

Forming Coalitions in Normal-Form Games

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How to cite this paper: Dwobeng, E., & Corley, H. (2022). Forming Coalitions in Normal-Form Games. *Theoretical Economics Letters*, 12, 1472-1488.

<https://doi.org/10.4236/tel.2022.125080>

Received: August 3, 2022

Accepted: October 23, 2022

Published: October 26, 2022

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Abstract

For a given n -person normal-form game G , we consider all possible sets of mutually exclusive and collective exhaustive coalitions of the n players. For each such set of coalitions, we define a coalitional semi-cooperative game Γ of G as one in which 1) the coalitions are taken as the players of this new game, 2) each coalition tries to maximize the sum of its individual players' payoffs, and 3) the players within a coalition cooperate to do so. The purpose of this paper is to determine an optimal set of coalitions for G for some relevant notion of optimality. To do so, for the payoff matrix of each possible Γ of G , we determine all Greedy Scalar Equilibria (GSEs), where a GSE is an analog of the Nash equilibrium but always exists in pure strategies. For each of these GSEs, we divide the total payoff for each coalition among its members in the same proportions as its members average over the entire payoff matrix of G . Doing so gives n modified individual player payoffs associated with each GSE of all the Γ s. For each of these GSEs, we then compute the geometric mean of its n modified payoffs. A set of coalitions associated with a GSE is deemed optimal for G if the corresponding geometric mean is a maximum among all the GSEs for all the Γ s. An optimal set of coalitions thus incorporates the selfishness of the coalitions via the GSE, while the geometric mean of the redistribution of the players' payoffs models the cooperation of and the fairness for the individual players.

Keywords

Game Theory, Normal-Form Game, Coalitions, Semi-Cooperative Game

1. Introduction

Game theory is the study of mathematical decision-making made by a finite number of players either as individuals or in coalitions. A player's choices are made according to his approach of treating both himself and others, as well as to

his expectation of the actions of the other players. In cooperative games, there is competition between groups of players, whereas in noncooperative games, there is competition between individual players. Games may also involve an arbitrator who selects the players' strategies in some hopefully reasonable way.

Historically, Borel informally introduced two-person zero-sum game theory beginning in 1921 as summarized in Borel (1941) and discussed by Fréchet in 1953. Borel did not establish the minimax theorem, which von Neuman later did in 1944, and also considered there cooperative coalitional games as well as non-cooperative zero-sum games. Subsequently in Nash (1950a), Nash assumed each player was selfish and established that an equilibrium for noncooperative games always exists in mixed strategies. In Nash (1950b), he further formulated and solved a game-theoretic bargaining problem. Much later, Bacharach (1987) formalized an axiomatic theory of solutions for normal-form games.

More specific to this work, von Neumann and Morgenstern (1944) considered cooperative games with coalitions in which the players within a coalition collaborate for their mutual benefit in competing with other coalitions. They defined the notion of a solution to such a game as the core, a certain set of undominated rewards for the game's players reminiscent of Pareto maxima. This definition was refined by Gillies in (2016). Later, Shapley provided an alternative solution concept for cooperative games, and the resulting Shapley value gives more equitable solutions than does the core (Shapley, 1951; Winston, 1987).

Although von Neumann and Morgenstern (1944) defined the core of a cooperative game, they provided no method for the players to agree on a particular member of the core. One method for doing so is for the players to agree to an arbitrator's choice. Raiffa (1953) defined an arbitration scheme for a given generalized two-person game to yield a unique mixed strategy for the arbitrator to enforce. Unlike Raiffa's mixed-strategy model for two-person games, Rosenthal (1976) presented a single Pareto-optimal pure strategy reflecting the relative strengths of the players for the arbitrator to select. Later, Kalai and Rosenthal (1978) devised schemes for an arbitrator to assign a fair outcome in the game where two players possess complete information.

More recently, Bacharach (1999) expressed the interactions between players who choose to act as individuals sometimes and as teams at other times with the possibility of being in more than one team. Unlike the model of Bacharach (1999), where players vacillate between acting as individuals and as teams, Bacharach et al. (2006) also considered a team reasoning approach where players made the optimal decisions for their respective teams without having determined the optimal teams and without considering themselves individually.

As opposed to Bacharach (1999) and Bacharach et al. (2006), our development involves games in which there is competition between players as individuals, as well as cooperation between the players in a coalition. With a somewhat similar goal, Kalai and Kalai (2013) decomposed a two-person normal-form game with transferable utility into cooperative and competitive components to determine

the original game's so called coco (competitive and cooperative) value that assigns payoffs to the two players.

In further related literature, Gerber (2000) interpreted a general nontransferable utility game as a collection of pure bargaining games that can be played by individual coalitions and developed a solution concept that predicts which coalitions are formed and how the payoffs are distributed. Similarly, Gomes (2022) proposed a new solution for coalition bargaining among n players. For any coalition, there is a fixed probability that it will form and known payoffs for the players if it does. Bloch (2003) considered the formation of coalitions as a consequence of spillovers, that is, the impact that seemingly unrelated events in one nation can have on the economy of another nation.

The notion of coalitional equilibria was considered by Gavan (2022) and Ray (1997) who gave conditions under which a Nash equilibrium exists for coalitions when the coalitions are taken as the players. Then in an approach applied in this paper to avoid the existence and computational issues of the Nash equilibrium, Corley (2017) defined the notion of a scalar equilibria that maximizes a scalar objective function. In addition, Corley (2017), as well as Nahhas and Corley (2018), discussed the difficulties with the interpretation and use of mixed strategies in games. As a consequence, we consider only pure strategies here.

In this paper, we consider an n -person normal-form game G with payoffs measured in the same units of some transferable utility allowing side payments among the players. The goal of the n players is to maximize their individual payoffs as much as jointly possible. Toward that end, we form all possible sets of mutually exclusive and collective exhaustive coalitions of the n players, including the set of singleton coalitions of the original game. For each set of coalitions, we define a coalitional semi-cooperative game of G as one in which coalitions are taken as the players of this new game, each coalition wants to maximize the sum of its individual players' payoffs among their possible pure strategies, and the players within a coalition may cooperate to do so with the inducement of side payments between them. For each coalitional semi-cooperative game, we determine its Greedy Scalar Equilibria, where a Greedy Scalar Equilibrium (GSE) is an analog of the Nash equilibrium modeling selfishness but always exists in pure strategies. Next we select an optimal set of coalitions for G by comparing the GSEs of its coalitional semi-cooperative according to a specified criterion, and finally we present examples.

The paper is organized as follows. We present preliminary notation, definitions, and results in Section 2. In Section 3 we formally define a coalitional semi-cooperative game and present an algorithm for selecting an optimal set of coalitions. Two examples are given in Section 4, and conclusions are stated in Section 5.

2. Preliminaries

Let $G = \langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be an n -player game where $I = \{1, \dots, n\}$ is the set

of players, $S_i = \{s_i^1, \dots, s_i^{m_i}\}$ is the finite set of $m_i \geq 2$ pure strategies for player i , and $u_i(s)$ is the transferable utility of player i for an action profile $s = (s_1, \dots, s_n) \in \times_{j \in I} S_j = S$. The $m_1 \times \dots \times m_n$ matrix of n -tuples $(u_1(s), \dots, u_n(s))$ for all $s \in S$ is called the payoff matrix for G , and a game given in terms of a payoff matrix is called a normal-form game as usual. As discussed in Corley (2017), we consider here only pure strategies (or actions) for the players since mixed strategies are known to be difficult to calculate, interpret, and implement except possibly in repeated games. The average payoff of player i (i.e., the arithmetic mean of his payoffs over S) is $A_i = \frac{\sum_{s \in S} u_i(s)}{\|S\|}$, where $\|S\|$ is the finite number of pure strategy profiles in S . An action profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash equilibrium (NE) if no player with a unilateral change of strategy can increase his payoff. While an NE always exists in mixed strategies for G , one may not exist in pure strategies.

For the game G , assume that each player is greedy and desires a payoff as high as jointly possible. Consider the scalar utility function $T_G : u(S) \rightarrow \mathbb{R}^1$ defined in Corley (2017) as

$$T_G[u(s)] = \prod_{i \in I} \frac{1}{M_i - u_i(s) + 1}, s \in S, \quad (1)$$

where $M_i = \max_{s \in S} u_i(s)$. A pure strategy profile s^* is called a Greedy Scalar Equilibrium (GSE) for G if and only if s^* maximizes $T_G[u(s)]$ over S . From (1), a GSE s^* has the property that each $u_i(s^*)$ is as close as jointly possible to the corresponding M_i . The reason is that the maximization of (1) is a discrete version of the continuous problem of maximizing $\prod_{i \in I} \frac{1}{x_i + 1}$ subject to the constraints $x_i \geq 0, i \in I$, for which the optimal x_i^* 's are 0. Setting $x_i = M_i - u_i(s) \geq 0$ in (1) establishes the relationship between the two problems. From the above discussion, it follows that the GSE may be construed as a scalar analog of the NE since each player is greedy and wants a payoff as high as possible. As opposed to an NE, however, a GSE always exists in pure strategies.

It should be noted that a GSE for one payoff matrix cannot be directly compared to a GSE for a different payoff matrix (i.e., a GSE for a different game). In addition, Corley (2017) proves that the GSE is Pareto maximal over all the strategies of G . In other words, the n -tuple of payoffs (u_1^*, \dots, u_n^*) associated with a GSE s^* is not dominated component wise by any other (u_1, \dots, u_n) in the payoff matrix.

3. Determining an Optimal Set of Coalitions for a Normal-Form Game

A coalition P of G is a nonempty subset of the set of n players I . A complete set Ω of coalitions for G is a set of Q mutually exclusive and collectively exhaustive subsets of I (i.e., a partition of I). For example, the coalition $P = I$ is called the

grand coalition, while $\{k\}, k = 1, \dots, n$, represent the complete set of singleton coalitions. A complete set Ω of coalitions is written with the coalitions enumerated in order beginning from the lowest numbered player in each coalition and with the individual coalitions enclosed within vertical bars. Moreover, each individual coalition is written in ascending order of its players' numbers separated by commas. In particular, coalition $k, k = 1, \dots, Q$ has C_k players denoted $k_j, j = 1, \dots, C_k$. We would thus write Ω as $|1_1, \dots, 1_{C_1} | 2_1, \dots, 2_{C_2} | \dots | Q_1, \dots, Q_{C_Q} |$.

Associated with G and an arbitrary complete set Ω of coalitions is a Q -person game Γ whose players are the Q coalitions of Ω . These players are called coalitional players. Γ is a distinct game from G , but a strategy for the coalitional player P of Γ is a tuple of the individual strategies in G for the players composing P . The payoff for the coalitional player P at a given strategy profile for Γ is the sum of the utilities for the players of P in the payoff matrix of G for the strategy profile of G corresponding to the given strategy profile in Γ . For a given Ω , Γ is called a Q -person coalitional semi-cooperative game of G if there is competition among the coalitions defining Ω and cooperation by the players within each coalition.

In particular, consider the coalitional semi-cooperative game Γ with a set of Q coalitional players. We can specify Γ more completely as

$\Gamma(|1_1, \dots, 1_{C_1} | 2_1, \dots, 2_{C_2} | \dots | Q_1, \dots, Q_{C_Q} |)$. A strategy profile t of

$\Gamma(|1_1, \dots, 1_{C_1} | 2_1, \dots, 2_{C_2} | \dots | Q_1, \dots, Q_{C_Q} |)$ is written as

$t = \left((s_{1_1}, \dots, s_{1_{C_1}}), \dots, (s_{Q_1}, \dots, s_{Q_{C_Q}}) \right)$ with a corresponding payoff vector $(u_{1_1} + \dots + u_{1_{C_1}}, \dots, u_{Q_1} + \dots + u_{Q_{C_Q}})$, where $u_{k_1}, \dots, u_{k_{C_k}}, k = 1, \dots, Q$, are the payoffs of players $1, \dots, C_k$ of coalition k , respectively, in the payoff matrix of G evaluated at the n players' individual strategies in t .

As an example, suppose that 5 coalitions are formed in a 12-player normal-form game G . Let players 1, 3, 4 form one coalition; players 2, 11 a second; players 5, 8, 9 a third; players 6, 10, 12 a fourth; and player 7 a fifth. The associated 5-person coalitional semi-cooperative game Γ is denoted

$\Gamma(|1,3,4 | 2,11 | 5,8,9 | 6,10,12 | 7 |)$ with the coalitions enclosed within vertical bars and the players in each coalition arranged in ascending order and separated by commas. Each strategy profile for $\Gamma(|1,3,4 | 2,11 | 5,8,9 | 6,10,12 | 7 |)$ is written as $t = \left((s_1, s_3, s_4), (s_2, s_{11}), (s_5, s_8, s_9), (s_6, s_{10}, s_{12}), (s_7) \right)$ with corresponding payoff $(u_1 + u_3 + u_4, u_2 + u_6, u_5 + u_8 + u_9, u_7)$ obtained from the payoff matrix for G .

Since an individual coalition P of G is a nonempty subset of I , there are $2^n - 1$ possible individual coalitions that can be formed from the players of G . Moreover, the number of complete sets of coalitions for the $n \geq 2$ players of G is precisely the number of partitions of I , which is known as the Bell number B_n (https://en.wikipedia.org/wiki/Bell_number, accessed 9/04/2022). For example, $B_2 = 5$, $B_3 = 15$, $B_4 = 52$, and $B_5 = 203$. More Bell numbers are found at (<https://oeis.org/>, accessed 9/04/2022). There is no known simple formula for the B_n . However, a recurrence relation is given in

(https://www.whitman.edu/mathematics/cgt_online/book/section01.04.html, accessed 9/04/2022), where it is also noted that B_n increases exponentially with n . Thus for large n , comparing the B_n possible complete sets of coalitions of G to obtain an optimal one, as next described, will be computationally intensive.

We now develop an approach for selecting an optimal complete set Ω^* of coalitions for the n players of G . For each possible Ω and associated coalitional semi-cooperative game $\Gamma(|1_1, \dots, 1_{C_1} | 2_1, \dots, 2_{C_2} | \dots | Q_1, \dots, Q_{C_Q} |)$ and for each payoff matrix cell of this game corresponding to a GSE of Γ , we divide the total payoff for each coalition among its members in the same proportions as its members average over the whole payoff matrix of G . Doing so gives n new individual player payoffs associated with each GSE. These new payoffs are called modified payoffs. Then for each GSE of Γ (its players being coalitions), we compute the geometric mean of the n modified payoffs associated with this GSE. Denote the largest of these geometric means for Γ as $GM(\Omega)$, which becomes a decision metric. We then designate Ω^* as an optimal complete set of coalitions for G if Ω^* maximizes $GM(\Omega)$ over the B_n possibilities. The associated coalitional semi-cooperative game is written Γ^* . There may be multiple Ω^* and Γ^* . Ties could be broken by the players or by an arbitrator. Since any Ω^* corresponds to a GSE of some Γ^* , then Ω^* models the selfishness of the coalitions. Moreover, Ω^* minimizes the variability between the modified payoffs of the n players by analogy with the fact that geometric mean of numbers x_1, \dots, x_n is maximized when x_1, \dots, x_n are equal. In other words, fairness is enforced. We now present Algorithm 1 below to determine an Ω^* for a given G .

Algorithm 1

- Step 1.** Compute the average payoff A_i for player $i, i = 1, \dots, n$, over the original game G .
- Step 2.** For a complete set Ω of coalitions of G not yet enumerated, consider the associated coalitional semi-cooperative game $\Gamma(|1_1, \dots, 1_{C_1} | 2_1, \dots, 2_{C_2} | \dots | Q_1, \dots, Q_{C_Q} |)$ of G . For each strategy profile of Γ , sum the payoffs of players in each coalition and form the payoff matrix for this coalitional game.
- Step 3.** Identify all GSEs of the game in Step 2.
- Step 4.** For each coalition j in Step 2, sum the averages A_{k_j} in Step 1 of all the players k_j in coalition j to obtain $V_j = \sum_{k_j=1}^{C_j} A_{k_j}$ for the coalitions $j = 1, \dots, Q$.
- Step 5.** For each player i that is a member of coalition j of Step 2, compute $R_{ij} = \frac{A_i}{V_j}$.
- Step 6.** For each GSE t of Step 3 and the payoff of each coalition j for this GSE, multiply R_{ij} by coalition j 's payoff from Step 2. For $i = 1, \dots, C_j$ and $j = 1, \dots, Q$, the resulting payoff is the modified payoff of player i in coalition j for the GSE t of the game of Step 2.
- Step 7.** For each GSE of Step 3 compute the geometric mean of the n modified payoffs obtained in Step 6, and then determine $GM(\Omega)$, the largest of these geometric means.
- Step 8.** Repeat Steps 2-7 for each of the remaining coalitional semi-cooperative games of G . If none remain, go to Step 9.
- Step 9.** Select an optimal complete set Ω^* of coalitions as one maximizing $GM(\Omega)$ over all iterations of Step 7. Ties may be broken by the players or an arbitrator.
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4. Examples

In this section, we present two examples to illustrate Algorithm 1. In Example 1, the GSE of the original game G is different from the GSE of the coalitional game for the optimal set of coalitions in Step 9. Unlike Example 1, in Example 2 the GSE of the coalitional game for an optimal set of coalitions in Step 9 coincides with the GSE of the original game G .

Example 1

Let G be a 3-player game in normal-form where the strategies for player i are s_i and t_i , $i = 1, 2, 3$, as shown in the payoff matrix of **Table 1**.

To determine an optimal set of coalitions to form using Algorithm 1, consider the coalitional semi-cooperative games of G . There are five possibilities. The first possibility consists of two coalitions: coalition I consisting of player 1 alone and coalition II consisting of players 2 and 3. We model the associated coalitional semi-cooperative game consisting of coalition I versus coalition II as a two-player normal-form game $\Gamma_{1|2,3}$, where coalition I is construed as the first player and coalition II as the second player. Similarly, the other possibilities are $\Gamma_{1,2|3}$, $\Gamma_{1,3|2}$, $\Gamma_{1,2,3}$, and $\Gamma_{1|2,3}$. In Step 1 of Algorithm 1, $A_1 = 3.5$, $A_2 = 1.75$ and $A_3 = 4.375$. Now consider $\Gamma_{1|2,3}$ with payoff matrix in **Table 2** as determined in Step 2.

Coalition II in $\Gamma_{1|2,3}$ has four strategies, which are simply all combinations of the strategies of players 2 and 3. For example, the cell $(s_1, (s_2, s_3))$ in **Table 2** has the payoff vector $(6, 10)$ where the payoff 10 for coalition II comes from cell (s_1, s_2, s_3) in **Table 1** by adding the payoffs 5 for player 2 and 5 for player 3. Note that in **Table 2**, none of the original strategies of G has been omitted. There are still 8 possibilities. **Table 3** represents the GSE matrix of $\Gamma_{1|2,3}$ of Step 3 with the bold value 0.2500 for the two GSEs $(s_1, (s_2, s_3))$ and $(t_1, (s_2, t_3))$ with their corresponding payoffs in $\Gamma_{1|2,3}$ as $(6, 10)$ and $(3, 13)$ respectively.

In Step 4, $V_1 = 3.5$ for player 1 in coalition I and the sum of the averages of players 2 and 3 in coalition II is $V_2 = 1.75 + 4.375 = 6.125$. In Step 5,

Table 1. Payoff Matrix for G .

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(6, 5, 5)	(1, -2, 4)	(6, 1, 3)	(4, 1, 4)
t_1	(4, 2, 3)	(3, 1, 4)	(3, 6, 7)	(1, 0, 5)

Table 2. Payoff Matrix for $\Gamma_{1|2,3}$.

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	(6, 10)	(1, 2)	(6, 4)	(4, 5)
t_1	(4, 5)	(3, 5)	(3, 13)	(1, 5)

$R_{11} = \frac{3.5}{3.5} = 1$, $R_{22} = \frac{1.75}{6.125} = 0.29$ and $R_{32} = \frac{4.375}{6.125} = 0.71$. In Step 6, the new payoffs corresponding to the first GSE of $G_{1|2,3}$ for player 1 is 6 because player 1's ratio of contribution to coalition I is $R_{11} = 1$. The payoff for coalition II corresponding to this GSE of $\Gamma_{1|2,3}$ is divided between players 2 and 3 using their ratios of contribution $R_{22} = 0.29$ and $R_{32} = 0.71$ respectively. Player 2's new payoff is $0.29 \times 10 = 2.9$ and the new payoff of player 3 is $0.71 \times 10 = 7.1$. The modified payoffs corresponding to this GSE of $\Gamma_{1|2,3}$ are $(6, 2.9, 7.1)$, and the geometric mean of this payoff in Step 7 is $\sqrt[3]{6 \times 2.9 \times 7.1} = 4.9805$. The new payoff corresponding to the second GSE of $\Gamma_{1|2,3}$ for player 1 is 3 since $R_{11} = 1$. The payoff for coalition II corresponding to the GSE of $\Gamma_{1|2,3}$ is again divided between players 2 and 3 using $R_{22} = 0.29$ and $R_{32} = 0.71$ respectively. Player 2's new payoff is $0.29 \times 13 = 3.8$ and the new payoff of player 3 is $0.71 \times 13 = 9.2$. The modified payoff corresponding to this GSE of $\Gamma_{1|2,3}$ is $(3, 3.8, 9.2)$, and the geometric mean of this payoff as in Step 7 is $\sqrt[3]{3 \times 3.8 \times 9.2} = 4.7159$.

In Step 8, we consider the remaining coalitional games and repeat Steps 2 to 7 for each of these games. Next is the game $\Gamma_{1,2|3}$ made of two coalitions where coalition I consists of players 1 and 2 and coalition II consists of player 3 alone. In Step 2, the payoff matrix for $\Gamma_{1,2|3}$ is **Table 4**. **Table 5** shows the GSE matrix

Table 3. GSE Matrix for $\Gamma_{1|2,3}$.

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	0.2500	0.0139	0.1000	0.0370
t_1	0.0370	0.0278	0.2500	0.0185

Table 4. Payoff Matrix for $\Gamma_{1,2|3}$.

	s_3	t_3
(s_1, s_2)	(11, 5)	(7, 3)
(t_1, s_2)	(6, 3)	(9, 7)
(s_1, t_2)	(-1, 4)	(5, 4)
(t_1, t_2)	(4, 4)	(1, 5)

Table 5. GSE Matrix for $\Gamma_{1,2|3}$.

	s_3	t_3
(s_1, s_2)	0.3333	0.0400
(t_1, s_2)	0.0333	0.3333
(s_1, t_2)	0.0192	0.0357
(t_1, t_2)	0.0313	0.0303

of $\Gamma_{1,2,3}$ of Step 3 with GSEs $((s_1, s_2), s_3)$ and $((t_1, s_2), t_3)$ with respective payoffs (11,5) and (9,7) in $\Gamma_{1,2,3}$.

The sum of the averages of players 1 and 2 in Step 4 is $V_1 = 3.5 + 1.75 = 5.25$ and player 3's in coalition II is $V_2 = 4.375$. In Step 5, $R_{11} = \frac{3.5}{5.25} = 0.67$, $R_{21} = \frac{1.75}{5.25} = 0.33$ and $R_{32} = \frac{4.375}{4.375} = 1$. In Step 6, player 1's new payoff corresponding to the first GSE $((s_1, s_2), s_3)$ is $0.67 \times 11 = 7.4$, and player 2's is $0.33 \times 11 = 3.6$. In coalition II, $R_{32} = 1$ since player 3 is the only player in it. Thus player 3's new payoff corresponding to this GSE of $\Gamma_{1,2,3}$ is 5. The modified payoff of all three players that corresponds to this GSE of $\Gamma_{1,2,3}$ is (7.4, 3.6, 5) in Step 6, and the geometric mean of this modified payoff in Step 7 is $\sqrt[3]{7.4 \times 3.6 \times 5} = 5.1070$. Again in Step 6, player 1's new payoff corresponding to the second GSE $((t_1, s_2), t_3)$ is $0.67 \times 9 = 6$, and player 2's is $0.33 \times 9 = 3$. In coalition II, $R_{32} = 1$ since player 3 is the only player in it. Hence player 3's new payoff corresponding to this GSE of $\Gamma_{1,2,3}$ is 7. The modified payoff of all three players that corresponds to this GSE of $\Gamma_{1,2,3}$ is (6, 3, 7), and the geometric mean of the modified payoff in Step 7 is $\sqrt[3]{6 \times 3 \times 7} = 5.0132$.

Now consider the game $\Gamma_{1,3,2}$ with two coalitions where coalition I comprises players 1 and 3 and coalition II comprises player 2 alone. The payoff matrix for $\Gamma_{1,3,2}$ in Step 2 is **Table 6**. Then in Step 3, **Table 7** is the GSE matrix for $\Gamma_{1,3,2}$ with GSEs $((s_1, s_3), s_2)$ and $((t_1, t_3), s_2)$ with respective payoffs (11,5) and (10,6).

The sum of the averages of players 1 and 3 in Step 4 is $V_1 = 3.5 + 4.375 = 7.875$

Table 6. Payoff Matrix for $\Gamma_{1,3,2}$.

	s_2	t_2
(s_1, s_3)	(11, 5)	(5, -2)
(t_1, s_3)	(7, 2)	(7, 1)
(s_1, t_3)	(9, 1)	(8, 1)
(t_1, t_3)	(10, 6)	(6, 0)

Table 7. GSE Matrix for $\Gamma_{1,3,2}$.

	s_2	t_2
(s_1, s_3)	0.5000	0.0159
(t_1, s_3)	0.0400	0.0333
(s_1, t_3)	0.0556	0.0417
(t_1, t_3)	0.5000	0.0238

and the sum of the average of player 2 in coalition II is $V_2 = 1.75$. In Step 5, $R_{11} = \frac{3.5}{7.875} = 0.44$, $R_{22} = \frac{1.75}{1.75} = 1$, and $R_{31} = \frac{4.375}{7.875} = 0.56$. In Step 6, player 1's new payoff corresponding to the first GSE $((s_1, s_3), s_2)$ is $0.44 \times 11 = 4.8$, player 2's new payoff is 5 since $R_{22} = 1$, and player 3's is $0.56 \times 11 = 6.2$. The modified payoff of all three players corresponding to this GSE of $\Gamma_{1,3|2}$ is $(4.8, 5, 6.2)$, and the geometric mean of the modified payoff in Step 7 is $\sqrt[3]{4.8 \times 5 \times 6.2} = 5.2991$. Again in Step 6, player 1's new payoff corresponding to the second GSE $((t_1, t_3), s_2)$ is $0.44 \times 10 = 4.4$, player 2's new payoff is 6 since $R_{22} = 1$, and player 3's is $0.56 \times 10 = 5.6$. The modified payoff of all three players corresponding to this GSE of $\Gamma_{1,3|2}$ is $(4.4, 6, 5.6)$, and the geometric mean of the modified payoff in Step 7 is $\sqrt[3]{4.4 \times 6 \times 5.6} = 5.2877$.

We next consider the grand coalition in which all players of G form the single coalition $\Gamma_{1,2,3}$. **Table 8** gives the corresponding payoff matrix from Step 2. Then the GSEs of $\Gamma_{1,2,3}$ in Step 3 are $((s_1, s_2, s_3))$ and $((t_1, s_2, t_3))$ with respective payoffs in **Table 9** as 16.

In Step 4, $V_1 = 3.5 + 1.75 + 4.375 = 9.625$. Moreover, $R_{11} = \frac{3.5}{9.625} = 0.36$, $R_{21} = \frac{1.75}{9.625} = 0.18$, and $R_{31} = \frac{4.375}{9.625} = 0.46$ in Step 5. In Step 6, Player 1's new payoff corresponding to the first GSE $((s_1, s_2, s_3))$ is $0.36 \times 16 = 5.8$, player 2's new payoff is $0.18 \times 16 = 2.9$, and player 3's is $0.46 \times 16 = 7.3$. The modified payoff of all three players corresponding to the first GSE of $\Gamma_{1,2,3}$ is $(5.8, 2.9, 7.3)$, and the geometric mean of the modified payoff in Step 7 is $\sqrt[3]{5.8 \times 2.9 \times 7.3} = 4.9703$. Similarly, the modified payoff and geometric mean corresponding to the second GSE $((t_1, s_2, t_3))$ are $(5.8, 2.9, 7.3)$ and 4.9703 respectively.

Finally, consider the game $\Gamma_{|2|3}$ with three coalitions in which each player constitutes a coalition unto himself. The payoff matrix of $\Gamma_{|2|3}$ in Step 2 is **Table 1**. In

Table 8. Payoff Matrix for $\Gamma_{1,2,3}$.

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	16	3	10	9
t_1	9	8	16	6

Table 9. GSE Matrix for $\Gamma_{1,2,3}$.

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	1.0000	0.1875	0.6250	0.5625
t_1	0.5625	0.5000	1.0000	0.3750

Table 10. GSE Matrix for $\Gamma_{1|2|3}$.

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	0.1667	0.0046	0.0333	0.0139
t_1	0.0133	0.0104	0.2500	0.0079

Step 3, the GSE matrix of $\Gamma_{1|2|3}$ is depicted in **Table 10** with GSE $((t_1), (s_2), (t_3))$ and corresponding payoff in **Table 11** as $(3, 6, 7)$. Thus each player's new payoff corresponding to the GSE of $\Gamma_{1|2|3}$ is the corresponding payoff in G .

In Step 4, $V_1 = 3.5, V_2 = 1.75$ and $V_3 = 4.375$. Since each player is a coalition onto himself, $R_{11} = R_{22} = R_{33} = 1$ from Step 5. The modified payoff of all three players that corresponds to the GSE of $\Gamma_{1|2|3}$ is $(3, 6, 7)$, and the geometric mean of this modified payoff in Step 7 is $\sqrt[3]{3 \times 6 \times 7} = 5.0133$.

The geometric means GM (Ω) of the game G are 4.9805 and 4.7159 for $\Gamma_{1|2,3}$, 5.1070 and 5.0132 for $\Gamma_{1,2|3}$, and 5.2991 and 5.2877 for $\Gamma_{1,3|2}$. In addition, the geometric mean of $\Gamma_{1|2|3}$ is 5.0132 and that of $\Gamma_{1,2,3}$ is 4.9703. Then according to Step 9, the best complete set of coalitions is given by $\Gamma_{1,3|2}$ since the associated geometric mean is the largest among all the geometric means for the coalitional semi-cooperative games of G . Hence the optimal complete set of coalitions is $\Omega^* = |1, 3|2|$.

Note in Example 1 that the three players get the modified payoffs $(4.8, 5, 6.2)$ from the coalitional game $\Gamma_{1,3|2}$ instead of the payoff $(3, 6, 7)$ from $\Gamma_{1|2|3}$, i.e., the original game G . These modified payoffs for players 2 and 3 are smaller than their payoffs in $\Gamma_{1,3|2}$. But as previously mentioned, fairness is enforced in the following ways. The R_{ij} 's in Step 5 of Algorithm 1 ensure that the players' new payoffs after joining their respective coalitions are in the same proportions as their average payoffs over the game G . In addition, the geometric mean minimizes the variability in the modified payoffs for the best coalition and gives the n players payoffs that are jointly closest together. These observations result from the Algorithm 1's trade-off between the selfish behavior of each coalition as a whole in Step 3, as well as from the cooperative behavior for each individual player within his coalition with the understanding that he will be fairly treated as in Steps 6 and Step 7. Finally, observe in Example 1 that the maximum GSE $((s_1, s_3), s_2)$ corresponding to $\Omega^* = |1, 3|2|$ and the maximum GSE $((t_1), (s_2), (t_3))$ of the original game G are different. We next present an example where they coincide.

Example 2

Let G be a 3-player game in normal form where the strategies for player i are s_i and t_i , $i = 1, 2, 3$, as shown in the payoff matrix of **Table 11**.

To determine a best complete set Ω^* of coalitions in the sense of Algorithm 1), consider the coalitional semi-cooperative games Γ of G . There are five possibilities, namely $\Gamma_{1|2,3}$, $\Gamma_{1,3|2}$, $\Gamma_{1,2,3}$, and $\Gamma_{1|2|3}$. In Step 1 of Algorithm 1, $A_1 = 3$,

$A_2 = 2.75$ and $A_3 = 2.375$. Now consider $\Gamma_{1|2,3}$ with payoff matrix **Table 12** as in Step 2. Then **Table 13** represents the GSE matrix of $\Gamma_{1|2,3}$ as in Step 3 with GSE $(t_1, (t_2, t_3))$ and corresponding payoffs $(5, 9)$.

In Step 4, $V_1 = 3$ for player 1 in coalition I, and the sum of the averages of players 2 and 3 in coalition II is $V_2 = 2.75 + 2.375 = 5.125$. In Step 5, $R_{11} = \frac{3}{3} = 1$, $R_{22} = \frac{2.75}{5.125} = 0.54$, and $R_{32} = \frac{2.375}{5.125} = 0.46$. In Step 6, the new payoff corresponding to the GSE of $\Gamma_{1|2,3}$ for player 1 is 5 because $R_{11} = 1$. The payoff for coalition II corresponding to the GSE of $\Gamma_{1|2,3}$ is divided between players 2 and 3 using $R_{22} = 0.54$ and $R_{32} = 0.46$ respectively. Player 2's modified payoff is $0.54 \times 9 = 4.86$, and the modified payoff of player 3 is $0.46 \times 9 = 4.14$. The modified payoffs corresponding to the GSE of $\Gamma_{1|2,3}$ are $(5, 4.86, 4.14)$, and the geometric mean in Step 7 is $\sqrt[3]{5 \times 4.86 \times 4.14} = 4.6509$.

In Step 8, we consider the remaining coalitional games and repeat Steps 2 to 7 for each of these games. Next is $\Gamma_{1,2|3}$ with coalition I consisting of players 1 and 2 and coalition II consisting of player 3 alone. In Step 2, the payoff matrix for $\Gamma_{1,2|3}$ is given in **Table 14**. **Table 15** then shows the GSE matrix of $\Gamma_{1,2|3}$ of Step 3 with GSE $((t_1, t_2), s_3)$ and corresponding payoff $(12, 2)$ in $\Gamma_{1,2|3}$.

The sum of the averages of players 1 and 2 in Step 4 is $V_1 = 3 + 2.75 = 5.75$ and player 2's in coalition II is $V_2 = 2.375$. In Step 5, $R_{11} = \frac{3}{5.75} = 0.52$, $R_{21} = \frac{2.75}{5.75} = 0.48$ and $R_{32} = \frac{2.375}{2.375} = 1$. Player 1's new payoff in Step 6 is $0.52 \times 12 = 6.24$, and player 2's is $0.48 \times 12 = 5.76$. For coalition II, $R_{32} = 1$

Table 11. Payoff Matrix for G .

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	(1, 1, 0)	(1, 2, 3)	(2, 4, 5)	(0, 1, 2)
t_1	(4, 3, 2)	(5, 7, 2)	(7, 3, 2)	(4, 1, 3)

Table 12. Payoff Matrix for $\Gamma_{1|2,3}$.

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	(1, 1)	(1, 5)	(2, 9)	(0, 3)
t_1	(4, 5)	(5, 9)	(7, 5)	(4, 4)

Table 13. GSE Matrix for $\Gamma_{1|2,3}$.

	(s_2, s_3)	(t_2, s_3)	(s_2, t_3)	(t_2, t_3)
s_1	0.0159	0.0286	0.1667	0.0178
t_1	0.0500	0.3333	0.2000	0.0417

since player 3 is the only player in it. Thus player 3's modified payoff corresponding to the unique GSE is 2. The modified payoffs for all three players are $(6.24, 5.76, 2)$, and the geometric mean of the modified payoffs of Step 7 is $\sqrt[3]{6.24 \times 5.76 \times 2} = 4.1579$.

Now consider the game $\Gamma_{1,3|2}$ with two coalitions, where coalition I has players 1 and 3 while coalition II has player 2 alone. The payoff matrix for $\Gamma_{1,3|2}$ in Step 2 is **Table 16**. Then **Table 17** for Step 3 shows the GSE matrix with GSE

Table 14. Payoff Matrix for $\Gamma_{1,2|3}$.

	s_3	t_3
(s_1, s_2)	(2, 0)	(6, 5)
(t_1, s_2)	(7, 2)	(10, 2)
(s_1, t_2)	(3, 3)	(1, 2)
(t_1, t_2)	(12, 2)	(5, 3)

Table 15. GSE Matrix for $G_{1,2|3}$.

	s_3	t_3
(s_1, s_2)	0.0152	0.1428
(t_1, s_2)	0.0147	0.0833
(s_1, t_2)	0.0333	0.0208
(t_1, t_2)	0.2500	0.0417

Table 16. Payoff Matrix for $G_{1,3|2}$.

	s_2	t_2
(s_1, s_3)	(1, 1)	(4, 2)
(t_1, s_3)	(6, 3)	(7, 7)
(s_1, t_3)	(7, 4)	(2, 1)
(t_1, t_3)	(9, 3)	(7, 1)

Table 17. GSE Matrix for $G_{1,3|2}$.

	s_2	t_2
(s_1, s_3)	0.0158	0.0278
(t_1, s_3)	0.0500	0.3333
(s_1, t_3)	0.0833	0.0179
(t_1, t_3)	0.2000	0.0476

$((t_1, s_3), t_2)$, with payoff (7, 7) in **Table 16**.

The sum of the averages of players 1 and 3 in Step 4 is $V_1 = 3 + 2.375 = 5.375$ and the sum of the average of player 2 in coalition II is $V_2 = 2.75$. In Step 5, $R_{11} = \frac{3}{5.375} = 0.56$, $R_{22} = \frac{2.75}{2.75} = 1$, and $R_{31} = \frac{2.375}{5.375} = 0.44$. In Step 6, player 1's new payoff is $0.56 \times 7 = 3.92$, player 2's new payoff is 7 since $R_{22} = 1$, and player 3's is $0.44 \times 7 = 3.08$. The modified payoffs of all three players corresponding to the GSE of $\Gamma_{1,3,2}$ are $(3.92, 7, 3.08)$, and the geometric mean of the modified payoffs in Step 7 is $\sqrt[3]{3.92 \times 7 \times 3.08} = 4.3885$.

We next consider the grand coalition of G and the associated $\Gamma_{1,2,3}$. **Table 18** gives the corresponding payoff matrix from Step 2. The GSE of $\Gamma_{1,2,3}$ in Step 3 is $((t_1, t_2, s_3))$ in **Table 19** with corresponding payoff 14 in **Table 18**.

In Step 4, $V_1 = 3 + 2.75 + 2.375 = 8.125$. Moreover, $R_{11} = \frac{3}{8.125} = 0.37$, $R_{21} = \frac{2.75}{8.125} = 0.34$, and $R_{31} = \frac{2.375}{8.125} = 0.29$ in Step 5. In Step 6, Player 1's new payoff corresponding to the GSE is $0.37 \times 14 = 5.18$, player 2's new payoff is $0.34 \times 14 = 4.76$, and player 3's is $0.29 \times 14 = 4.06$. The modified payoffs of all three players corresponding to the unique GSE of $\Gamma_{1,2,3}$ are $(5.18, 4.76, 4.06)$, and the geometric mean of the modified payoffs in Step 7 is $\sqrt[3]{5.18 \times 4.76 \times 4.06} = 4.6432$.

Next consider $\Gamma_{1|2|3}$ with three coalitions in which each player constitutes a coalition unto himself. The payoff matrix of $\Gamma_{1|2|3}$ in Step 2 is **Table 11**. In Step 3, the GSE matrix of $\Gamma_{1|2|3}$ is depicted in **Table 20** with GSE $((t_1), (t_2), (s_3))$ and corresponding payoff in **Table 11** as $(5, 7, 2)$. Each player's modified payoff corresponding to the GSE is the corresponding payoff for G .

In Step 4, $V_1 = 3 + 2.75 + 2.375 = 8.125$. Moreover, $R_{11} = \frac{3}{8.125} = 0.37$, $R_{21} = \frac{2.75}{8.125} = 0.34$, and $R_{31} = \frac{2.375}{8.125} = 0.29$ in Step 5. In Step 6, Player 1's new

Table 18. Payoff Matrix for $\Gamma_{1,2,3}$.

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	2	6	11	3
t_1	9	14	12	8

Table 19. GSE Matrix for $\Gamma_{1,2,3}$.

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	0.0769	0.1111	0.2500	0.0833
t_1	0.1667	1.000	0.3333	0.1429

Table 20. GSE Matrix for $\Gamma_{1|2|3}$.

	s_3		t_3	
	s_2	t_2	s_2	t_2
s_1	0.0034	0.0079	0.0417	0.0045
t_1	0.0125	0.0833	0.0500	0.0119

payoff corresponding to the GSE is $0.37 \times 14 = 5.18$, player 2's new payoff is $0.34 \times 14 = 4.76$, and player 3's is $0.29 \times 14 = 4.06$. The modified payoffs of all three players corresponding to the unique GSE of $\Gamma_{1,2,3}$ are $(5.18, 4.76, 4.06)$, and the geometric mean of the modified payoffs in Step 7 is

$$\sqrt[3]{5.18 \times 4.76 \times 4.06} = 4.6432.$$

Next consider $\Gamma_{1|2|3}$ with three coalitions in which each player constitutes a coalition unto himself. The payoff matrix of $\Gamma_{1|2|3}$ in Step 2 is **Table 11**. In Step 3, the GSE matrix of $\Gamma_{1|2|3}$ is depicted in **Table 20** with GSE $((t_1), (t_2), (s_3))$ and corresponding payoff in **Table 11** as $(5, 7, 2)$. Each player's modified payoff corresponding to the GSE is the corresponding payoff for G .

In Step 4, $V_1 = 3, V_2 = 2.75$, and $V_3 = 2.375$. Since each player is a coalition unto himself, $R_{11} = R_{22} = R_{33} = 1$ from Step 5. The modified payoffs corresponding to the GSE of $G_{1|2|3}$ in Step 6 are thus $(5, 7, 2)$, and the geometric mean of the modified payoffs in Step 7 is $\sqrt[3]{5 \times 7 \times 2} = 4.1212$.

In summary, the geometric means of all coalitional semi-cooperative games for G are 4.6509, 4.1579, 4.3885, 4.6432, 4.1212 for $G_{1|2,3}$, $G_{1,2|3}$, $G_{1,3|2}$, $G_{1,2,3}$, $G_{1|2|3}$, respectively. According to Step 9, the optimal complete set of coalitions is $\Omega^* = |1|2,3|$.

5. Conclusion

The principle contribution of this work is the development of a simple approach for forming coalitions in n -person normal-form games where only pure strategies for the players are considered. In particular, the notion of a coalitional semi-cooperative game studied here models a normal-form game with both cooperative and competitive aspects. An optimal set of coalitions is then obtained via Algorithm 1. In this approach, we use the GSE, which gives players their largest individual payoffs jointly possible, always exists in pure strategies, and can be readily computed. We note that in the examples of section 4, Algorithm 1 determines coalitions that, in fact, seemed intuitively optimal. In Example 1, the optimal set of coalitions is $|1,3|2|$ with the strategies (s_1, s_2, s_3) yielding the associated GSE for $\Gamma_{1,3|2}$. The original payoffs for G were $(6, 5, 5)$, which had the largest sum in the payoff matrix. Because of the cooperation required to form these coalitions, however, the players would actually receive $(4.8, 5, 6.2)$ since players 2 and 3 usually had lower payoffs in the payoff matrix than 5 and 5, respectively. Similar remarks can be made about Example 2 with an optimal set of coalitions $|1|2,3|$.

There is an immediate conceivable extension to this paper. The coalitions formed here are based on the assumption that the players want their respective coalitions to obtain the largest possible payoffs. We used the GSE to model this greediness. However, the coalitions need not be greedy. Future work might define optimal coalitions using other scalar equilibria than the GSE. Alternatives can be found in Corley (2017).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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