it is necessary to construct an innovative formalism of quantum mechanics with the potential of solving the interpretation problem. Thus, we have presented an alternative version of quantum mechanics in phase space as an independent and inclusive theory.

To emphasize the significance of the subject of our research, it is necessary to cite the following description written by Auletta [1]:

In spite of this remarkable success, quantum mechanics remains mysterious. It is not only the problem of explaining its meaning without using advanced mathematics that forbids a simple exposition of its properties to the layman. The rules are weird: the fundamental objects are complex amplitudes and probabilities are the modulus square. A scientist like Feynman who contributed to a new formulation of quantum mechanics and made some of the crucial steps to extend quantum mechanics into the relativistic domain, wrote once nobody understands quantum mechanics. Also Poliakov, one of the greatest living theoretical physicists said in a lecture that eventually someone has to explain why the probability is the modulus square of complex amplitude.

According to Auletta's statement, we can see that the subject of our investigation is perfectly legitimate and the obtained result becomes a significant contribution.

It is necessary to consider why the subject under consideration can be solved just based on a new formulation which is fundamentally different from the existing ones. A fundamental theory of physic deals with the nature of real existences. This problem is particularly acute and intractable in quantum mechanics. The most important conception of quantum mechanics is the wave function which plays the crucial role of theoretical construction in determining the probabilities of later events. According to the standard theory, quantum mechanics proves to be constructed based on the wave function, which has no direct physical meaning. Thus, the problem is raised that asks: if the wave function is not physical, does something that is a reality associated with the wave function exist or not?

All the preceding formulations cannot give a straightforward answer to this question because of the limitation relevant to the used configuration formulation, but our investigation provides a perfect solution to this problem with the help of methodology based on phase space.

Our formalism is essentially distinguished from the formulations starting with the wave function. The latter inevitably needs the assumption about the relation between the wave function and the probability density to connect the wave function to a physically meaningful object.

We can represent the case with the help of the diagram shown in **Figure 3**.

It is impossible to deduce the wave equation starting with the probability density in configuration space so that the procedure can carry the physical meaning of the wave function. That is because the procedure dealing with ensemble in configuration space cannot lead to conceiving such an alternative equation of motion with regard to the probability density in configuration space as the Liouville theorem which is established in phase space.

The phase-space formulations using the Wigner function and the Weyl map is a theory dealing with the mapping of the wave functions in configuration space to the probability density in phase space. In the conventional phase-space formulation, one determines the phase-space distribution function, *i.e.*, the Wigner function in use of the wave function obtained by solving the Schrödinger equation. The Wigner function

$$W\left(x,p\right)=\frac{1}{2\pi\hbar}\int \mathrm{d}y\psi^{*}\left(x+\frac{r}{2}\right)\mathrm{e}^{-ipr/\hbar}\psi\left(x-\frac{r}{2}\right)$$

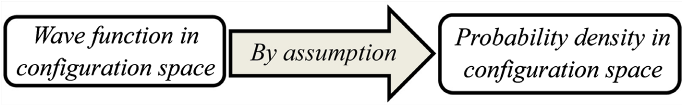

Figure 3. The standard theory of quantum mechanics transforms the wave function in configuration space into the probability density in configuration space based on an assumption.

satisfies just the normalization condition. With the help of a formal method devised purely in a mathematical way, the Wigner function has such a physical meaning as the Boltzmann distribution function in statistical physics. However, it is not always that the Wigner function satisfies the positive-value property of distribution function. In fact, the Wigner function is not derived from a certain physical foundation but is assumed in view of the normalization condition. This tells us that the Wigner function, in the true sense, is not a distribution function as determined uniquely subject to a definite physical condition, and thus there may be alternatives.

On the other hand, this formulation does not assume the phase of quantum state and the wave function which contains the phase information. Hence, we can see the limitation which is due to the formal methodology of the preceding phase-space formulation. In fact, to overcome such shortcomings, the tomography theory has been developed. But this formulation also cannot transcend the limitation pertaining to the phase of quantum state and the introduction of the wave function as the container of phase information.

The relationship between the wave function in configuration space and the distribution function in phase space can be represented by the diagram shown in **Figure 4**.

Since the previous phase-space formulations commonly use the wave function in position space to obtain the phase-space distribution function, they also are the formulations starting with the wave function given by the Schrödinger equation instead of the probability density. Therefore, the previous phase-space formulations cannot explain the relation between the wave function and the probability density.

On the contrary, our formalism takes the advantage of developing from the probability density to logically resolve the problem of the relation between the wave function and the probability density without the aid of assumption. The diagram depicted in **Figure 5** shows the above explanation.

Unlike the previous formulations of quantum mechanics in phase space, our formulation introduces the wave function to treat the phase information of quantum state, and makes use of observables and the corresponding operators together.

An essential feature is that our formulation adopts the procedure that starts with the probability density to yield the wave function. Thus, our formulation is

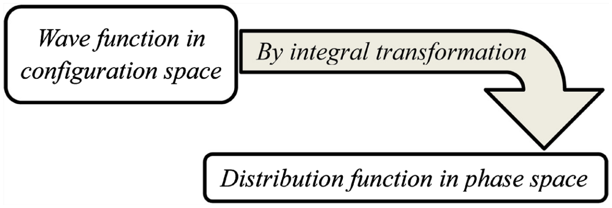

Figure 4. Conventional quantum mechanics in phase space transforms the wave function in configuration space into the distribution function in phase space in terms of integral transformation.

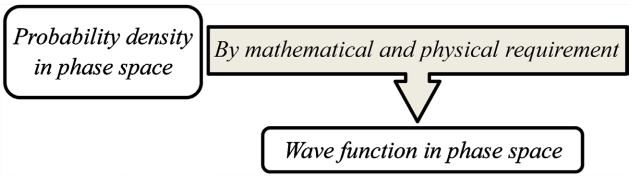

Figure 5. Our phase-space formalism determines the wave function in phase space in terms of the probability density in phase space in compliance with mathematical and physical requirements instead of assumption.

clearly distinguished from both the previous formulations of quantum mechanics in phase space and the standard theory of quantum mechanics.

Quantum theory must be generalized to apply to all physical worlds irrespective of microscopic and macroscopic worlds, for instance, cosmology and quantum spacetime. Our formulation rests on phase space, so it is a deterministic theory in the sense of statistical mechanics. It means that our formalism has raised the possibility that quantum theory is consistently connected with physics of macroscopic scale and the relativity theory. In fact, the deduction of the relativistic wave equation based on our formulation has illustrated that the connection between quantum theory and relativity theory can be made in a usual and smooth way.

In this way, our formalism starting with phase space fundamentally has explained the relation between the wave function and the probability density thanks to the potential and advantage of statistical perspective. Finally, the goal of our research has been achieved by adopting a new approach to quantum mechanics in phase space.

5. Conclusions

We have constructed an alternative formalism of quantum mechanics in phase space distinguished from the previous formulations which keep to formal technique of mathematical operation. The best result the presented formalism provides is the explanation of the relation between the wave function and the probability density, thus withdrawing the most important assumption of quantum mechanics. Without considering statistical ensemble in phase space and adopting the statistical perspective on quantum mechanics, it is impossible to reach the goal. Our work has shown that it is possible to establish an alternative autonomous formalism of quantum mechanics in phase space, starting with statistical ensemble in phase space. This formalism independent of the previous formulations of quantum mechanics clarifies the assumed Born's rule as a direct result of its theory.

Based on this formalism, we have confirmed that phase space is more suitable than configuration space for the construction of consistent quantum theory. As the investigation of quantum mechanics on the basis of configuration space showed, the approach starting with the wave function cannot help introducing the assumption about the relation between the wave function and the probability density in order to connect the wave function with a certain physically meaningful quantity.

On the contrary, our approach starting with the probability density in phase space naturally elucidates as a result of logical deduction for formulating the theory why the probability density has to be the modulus square of the wave function. In the end, the solution of the open question at issue turns out to be a natural and inevitable result of our formalism. This fact shows that quantum mechanics should be constructed in phase space as in classical mechanics.

Up to now, the standard formulation of quantum mechanics has been certainly most successful, so it is still playing the leading role in developing the science for microscopic world, pursuing the picture essentially different from that of classical mechanics. Nevertheless, this formulation cannot overcome limitations in solving several open questions, for instance, including the relation between the wave function and the probability density. Evidently, within the confines of the standard theory of quantum mechanics, it is impossible to construe the relation between the two concepts under consideration, since with the probability density in configuration space, it is impossible to arrive at the concept of wave function. In fact, it is in configuration space that one cannot apply the Liouville theorem to connect the probability density with the wave function. The previous formulations in phase space begin with configuration space to go over to phase space, so they also cannot transcend the limitation of configuration formulation. The understanding of such limitations of the previous formulations makes us emphasize the importance and significance of our formulation that can make a natural and logical connection between the concepts of the wave function and the probability density.

In conclusion, our work confirms that the presented phase-space formalism of quantum mechanics is consistent with the fundamentals of quantum mechanics, and its superiority to the other formulations enables a clear explanation of the relation between the wave function and the probability density as the most important assumption of quantum mechanics. The aspect characterizes our formalism that the formalism is built in phase space and develops from the probability density. Moreover, our work confirms that there is sufficient ground for the view that quantum mechanics has a structure akin to statistical mechanics [77] [78].

In this paper, we referred to some parts of our previous papers [73] to systematically explain a new theme distinct from the previous subjects: Born's probabilistic interpretation. Nevertheless, we look upon this paper as having an independent significance, since it treats the most important open question of quantum mechanics as a particular subject.

Our research offers good prospects of resolving a wide range of subjects relevant to the foundations of quantum mechanics. Based on the perspective and theory presented in this paper, the authors plan to propose in the nearest future some issues obtained in the research on the foundation of quantum mechanics including the interpretation of the correspondence principle and the measurement problem.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A: Relativistic Phase Velocity

It is currently accepted that the phase velocity has not physical meaning for no other reason than for the relativistic case the phase velocity determined by definition exceeds the speed of light. Meanwhile, it is inconsistent with our common sense in physics to accept the very fact that the phase velocity is an important concept which characterizes the probability wave, while it has no physical meaning. However, we can confirm that even for the relativistic case the phase velocity cannot exceed the speed of light, provided that the rest energy of a particle is considered as the origin of energy.

Now, let us examine points in question of the present definition of the phase velocity and interpretation. According to the standard theory of quantum mechanics, the phase velocity determined by definition exceeds the light speed. The case is obvious because the phase velocity is computed as

$$v_{phase} = \frac{E}{p} = \frac{\sqrt{c^2p^2 + m^2c^4}}{p} = c\sqrt{1 + \frac{m^2c^2}{p^2}} > c\,.$$

From this, it follows that the phase velocity is always higher than the light speed.

On the other hand, according to the special theory of relativity, the limit of speed in nature is the light speed. For this reason, most of physicists consider that the phase velocity is meaningless in the physical respect [25] [79] [80] [81].

In this connection, the problem of the interpretation of the foundation of quantum theory inevitably is raised. If the phase velocity were to be physically meaningless, it would be problematic to define this quantity itself. Physic should adopt only such quantities that are related exactly to physical realities. If not so, the foundations of such a theory are not to be trusted. If we define physically meaningful quantities with the help of the phase velocity and obtain certain results related to those quantities, it means that certainly, we cannot avoid some errors involved in the definition and interpretation.

In the next place, the following question is raised. Does the phase velocity of the matter wave give physical meaning at least in the nonrelativistic case? It is not possible to give any positive answer to this question as well. We can explain through the following review that the phase velocity is physically meaningless even in the nonrelativistic case. In the nonrelativistic case, the energy of a free particle is expressed as

$$E = \sqrt{c^2p^2 + m^2c^4} \approx mc^2 + \frac{1}{2}mv^2. \tag{A.1}$$

Taking mc^2 in Equation (A.1) for the origin of energy, we obtain the energy in the nonrelativistic case:

$$E \approx \frac{1}{2}mv^2.$$

Provided this relation is used to define the phase velocity, the result proves to be reasonable. On the other hand, if Equation (A.1) is employed to do so, the result is given as

$$v_{phase} = \frac{E}{p} \approx \frac{mc^2}{p} + \frac{1}{2}\frac{mv^2}{p} = \frac{mc^2}{p} + \frac{v}{2}. \tag{A.2}$$

From the above expression, it follows that when momentum vanishes, the phase velocity always diverges and generally, the phase velocity is higher than the group velocity. Therefore, we can see that even in the nonrelativistic case the phase velocity is meaningless unless the origin of kinetic energy is taken correctly.

Some physicists who take the position of compromise affirm that the phase velocity exceeding the light speed is physically possible. Meanwhile, we understand that according to the special theory of relativity the light speed is identified with the maximum transmission rate of interaction in nature. If so, the question on what physical reality the phase velocity expresses is raised. In case the phase velocity is related to a certain interacting physical reality, the phase velocity cannot be free from the constraint on the limit speed. From the above argument, it is evident that the definition of the phase velocity still contains ambiguous points.

In relativistic mechanics, we define as the kinetic energy the part of the particle's energy that turns into zero as its velocity vanishes.

Thus, we take the kinetic energy of a particle for

$$K = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2. \tag{A.3}$$

Then the phase velocity is determined by

$$v_{phase} = \frac{K}{p} = \frac{\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2}{p} = \frac{\frac{m_0 c^2 - m_0 c^2\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}}{\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{c^2\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)}{v}. \tag{A.4}$$

From this, it follows that as $v \ll c$, $v_{phase} \approx \frac{1}{2} v_{group}$, while as $v \to c$, $v_{phase} \to c$. Meanwhile, as $v \to 0$, $v_{phase} \to 0$ according to L' Hospital rule. Hence, we can see that the phase velocity varies from zero to the light speed and cannot exceed the light speed. Therefore, the inconsistency that the phase velocity is infinite disappears and it is no wonder that the phase velocity is a physically meaningful quantity. In the end, we reach the conclusion that the conventional definition of the phase velocity to be wrong.

In relation to characteristics of the phase velocity, it is necessary to consider the case when the phase velocity equals the group velocity. For the two velocities to be equal, from Equation (A.4) the following relation should hold.

$$v_{phase} = \frac{c^2\left(1-\sqrt{1-\left(v^2/c^2\right)}\right)}{v} = v. \tag{A.5}$$

Hence, we go through

$$c^2\left(1-\sqrt{1-\left(v^2/c^2\right)}\right)=v^2,$$

$$1-\frac{v^2}{c^2}=\sqrt{1-\left(v^2/c^2\right)},$$

$$\left(1-\frac{v^2}{c^2}\right)^2-\left(1-\frac{v^2}{c^2}\right)=0,$$

$$-\left(1-\frac{v^2}{c^2}\right)\cdot\frac{v^2}{c^2}=0. \tag{A.6}$$

From Equation (A.6), it follows $v=c$. This fact tells us that since a velocity of a particle cannot arrive at the light speed, the phase velocity cannot reach the group velocity.

Meanwhile, it is necessary to compare the phase velocity with the group velocity. Starting with the phase velocity:

$$v_{phase}=\frac{c^2\left(1-\sqrt{1-\left(v^2/c^2\right)}\right)}{v}, \tag{A.7}$$

our calculation goes through

$$1-\frac{v_{phase}v}{c^2}=\sqrt{1-\left(v^2/c^2\right)},$$

$$-2\frac{v_{phase}v}{c^2}+\left(\frac{v_{phase}v}{c^2}\right)^2=-v^2/c^2,$$

$$-2v_{phase}+\frac{v_{phase}^2v}{c^2}=-v.$$

Hence, we get

$$v=\frac{2v_{phase}}{\frac{v_{phase}^2}{c^2}+1}. \tag{A.8}$$

Since v_{phase} cannot exceed the speed of light, taking into consideration the relation $v_{phase}^2/c^2<1$, we conclude that the relation $v_{phase}<v$ always holds. From this, it follows that the phase velocity is always lower than the group velocity.

In this connection, we can analyze the inconsistent respects that the conventional definition causes. In the nonrelativistic case, the phase velocity is a half of the group velocity. But in the relativistic case, the phase velocity is always higher than the light speed, so that we feel as if the logic connection between relativistic and nonrelativistic theory were broken.

However, if the phase velocity is defined in terms of Eq. (A.4), there is not such an inconsistency. It turns out that the phase velocity satisfies the relativistic requirement.

Appendix B: Phase-Space Formalism Containing the Schrödinger Equation as Its Special Case

To demonstrate that our phase-space formalism contains the Schrödinger equation as its special case, let us start with the momentum operator. For a given wave function, $\psi = \varphi \exp\left(\mathrm{i}\frac{S}{\hbar}\right)$, the momentum operator, $\hat{\mathbf{p}}$ should satisfy the following operator equation:

$$\hat{\mathbf{p}}\psi = \mathbf{p}\psi + \frac{1}{\varphi}\hat{\mathbf{p}}\varphi \cdot \psi = \left(\mathbf{p} + \frac{1}{\varphi}\hat{\mathbf{p}}\varphi\right)\psi = \tilde{\mathbf{p}}\psi,$$

where ψ_o is a real-valued function and in general, $\tilde{\mathbf{p}}$, a complex-valued function. In the above equation, we took into consideration $\hat{\mathbf{p}}\exp\left(\mathrm{i}\frac{S}{\hbar}\right) = \mathbf{p}\cdot\exp\left(\mathrm{i}\frac{S}{\hbar}\right)$.

Therefore, we can imagine the following correspondence:

$$\mathbf{p}_R = \mathrm{Re}\left(\frac{\hat{\mathbf{p}}\psi}{\psi}\right) = \mathbf{p},$$

$$\mathbf{p}_I = \mathrm{Im}\left(\frac{\hat{\mathbf{p}}\psi}{\psi}\right) = \frac{-\mathrm{i}}{\varphi}\hat{\mathbf{p}}\varphi.$$

Without loss of generality, we have $\tilde{\mathbf{p}} = \mathbf{p} + \frac{1}{\varphi}\hat{\mathbf{p}}\varphi = \mathbf{p}_R + \mathrm{i}\mathbf{p}_I$.

The above operator equation shows how the operator corresponding to a given dynamical quantity should yield the corresponding dynamical quantity as a result of its application to the wave function. In general, dynamical quantities in quantum mechanics are represented not by an eigenvalue, but by a function. In the above expression, $\mathbf{p}$ generally is not an eigenvalue but a function pertaining to observables. Therefore, the function, $\mathbf{p}$ can be referred to as the function on momentum. On the other hand, $\mathbf{p}_I$ can be considered to be relevant to the wave-like characteristics.

Now, we consider the approximation of the fundamental equation to the Schrödinger equation. Introducing the momentum instead of the phase velocity in Equation (50) gives

$$-\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{f}\left[\frac{p_i}{m_{(p_i)}}\frac{\partial\Psi(\mathbf{q},\mathbf{p},t)}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial\Psi(\mathbf{q},\mathbf{p},t)}{\partial p_i}\right] = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\mathbf{q},\mathbf{p},t). \tag{B.1}$$

Here, Φ is the wave function and $m_{(p_i)}$ denotes the mass of a particle having the momentum component, p_i.

Corresponding $-\mathrm{i}\hbar\frac{\partial}{\partial q_i}$ to the momentum operator, $\hat{p}_i$ leads to

$$\frac{1}{2}\sum_{i=1}^{f}\left[\frac{p_i}{m_{(p_i)}}\hat{p}_i + \mathrm{i}\hbar\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right]\Psi(\mathbf{q},\mathbf{p},t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\mathbf{q},\mathbf{p},t). \tag{B.2}$$

By assumption, we write $\Psi(\mathbf{q},\mathbf{p},t) = \varphi(\mathbf{q},\mathbf{p},t)\cdot\exp\left(\frac{\mathrm{i}}{\hbar}S\right)$. Here $\varphi(\mathbf{q},\mathbf{p},t)$ is a real-valued function. Taking into account the commutation relation between momentum and momentum operator,

$$p_i\hat{p}_i - \hat{p}_i p_i = -\mathrm{i}\hbar\left(p_i\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i}p_i\right) = 0,$$

and the operator equation,

$$\hat{p}_i\Psi(\mathbf{q},\mathbf{p},t) = p_i\Psi(\mathbf{q},\mathbf{p},t) + \frac{1}{\varphi}\hat{p}_i\varphi\cdot\Psi,$$

we get the following equation:

$$\left[\frac{1}{2}\sum_{i=1}^{f}\frac{\hat{p}_i^2}{m_{(p_i)}} - \frac{1}{2}\sum_{i=1}^{f}\frac{1}{m_{(p_i)}}\hat{p}_i\left(\frac{1}{\varphi}\hat{p}_i\varphi\right) + \hat{U}\right]\Psi(\mathbf{q},\mathbf{p},t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\mathbf{q},\mathbf{p},t). \tag{B.3}$$

For the sake of convenience, we introduce following notation:

$$\hat{U}_o = -\frac{1}{2}\sum_i\frac{1}{m_{(p_i)}}\hat{p}_i\left(\frac{1}{\varphi}\hat{p}_i\varphi\right) + \hat{U}. \tag{B.4}$$

To transform Equation (B.3) into Schrödinger equation, it is necessary to perform the following variable separation:

$$\Psi(\mathbf{q},\mathbf{p},t) = \psi(\mathbf{q},t)\phi(\mathbf{p}). \tag{B.5}$$

Inserting the above function into Equation (B.3), we get

$$\left(\sum_{i=1}^{f}\frac{\hat{p}_i^2}{2m_{(p_i)}} + \hat{U}_o\right)\left[\psi(\mathbf{q},t)\phi(\mathbf{p})\right] = \mathrm{i}\hbar\frac{\partial}{\partial t}\left[\psi(\mathbf{q},t)\phi(\mathbf{p})\right].$$

Multiplying both sides of the above equation by $\phi^*(\mathbf{p})$ and integrating it over $\mathbf{p}$, we obtain the following equation:

$$\left(\sum_{i=1}^{f}\frac{\hat{p}_i^2}{2m_{(p_i)}} + \int\phi^*\hat{U}_o\phi\mathrm{d}\mathbf{p}\right)\psi(\mathbf{q},t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{q},t). \tag{B.6}$$

In the above calculation, we made use of the following normalization condition:

$$\int\phi^*\phi\mathrm{d}\mathbf{p} = 1.$$

The approximation of the integral expression relevant to the potential operator, $\int\phi^*\hat{U}_o\phi\mathrm{d}\mathbf{p}$ to the potential function,

$$\int\phi^*\hat{U}_o\phi\mathrm{d}\mathbf{p} = U$$

yields the Schrödinger equation in configuration space,

$$\left(\sum_{i=1}^{f}\frac{\hat{p}_i^2}{2m_{(p_i)}} + U\right)\psi(\mathbf{q},t) = \mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{q},t). \tag{B.7}$$

Thus, we explain that the fundamental equation of QMPS approximates to the Schrödinger equation as a special case. This corroborates the generality and validity of the present formalism of quantum mechanics in phase space.

Appendix C: Assessment of Exactitude of the Schrödinger Equation

The Schrödinger equation, which was essentially assumed, embodies the fundamental concepts and methodologies of quantum mechanics in configuration space. The best result which the Schrödinger equation gives is the conception of operators corresponding to observables. The Klein-Gordon equation and Dirac's equation as the relativistic equations were made with the help of the momentum operator inferred from the Schrödinger equation. Consequently, the Schrödinger equation amounts to the basic premise for quantum mechanics in all respects and the operators, rather than the equation, have general meaning.

However, it is necessary to review the problem of whether the Schrödinger equation is mathematically rigorous, since it essentially was assumed. Starting with the definition of mean value, we can explicitly demonstrate that the Schrödinger equation makes some approximations besides non-relativistic one [73]. For convenience, we consider the Schrödinger equation for one particle. By definition, the mean value of momentum component p_x reads

$$\langle p_x \rangle = \int \psi^* \hat{p}_x \psi \mathrm{d}v = \int \psi^* \left(\frac{\hat{p}_x \psi}{\psi} \right) \psi \mathrm{d}v. \tag{C.1}$$

Obviously, the real part of $\frac{\hat{p}_x \psi}{\psi}$ is *x*-component of momentum. Therefore, we can write $\frac{\hat{p}_x \psi}{\psi}$ as

$$\frac{\hat{p}_x \psi}{\psi} = \tilde{p}_x = p_{x-real} + \mathrm{i} p_{x-imag}, \tag{C.2}$$

where p_{x-real} and p_{x-imag} are the real and imaginary part of $\frac{\hat{p}_x \psi}{\psi}$, respectively. Generally, $\tilde{p}_x = \frac{\hat{p}_x \psi}{\psi}$ is a complex function dependent on coordinates, since ψ is not the eigen function of $\hat{p}_x$. As a result, we easily arrive at

$$\begin{aligned} \hat{p}_x^2 \psi &= \hat{p}_x \hat{p}_x \psi = \hat{p}_x \left(\tilde{p}_x \psi \right) = \tilde{p}_x \hat{p}_x \psi + \psi \hat{p}_x \tilde{p}_x \\ &= \tilde{p}_x^2 \psi + \psi \hat{p}_x \tilde{p}_x \neq \left(\operatorname{Re} \tilde{p}_x \right)^2 \psi = p_{x-real}^2 \psi, \end{aligned} \tag{C.3}$$

where Re denotes the real part of complex number. On the other hand, the Schrödinger equation is obtained in terms of the energy relation

$$E = \frac{\mathbf{p}^2}{2m} + U. \tag{C.4}$$

The operator relation corresponding to Equation (C.4) reads

$$\hat{E} = \frac{\hat{\mathbf{p}}^2}{2m} + U. \tag{C.5}$$

Consequently, the wave equation for this operator is written as

$$\hat{E} \psi = \left(\frac{\hat{\mathbf{p}}^2}{2m} + U \right) \psi. \tag{C.6}$$

It is this equation that is the Schrödinger equation. According to Equation (C.3), it is well-grounded that Equation (C.6), *i.e.*, the Schrödinger equation is assessed as comprising some terms violating the correspondence principle. It is obvious that only when these terms are negligible, the Schrödinger equation gives reasonable solutions. Consequently, the requirement for approximation to the Schrödinger equation is that $\tilde{p}_x$ can approximate to a real constant. From the above argument, it follows that the double application of differential operator such as the momentum operator to wave function violates the exact correspondence relation between operator and dynamical quantity. Especially, for the case of real-valued wave function we encounter an intractable problem. In this case, $\frac{\hat{p}_x\psi}{\psi}$ via a single application of $\hat{p}_x$ to the wave equation becomes a purely imaginary number. This indicates that the momentum vanishes. On the other hand, the calculation of $\frac{1}{2m}\frac{\hat{p}_x^2\psi}{\psi}$ via a double application of $\hat{p}_x$ to the wave function gives a purely real number which means nonzero kinetic energy. This result shows that in despite of zero momentum, the corresponding kinetic energy may have a nonzero value. It is this fact that demonstrates the approximate aspect of Schrödinger equation.

The aforementioned argument shows that the correspondence principle generally does not hold for arbitrary operators. To recall the fact that there does not exist the isomorphic mapping between physical quantities and the corresponding quantum-mechanical operators, it is enough to understand this context.

Now, we shall review this matter in detail. Let us denote observables by $a,b,c,\cdots$ and the corresponding operators of quantum mechanics by $\hat{A},\hat{B},\hat{C},\cdots$. Then the isomorphic relations between physical quantities and operators can be represented as

$$\begin{gathered} a = M\left(\hat{A}\right), \\ b = M\left(\hat{B}\right), \\ c = M\left(\hat{C}\right), \\ \vdots \\ F\left(a,b,c,\cdots\right) = M\left[F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\right], \end{gathered} \tag{C.7}$$

where F refers to the algebraic expression for observables or operators and M, the mapping from operator to observable. It is currently accepted that the isomorphic mapping M via the wave function should satisfy

$$\begin{gathered} a\psi = \hat{A}\psi, \\ b\psi = \hat{B}\psi, \\ c\psi = \hat{C}\psi, \\ \vdots \end{gathered}$$

$$F(a,b,c,\cdots)\psi = F(\hat{A},\hat{B},\hat{C},\cdots)\psi. \tag{C.8}$$

From this, the isomorphic mapping is represented as

$$a = M(\hat{A}) = \frac{\hat{A}\psi}{\psi},$$

$$b = M(\hat{B}) = \frac{\hat{B}\psi}{\psi},$$

$$c = M(\hat{C}) = \frac{\hat{C}\psi}{\psi},$$

$$\vdots$$

$$F(a,b,c,\cdots) = M\left[F(\hat{A},\hat{B},\hat{C},\cdots)\right] = \frac{F(\hat{A},\hat{B},\hat{C},\cdots)\psi}{\psi}. \tag{C.9}$$

Hence, it follows that for the algebra and mapping between observables and operators, we should adopt as a general rule

$$F\left[M(\hat{A}),M(\hat{B}),M(\hat{C}),\cdots\right] = M\left[F(\hat{A},\hat{B},\hat{C},\cdots)\right] \tag{C.10}$$

or

$$F\left(\frac{\hat{A}\psi}{\psi},\frac{\hat{B}\psi}{\psi},\frac{\hat{C}\psi}{\psi}\right) = \frac{F(\hat{A},\hat{B},\hat{C},\cdots)\psi}{\psi}. \tag{C.11}$$

It is obvious that for arbitrary operators including multiplication and differential operators, and the wave function, the mathematical relation such as Equation (C.11) in general does not hold. Therefore, it is concluded that there is not the mathematically isomorphic relation between observables and operators.

Now, let us examine the algebraic isomorphism between observables and operators from a different aspect. There is no doubt for the fact that the definition of the mean value for quantum mechanics is true, since it is rudimentary knowledge of quantum mechanics.

By definition, for an arbitrary quantum operator $\hat{L}$, we can represent its mean value as

$$\langle \hat{L} \rangle = \int \psi^* \hat{L}\psi \mathrm{d}v = \int \psi^* \psi \frac{\hat{L}\psi}{\psi} \mathrm{d}v = \int \rho \frac{\hat{L}\psi}{\psi} \mathrm{d}v.$$

On the other hand, since $\hat{L}$ is a Hermitian operator, the mean value always is real. Therefore, the meaningful part of the integrand in the above integration is $\mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right)$. Thus, we represent the observable L corresponding to an operator $\hat{L}$ as

$$L = \mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right). \tag{C.12}$$

As an example, the observable of momentum is written by the corresponding operator as

$$p = \mathrm{Re}\left(\frac{p\psi}{\psi}\right).$$

Therefore, according to Equation (C.12) the mapping from operators to observables can be adopted as

$$L = M\left(\hat{L}\right) = \mathrm{Re}\left(\frac{\hat{L}\psi}{\psi}\right).$$

Here is not any assumption. Actually, the above relation expresses the mapping relation between observable and operator. According to Equation (C.12), if the correspondence principle is justified, then the mapping relations must be written as

$$a = M\left(\hat{A}\right) = \mathrm{Re}\left(\frac{\hat{A}\psi}{\psi}\right),$$

$$b = M\left(\hat{B}\right) = \mathrm{Re}\left(\frac{\hat{B}\psi}{\psi}\right),$$

$$c = M\left(\hat{C}\right) = \mathrm{Re}\left(\frac{\hat{C}\psi}{\psi}\right),$$

$$\vdots$$

$$F\left(a,b,c,\cdots\right) = \mathrm{Re}\left[\frac{F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi}{\psi}\right] = M\left[F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\right]. \tag{C.13}$$

The comparison of Equation (C.9) with Equation (C.13) leads to the conclusion that these mapping relations are not identical. Therefore, we should abandon either of two mapping rules. The mapping relations in terms of Equation (C.9) should be rejected because Equation (C.13) is more tenable than Equation (C.9) according to Equation (C.12).

From Equation (C.13), we have

$$\begin{aligned} F\left(a,b,c,\cdots\right) &= F\left[M\left(\hat{A}\right),M\left(\hat{B}\right),M\left(\hat{C}\right),\cdots\right] \\ &= F\left[\mathrm{Re}\left(\frac{\hat{A}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{B}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{C}\psi}{\psi}\right),\cdots\right], \end{aligned} \tag{C.14}$$

and at the same time

$$F\left(a,b,c\right) = M\left[F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\right] = \mathrm{Re}\left[\frac{F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi}{\psi}\right]. \tag{C.15}$$

From Equation (C.14) and Equation (C.15) follows

$$\mathrm{Re}\left[\frac{F\left(\hat{A},\hat{B},\hat{C},\cdots\right)\psi}{\psi}\right] = F\left[\mathrm{Re}\left(\frac{\hat{A}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{B}\psi}{\psi}\right),\mathrm{Re}\left(\frac{\hat{C}\psi}{\psi}\right),\cdots\right]. \tag{C.16}$$

Clearly, Equation (C.16) is not possessed of generality. Of course, for multiplication operators, Equation (C.16) holds. Actually, we can verify through a simple examination that for differential operators and arbitrary wave functions, Equation (C.16) in general is not valid. Thus, we arrive at the conclusion that the correspondence principle is not justified, so far as we review the relation of isomorphic mapping merely from the mathematical point of view.

Meanwhile, it is necessary to examine whether there necessarily exists a definite operator corresponding to every observable. The correspondence principle is of important significance for the standard theory of quantum mechanics. In fact, it is not too much to say that the correspondence principle had formulated the mathematical structure of quantum mechanics because all the operators of quantum mechanics were obtained with the help of the correspondence principle. We would be justified in saying that this principle actually has commanded quantum mathematical operations.

Based on the above argument, we can conclude that it is not mathematically rigorous to apply in general the correspondence principle to obtain quantum-mechanical operators. Such a situation significantly emphasizes the necessity of accepting the phase-space formalism whose fundamental equation can be derived without applying the correspondence principle.

Appendix D: Proof of Uncertainty Relations for Position and Momentum in Terms of Commutation Relation

Let us prove the uncertainty relation with regard to the wave function in phase space. Let the domain of variability of p and x be $p \in [p_1, p_2]$ and $x \in [x_1, x_2]$. As usual, we may take the variability as $p_1 = -\infty$, $p_2 = \infty$, $x_1 = -\infty$, $x_2 = \infty$. At the boundaries, the values of the wave function vanish.

With the help of the commutation relation between position and momentum operator,

$$(x\hat{p} - \hat{p}x) = -\mathrm{i}\hbar\left(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x\right) = \mathrm{i}\hbar,$$

we calculate

$$\begin{aligned}&\int_{p_1}^{p_2}\mathrm{d}p\int_{x_1}^{x_2}\mathrm{d}x\left|\alpha x\psi + \frac{\partial\psi}{\partial x}\right|^2\\ &= \int_{p_1}^{p_2}\mathrm{d}p\int_{x_1}^{x_2}\mathrm{d}x\left(\alpha x\psi^* + \frac{\partial\psi^*}{\partial x}\right)\left(\alpha x\psi + \frac{\partial\psi}{\partial x}\right)\\ &= A\alpha^2 + B\alpha + C \ge 0.\end{aligned} \tag{D.1}$$

Hence, we find

$$B^2 - 4AC \le 0 \to \frac{B^2}{4} \le AC.$$

Then the calculated A, B and C are as follows:

$$A = \int_{p_1}^{p_2}\mathrm{d}p\int_{x_1}^{x_2}\mathrm{d}x x^2|\psi|^2 = \langle x^2\rangle, \tag{D.2}$$

$$\begin{aligned}B &= \int_{p_1}^{p_2} \mathrm{d}p \int_{x_1}^{x_2} \mathrm{d}x \left(\psi x \frac{\partial}{\partial x} \psi^* + \psi^* x \frac{\partial}{\partial x} \psi \right) \\ &= \int_{p_1}^{p_2} \mathrm{d}p \int_{x_1}^{x_2} \mathrm{d}x \psi^* \left(x \frac{\partial}{\partial x} - \frac{\partial}{\partial x} x \right) \psi \\ &= \frac{-1}{\mathrm{i}\hbar} \int_{p_1}^{p_2} \mathrm{d}p \int_{x_1}^{x_2} \mathrm{d}x \psi^* \left(x\hat{p} - \hat{p}x \right) \psi = -1,\end{aligned} \tag{D.3}$$

$$C = \int_{p_1}^{p_2} \mathrm{d}p \int_{x_1}^{x_2} \mathrm{d}x \frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x} = -\int_{p_1}^{p_2} \mathrm{d}p \int_{x_1}^{x_2} \mathrm{d}x \psi^* \frac{\partial^2 \psi}{\partial x^2} = \frac{\langle p^2 \rangle}{\hbar^2}. \tag{D.4}$$

Here we took into consideration that the wave functions at the boundary points vanish and the wave function satisfies the normalization condition in phase space,

$$\int_{p_1}^{p_2} \mathrm{d}p \int_{x_1}^{x_2} \mathrm{d}x |\psi|^2 = 1. \tag{D.5}$$

Hence, it follows that relation

$$\frac{\langle x^2 \rangle \langle p^2 \rangle}{\hbar^2} \geq \frac{1}{4} \Rightarrow \sqrt{\langle x^2 \rangle} \cdot \sqrt{\langle p^2 \rangle} \geq \frac{\hbar}{2} \tag{D.6}$$

should hold. This is the uncertainty relation for position and momentum in the sense of the standard deviation.

The action is defined with respect to generalized coordinates and momenta, so that the proof of the uncertainty relations naturally extends to the case of angle and angular momentum. Consequently, the uncertainty relation for the angle and angular momentum operator is equally obtained from commutation relation

$$\left(\varphi \hat{L}_\varphi - \hat{L}_\varphi \varphi \right) = -\mathrm{i}\hbar \left(\varphi \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \varphi} \varphi \right) = \mathrm{i}\hbar.$$

Thus, the uncertainty relation has been generally proved by means of the commutation relation for position and momentum operator in our formalism. It is obvious that similar calculation for the position operator $-\mathrm{i}\hbar \frac{\partial}{\partial p}$ and momentum p gives the identical uncertainty relation. This is because these operator and observable satisfy commutation relation

$$\left(p\hat{x} - \hat{x}p \right) = -\mathrm{i}\hbar \left(p \frac{\partial}{\partial p} - \frac{\partial}{\partial p} p \right) = \mathrm{i}\hbar.$$

Evidently, the proof shows the uncertainty relation in the sense of statistical ensemble, which does not reflect the disturbance due to measurement. Therefore, our description vindicates the statistical formulation of the uncertainty principle.

Appendix E: Proof of Uncertainty Relations for Time and Energy in Terms of Commutation Relation

It is interesting to prove the uncertainty relation for energy and time by using the commutation relations between them,

$$\left(t\hat{E}-\hat{E}t\right)=\mathrm{i}\hbar\left(t\frac{\partial}{\partial t}-\frac{\partial}{\partial t}t\right)=-\mathrm{i}\hbar, \tag{E.1}$$

$$\left(E\hat{t}-\hat{t}E\right)=\mathrm{i}\hbar\left(E\frac{\partial}{\partial E}-\frac{\partial}{\partial E}E\right)=-\mathrm{i}\hbar,$$

where the energy operator and the time operator are represented respectively as

$$\hat{E}=\mathrm{i}\hbar\frac{\partial}{\partial t},\quad \hat{t}=\mathrm{i}\hbar\frac{\partial}{\partial E}. \tag{E.2}$$

The probability density in phase space can be considered to be the same as that in energy-time space, assuming that the equivalence relation $\Delta\mathbf{r}\cdot\Delta\mathbf{p}\Leftrightarrow\Delta E\cdot\Delta t$ holds. For this reason, the integral of the probability density in energy-time space is identical to that in phase space. Thus, we have good reason for conceiving the integral of the probability density in energy-time space as the meaningful one.

To prove the uncertainty relation for energy and time similar to the case of the momentum operator and position, we perform the following integral in energy-time space.

$$\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}\mathrm{d}t\left(\alpha t\psi^*+\frac{\partial\psi^*}{\partial t}\right)\left(\alpha t\psi+\frac{\partial\psi}{\partial t}\right)=A\alpha^2+B\alpha+C\geq 0. \tag{E.3}$$

Here $E\in\left[E_1,E_2\right]$ and $t\in\left[t_1,t_2\right]$ represent the domain of variability of energy and time. As usual, we may take the variability as $E_1=0$, $E_2=\infty$, $t_1=0$, $t_2=\infty$. At the boundaries the values of the wave function vanish.

From Equation (E.3), we get

$$B^2-4AC\leq 0\rightarrow B^2/4\leq AC. \tag{E.4}$$

In the next place, we determine A,B,C as follows.

$$A=\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}t^2\left|\psi\right|^2\mathrm{d}t=\left\langle t^2\right\rangle, \tag{E.5}$$

$$\begin{aligned}B&=\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}\mathrm{d}t\left(\psi t\frac{\partial\psi^*}{\partial t}+\psi^*t\frac{\partial\psi}{\partial t}\right)\\&=\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}\mathrm{d}t\psi^*\left(t\frac{\partial}{\partial t}-\frac{\partial}{\partial t}t\right)\psi\\&=\frac{1}{\mathrm{i}\hbar}\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}\mathrm{d}t\psi^*\left(t\hat{E}-\hat{E}t\right)\psi=-1,\end{aligned} \tag{E.6}$$

$$C=\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}\mathrm{d}t\frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial t}=-\int_{E_1}^{E_2}\mathrm{d}E\int_{t_1}^{t_2}\mathrm{d}t\psi^*\frac{\partial^2\psi}{\partial t^2}=\frac{\left\langle E^2\right\rangle}{\hbar^2}, \tag{E.7}$$

where we took into account that the wave functions at the boundary points vanish and the wave function satisfies the normalization condition in energy-time space,

$$\int_{t_1}^{t_2}\mathrm{d}t\int_{E_1}^{E_2}\mathrm{d}E\left|\psi\right|^2=1. \tag{E.8}$$

This stands for the fact that any particular state $\left(q,p\right)$ in phase space is necessarily found over the whole process of time and energy. Hence, we obtain

$$\frac{\left\langle E^2 \right\rangle \cdot \left\langle t^2 \right\rangle}{\hbar^2} \geq \frac{1}{4}, \tag{E.9}$$

namely,

$$\sqrt{\left\langle E^2 \right\rangle} \cdot \sqrt{\left\langle t^2 \right\rangle} \geq \frac{\hbar}{2}. \tag{E.10}$$

In doing so, we reach the uncertainty relation for energy and time based on the statistical formulation.

Likewise, we can obtain the uncertainty relation for energy and time, performing the calculation in energy-time space in terms of commutation relation between the energy operator and the time operator,

$$\left(E\hat{t} - \hat{t}E \right) = \mathrm{i}\hbar \left(E \frac{\partial}{\partial E} - \frac{\partial}{\partial E} E \right) = -\mathrm{i}\hbar. \tag{E.11}$$

From this, it is evident that the same uncertainty relation as Equation (E.10) is obtained.