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Improving Portfolio Selection by Balancing Liquidity-Risk-Return: Evidence from Stock Markets

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Abstract

The Modern Portfolio Theory was mathematically structured on the basis of the risk-return tradeoff: in other words, the riskier the investment, the greater the required potential return. Traditional portfolio optimization models, however, implicitly consider that all assets can be traded at any time and in any quantity, which is unrealistic. The aim was to propose a two-stage method that includes the prior classification of liquidity based on the bid-ask spread and a mathematical optimization model that uses liquidity as a defined participation constraint. Simulations were carried out using twenty years of data from the American (NYSE) and Brazilian (B3) stock exchanges. The results showed that the method developed offers a broader range of the alternatives that comprise the MV model with a more realistic approach to liquidity. The proposed method can form portfolios that respect the risk-return rules once the investor's risk profile has been defined, making it a useful recommendation tool for institutional investors. From a conservative point of view, the developed method also showed the potential for reducing uncertain sales by 10.3% on average.

Keywords

Asset Allocation, Bid-Ask Spread, Investor Preference, Liquidity, Portfolio Optimization

1. Introduction

Because of the importance of asset allocation decisions, portfolio optimization is one of the most significant problems in financial engineering. The main idea behind portfolio optimization is to find an optimal combination of assets for a specific level of risk that maximizes the return or minimizes the portfolio risk for a defined target return. The first portfolio optimization model was proposed by Markowitz (1952), and formed the basis of Modern Portfolio Theory. Markowitz's proposal, also known as the mean-variance model (MV), is based on variance as the adopted risk measure, which is usually obtained by analyzing historical data. Despite having been developed a long time ago, its principles are still a significant reference in the theory and practice of finance (Abensur, Moreira, & De Faria, 2020; Hung, Yang, Zhao, & Lee, 2018; Li, Yao, & Li, 2010; Qin, 2015).

Over time, the MV model has influenced other optimization approaches by replacing, for example, variance as the risk measure. In this particular regard, we can cite value at risk (VaR) (Feng, Wächter, & Staum, 2015; Wang, Xie, Jiang, Wu, & He, 2017), conditional value at risk (CVaR) (Noyan & Rudolf, 2013; Rockafellar & Uryasev, 2000), mean absolute deviation (Konno & Yamazaki, 1991), semivariance (Chen, Peng, Zhang, & Rosyida, 2017), and downside risk models (Bawa & Lindenberg, 1977) among others. Nonetheless, they all maintain the relevant MV characteristics that consider all analyzed assets to be equally liquid. In other words, all assets can be traded at any time and in any quantity, which is unlikely.

The emerging market's degree of integration with the global economy has been changing gradually but significantly due to domestic factors such as deregulation and international factors such as financial innovation and technology which enable investors to invest internationally more easily (Batten & Vo, 2014). The Brazilian stock market (B3) is the largest Latin American financial market whose stocks are traded simultaneously in important markets like the New York Stock Exchange (NYSE). According to the International Monetary Fund (IMF), before the coronavirus pandemic, the Brazilian economy was one of the ten largest in the world, so Brazil is a relevant alternative market for portfolio diversification (IMF, 2019). As previous empirical evidence has shown, however, stock returns, marketability, and market volatility in emerging markets like Brazil are significantly different than the performance we see in developed countries (Atilgan, Demirtas, & Simsek, 2015; Batten & Vo, 2014; Sarwar & Khan, 2017). In particular, liquidity is challenging (Abensur, Saigal, Zhang, Song, & Yu, 2020). There are similarities, however, between mature liquid markets, such as the US, and emerging and less liquid markets, like Brazil.

Figure 1 and **Figure 2** show that there is an interesting similarity in the liquidity between the two markets using the cumulative financial participation of the stocks traded every day. Sixty percent (60%) of Brazilian shares had an accumulated financial participation of less than 20% in 2019. This percentage was even higher in the USA, where it reached 70% of the shares in 2014. Therefore, considering the financial volume traded in both markets, a relevant number of stocks are traded on a daily basis but in small quantities.

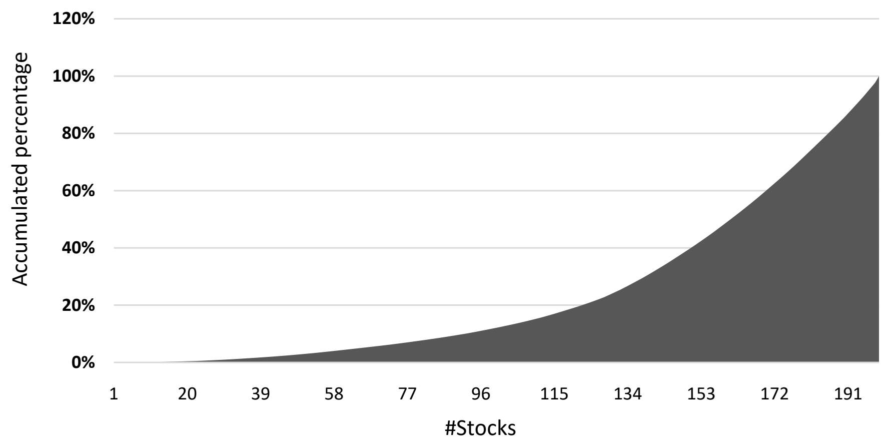

Figure 1. B3 most traded stocks in 2019. Source: prepared by the authors.

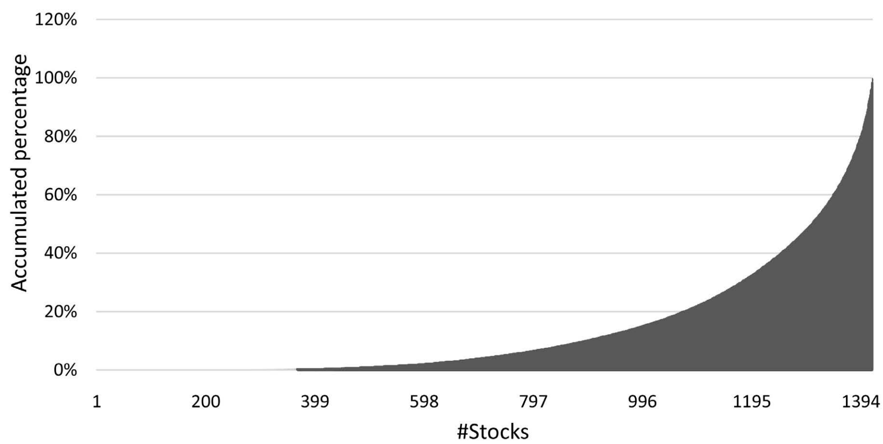

Figure 2. NYSE most traded stocks in 2014. Source: prepared by the authors.

Main Goals and Contribution

Mathematical models for forming portfolios that simultaneously balance return, risk, and liquidity are important tools for investors and for the stability of society as a whole. Besides the risk-return tradeoff, this study also incorporates liquidity (or illiquidity) in the portfolio allocation decision using an accessible method that includes a mathematical model for investors, and particularly institutional investors (brokers, investment banks, pension funds). The main points and contributions of this work are:

- It expands on the traditional portfolio approaches by using the defined liquidity participation of the analyzed assets throughout the optimization process which is tested in two different and relevant stock markets.
- It expands the role of the bid-ask spread as a workable liquidity factor for portfolio optimization models.
- It proposes an alternative way of estimating the liquidity premium or illiquidity penalty.
- The study shows that the developed liquidity approach comprises the MV portfolios according to the adopted assumptions.

This study is structured as follows. Section 2 outlines a brief review of the literature on the mean-variance optimization model and bid-ask spread. Section 3 describes the applied methodology, including the assessment of formed portfolios and the proposed method. Section 4 shows the results, while the last section provides the conclusions and suggests possible future work.

2. Literature Review

This section describes some of the widely used liquidity measures that support selection of the bid-ask-spread (Bid) as the appropriate liquidity proxy for this study. The notion of liquidity and the fundamentals of the MV model are also introduced. Besides selecting Bid as a liquidity measure, this section shows that the liquidity of stocks has not been adequately considered in traditional portfolio optimization models. These points provided the basis for data collection and the mathematical model presented in Section 3.

2.1. Liquidity and the Bid-Ask Spread (Bid)

The Finance literature describes many liquidity proxies that have different characteristics, complex formulations, and contradictory performances, making it almost impossible to choose the best liquidity measure.

The concept and impacts of liquidity on the financial system were studied and defined by Keynes (1936). He described the concept of liquidity due to the impossibility of estimating interest rate trends based on the relationship between money demand and supply. Instead, liquidity would represent the investor's preference for easily traded assets. He examined this characteristic from the monetary perspective and related it to the uncertain the future of the interest rate, implying that the greater the uncertainty, the greater the preference for liquidity, which in this specific case means that investors would prefer to hold cash. Therefore, liquidity was identified as a factor that investors consider in their decisions in order to avoid trading problems.

The concept of liquidity has been interpreted in different ways over time. The liquidity level of assets has been assessed by how fast they can be transformed into the most liquid asset: cash. Thus, all else being equally constant (risk and return), investors prefer assets that are more liquid rather than less liquid.

There is no appropriate and, consistent liquidity measure for all markets despite abundant theoretical and empirical literature on liquidity and related issues (Le & Gregoriou, 2020). To a great extent, this is because the liquidity concept is a multi-dimensional attribute that includes aspects such as the amount of trade, trading time, reputation, experience, and price impact. According to Tavana et al. (2018), the term "liquidity" can refer to different dimensions simultaneously, particularly when they are combined with market liquidity risk or systemic liquidity risk. Moreover, there is still controversy regarding the definition of a liquid market or liquid/illiquid assets and their appropriate liquidity measures (Díaz & Escribano, 2020; Goyenko, Holden, & Trzcinka, 2009; Le & Gregoriou, 2020;

Ramos & Righi, 2020).

Over time, several measures have been developed to estimate liquidity (Leirvik, Fiskerstrand, & Fjellvikås, 2017; Næs, Skjeltorp, & Ødegaard, 2011). For example, Liu (2006) defined liquid stocks as those that can quickly trade large volumes at low cost with little impact on price. In general, therefore, liquidity measures can be divided into 1) liquidity based on frequency such as low-frequency (daily measures) or high frequency (intraday); 2) liquidity based on spread, and; 3) liquidity based on traded financial volume.

As an example of a liquidity frequency proxy, Lesmond et al. (1999) introduced the Zeros, which captures the frequency of zero return days. This is defined as the number of days when returns are zero divided by the number of observable days. A limitation of this category of indicator is that a zero return may be the result of a market trend rather than attributable to the behavior of an individual stock.

Trading volume and turnover ratio are categories of commonly traded financial volumes. The trading volume proxy is calculated by adding the financial volumes of a traded stock during a specified period. The turnover ratio is defined by dividing the number of traded shares by the number of outstanding shares. Volume-based liquidity measures fail to show how price changes with the arrival of a sudden order, but are useful measures as a starting point in the analysis process (Le & Gregoriou, 2020).

Amihud & Mendelson (1986) used the Bid to investigate the return-liquidity relationship. The bid-ask spread is the price investors must pay for liquidity in the form of a quick execution of the negotiation. Since then, the Bid has become the most popular spread liquidity estimator (Będowska-Sójka, 2018). Literature also shows that the bid-ask spread is the most widely-used measure of trading costs as it captures almost all of the costs associated with stock trading (Poufinas & Pappas, 2021; Sarr & Lybek, 2002). For simplicity and according to publicly available data, this study calculated the Bid using its original form that is presented in Equation (1) as follows (Qiu, Chen, Zhong, & Wu, 2012):

$$\text{Bid Ask Spread}(\%) = \frac{\text{Best ask price}(\text{BAP}) - \text{Best bid price}(\text{BBP})}{\text{Best ask price}(\text{BAP})} \tag{1}$$

where:

BAP = The lowest selling price;

BBP = The highest buying price.

As shown in Equation (1), the higher the Bid, the greater the trading difficulty, and therefore the less the liquidity. Another possible understanding to do with the immediate closing of the transaction: the greater the difference, the greater the amount that must be lost (by reducing the sale price or increasing the purchase price) in order to carry out the negotiation at the exact moment.

Investing in illiquid stocks should be compensated for by higher gross returns, a liquidity premium, which is defined as the extra return that an illiquid stock must earn (Díaz & Escribano, 2020; Pereira & Zhang, 2010). According to this line of thinking, a portfolio containing a significantly high proportion of illiquid (or with reduced liquidity) stocks does not have an estimated date for being bought or sold, and no estimated date for achieving the expected gains. Assessing the liquidity aspect during the optimization process is, therefore, crucial.

Figure 3 shows the B3 Bid histogram of the stocks traded every day from January 2018 to June 2019, while Figure 4 presents a similar assessment of the NYSE in 2014. Despite the difference in the scale of shares traded in the two markets, there are some observable similarities. According to Figure 3 and Figure 4,

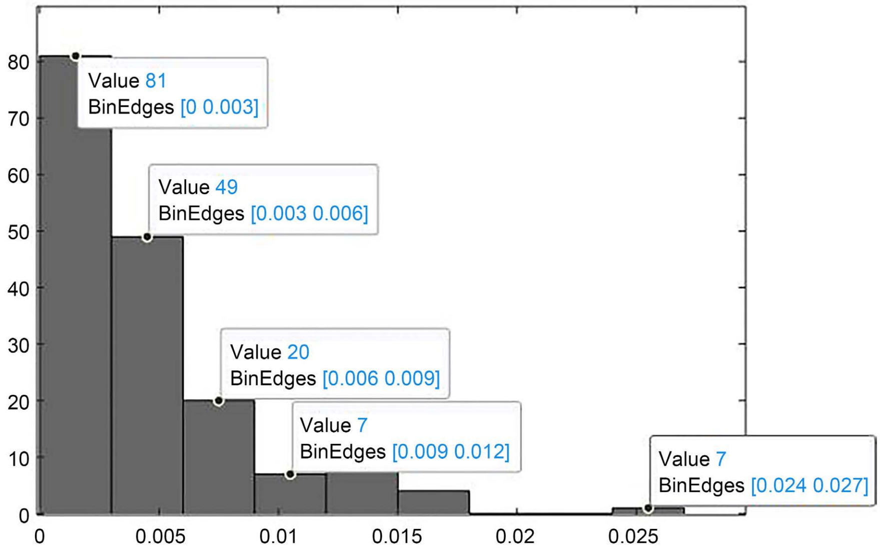

Figure 3. B3 bid histogram (2018-2019). Source: prepared by the authors.

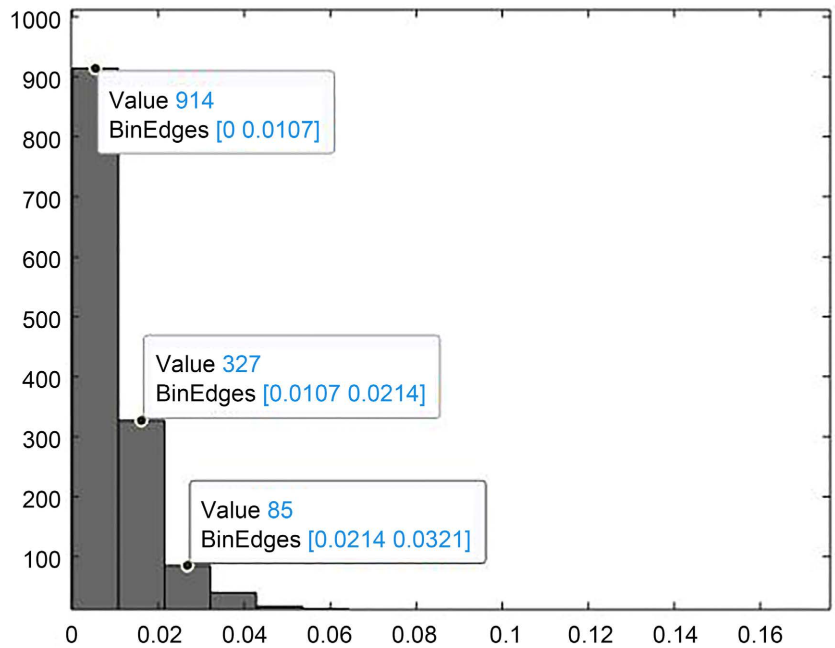

Figure 4. NYSE bid histogram of 2014. Source: prepared by the authors.

we can define four liquidity classes in B3 (high liquidity—HL, medium liquidity—ML, low-liquidity—LL, and extremely low liquidity—NL) and three categories in the NYSE (HL, ML, LL). In terms of the Bid, it is clear that there is a significant difference of around 200% between the most heavily traded stocks in the two markets.

2.2. Mean-Variance Optimization Model (MV)

The mean-variance model is, essentially, an investment framework for selecting and constructing investment portfolios based on maximizing the expected returns of the portfolio while simultaneously minimizing the investment risk (Fabozzi, Gupta, & Markowitz, 2002). "Mean" is an abbreviation for the average expected return, and "variance" is the adopted risk. The MV model is shown below. Equation (2) minimizes the risk of the portfolio. In Equation (3), portfolio profitability must exceed a defined target, while Equation (4) shows that the total percentage of the available investment must be allocated. Only positive values are allowed.

$$\min Z = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} \tag{2}$$

Subject to:

$$\sum_{i=1}^{N} x_i \mu_i \geq \rho \tag{3}$$

$$\begin{gathered} \sum_{i=1}^{N} x_i = 1 \\ x_i \geq 0 \quad i = 1, 2, \cdots, N \end{gathered} \tag{4}$$

where:

N—Number of stocks evaluated in the portfolio;

x_i—Percentage of capital to be invested in stock i;

σ_{ij}—Covariance between stocks i and j, where σ_{ii} is the variance of stock i;

μ_i—The expected rate of return of stock i;

ρ—Minimum rate of return defined by the investor.

The understanding and widespread dissemination of the MV principles have increased rapidly. As a result, some of its pros and cons have been identified and technically discussed, such as symmetrical risk assessment. This problem occurs because variance, as a risk measure, penalizes in equal measure both positive deviations (favorable to investors) and negative deviations (unfavorable to investors) (Ayub, Shah, & Abbas, 2015; Jarrow & Zhao, 2006; Nawrocki, 1999). The MV model results lie on a geometric curve called the efficient frontier. According to Fabozzi et al. (2002), this frontier is efficient because every point on it is a portfolio that results in the greatest possible expected return for that level of risk or results in the smallest potential risk for that level of expected return. **Figure 5** presents an example of the efficient frontier using only B3 shares from January 2018.

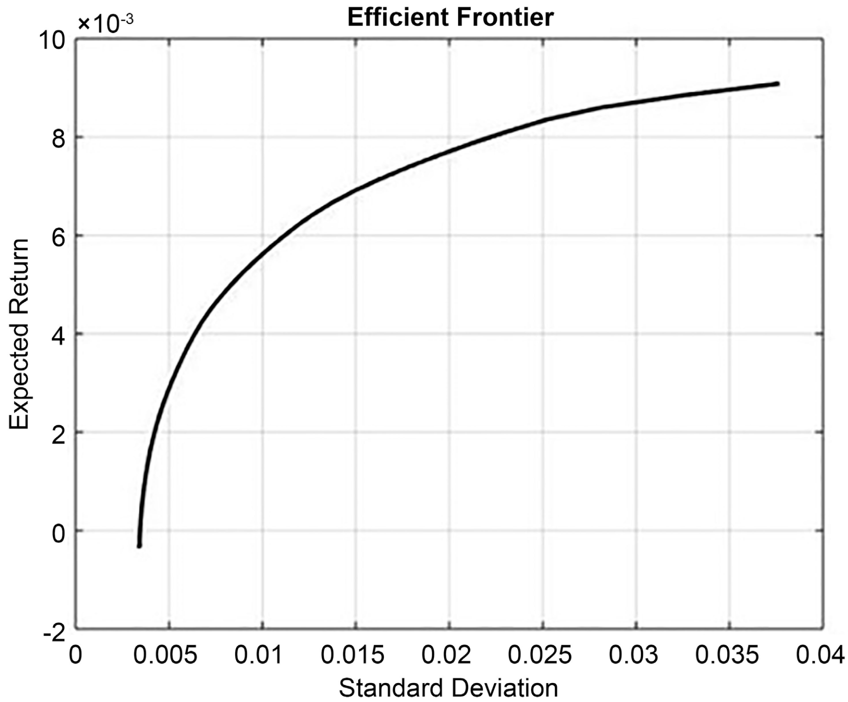

Figure 5. Efficient frontier of B3 shares (2018/01). Source: prepared by the authors.

As shown in **Figure 5**, the MV model assumes that portfolio selection can be reduced to only two main dimensions: 1) the variance or risk of the portfolio and 2) its expected return. Thus, it is primarily risk-return duality balance that treats all assets equally in liquidity. In reality, however, investors face liquidity constraints in virtually all financial markets; in other words, they are incapable of quickly changing portfolio positions when they need to (Longstaff, 2001).

3. Methodology

We have used twenty years of historical data from the B3 and NYSE. The B3 data were obtained from http://www.b3.com.br/pt_br/market-data-e-indices/servicos-de-dados/market-data/historico/mercado-a-vista/series-historicas/. We collected the NYSE data from an available Bloomberg database. All simulations were carried out using MATLAB of MathWorks®. The basic statistics used in this study are shown in the **Appendix** (**Table A1** and **Table A2**), while **Table A3** presents data samples.

We collected the daily traded data of the B3 for all assets from 2010 to 2019 and of the NYSE from 2005 to 2014. Initially, the data were organized by calendar year. Altogether, 8.2×10^6 records of two thousand (2300) stocks were compiled. We consider only essential information such as date, tickers (stock codes), price, traded volume, traded stocks, BAP, and BBP. We subsequently excluded negative bid records, or records that had no price or traded volume observations to ensure the integrity of the results. The Bid was assumed to follow a normal distribution behavior. We also excluded extreme values beyond the mean (μ)

plus three times the standard deviation (3σ). Finally, only stocks traded on at least 99% of the respective year's business days were considered in the analysis to ensure feasible portfolios.

The bid histograms were constructed by calculating the median of the Bid values using the data we collected primarily organized by date, ticker, and Bid. Once the median Bid had been estimated for each stock, they were classified by type of liquidity by way of visual observation, as shown in **Figure 3** and **Figure 4**. After the filters had been applied, one thousand five hundred stocks remained in the process, on average.

Once the stocks had been classified by liquidity category (HL, ML, LL, NL), the MV and the Bid mean-variance portfolios were formed as follows:

- Only samples composed of stocks traded on at least 99% of the respective year's business days were considered in the analysis.
- The data were sorted by the median Bid in ascending order. This enabled us to organize the shares from the smallest (most liquid) to the biggest (less liquid) Bid value.
- The independent effect of the stock prices was ensured by dividing the samples into their respective calendar years.
- As shown in the **Appendix** (**Table A1** and **Table A2**), the Brazilian data are more recent. We therefore prepared a pilot sample for the B3 period from January 2018 to June 2019 to estimate the composition of the formed portfolios that were formed using the MV and the Bid mean-variance models (Anderson, Sweeney, & Williams, 2007).

Altogether, we put together 240 portfolios a 99% confidence level and a 1.3% margin of error, as shown in Equation (5) below. This error level was considered to be entirely acceptable given the values involved.

$$N = \frac{\left(z_{\alpha/2}\right)^2 \sigma^2}{E^2} \tag{5}$$

where:

N—number of portfolios;

σ—standard deviation of the pilot sample;

α—confidence level;

E—error margin

The Bid mean-variance portfolios (Bid-MV) were formed using Equations (6) to (9). Equation (8) shows that all of the available money will be allocated respecting the classification of liquidity of the assets (bid histograms). According to investor profiles (conservative, moderate, aggressive), Equation (9) limits the participation of less liquid stocks to a defined percentage. Finally, no negative results are allowed.

$$\min Z = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \quad \text{portfolio risk} \tag{6}$$

Subject to:

$$\sum_{i=1}^{N} x_i \mu_i + \sum_{i=1}^{N} y_i \mu_i \geq \rho \quad \text{expected target return} \tag{7}$$

$$\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} y_i = 1 \quad \text{investment allocation} \tag{8}$$

$$\sum_{i=1}^{N} y_i \leq p \quad \text{low liquidity target} \tag{9}$$

$$x_i \geq 0 \quad i = 1, 2, \cdots, N \quad \text{positive rates}$$

where:

N—Number of stocks evaluated in the portfolio;

x_i—Percentage of capital to be invested in liquid stock i;

y_i—Percentage of money to be invested in low liquid stock i;

w_i—Liquid (x_i) and low liquidity assets (y_i);

σ_{ij}—Covariance between stocks i and j, where σ_{ii} is the variance of stock i;

μ_i—The expected rate of return on stock i;

ρ—Minimum rate of return defined by the investor;

p—Percentage' participation of low liquidity stocks in the portfolio

Using both models, we put together 240 portfolios for the period between 2005/01/01 and 2019/12/31. This period includes several financial crises, such as the subprime crisis in 2008, the Brazilian presidential impeachment process in 2016, and the trade dispute between China and the USA in 2019. These events led to considerable variation in the financial markets that was caused by the consequent volatility. We, therefore, varied the participation (p) of low liquidity stocks from zero (0) to forty (0.40) to assess the return-risk impact.

4. Results

For each analyzed year, eleven Bid-MV portfolios and one MV portfolio were formed. The Brazilian stock market index (IBV) and the S&P 500 were assumed as yearly expected return and risk references of both markets. **Figure 6** and **Figure 7** show the percentage of low liquidity (LL) shares in the MV portfolios, while **Figures 8-11** show that the Bid-MV and the MV portfolios have a similar return-risk performance. As expected, **Figure 6** and **Figure 7** show that the MV portfolios have a large proportion of low liquidity stocks because all stocks were considered to be equally liquid. The average proportion (10.30%) of low liquidity stocks is a liquidity penalty because there is no estimated date to trade them.

Table 1 shows a more accurate risk-return assessment. At a 1% significance level, there is significant evidence of equivalent results (mean difference equal to zero) for the return between the models. There is also important evidence of risk difference (standard deviation) equal to 0.003% for the B3 and zero (0.00) for the NYSE for the same significance level. In other words, the Bid-MV portfolios balanced liquidity offered a similar return-risk performance as the MV model, but with better liquidity (**Figure 6** and **Figure 7**) because liquidity can be adjusted from 0% to 40%, as shown in Equation (9). Both of the tested models also had a better performance than the stock market proxies (**Figures 8-11**).

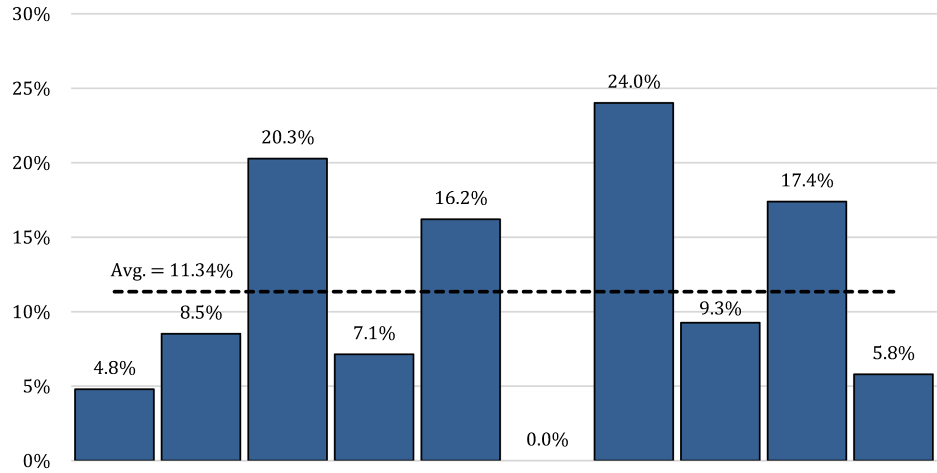

Figure 6. The low liquidity proportion of the NYSE MV portfolios (2005-2014). Source: prepared by the authors.

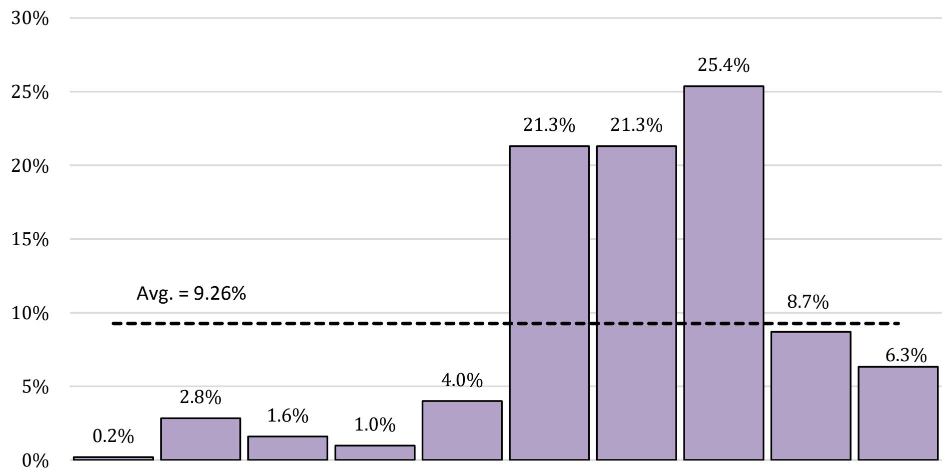

Figure 7. The low liquidity proportion of the B3 MV portfolios (2010-2019). Source: prepared by the authors.

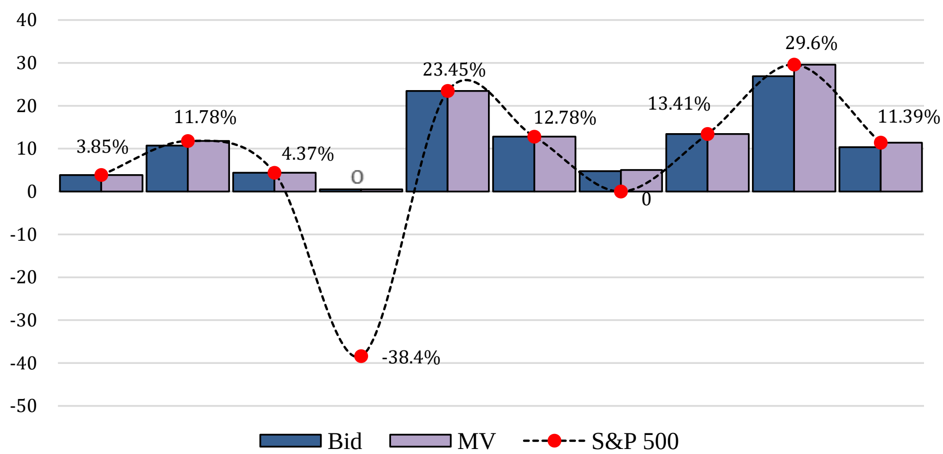

Figure 8. US portfolio returns (Bid versus MV) from 2005 to 2014. Source: prepared by the authors.

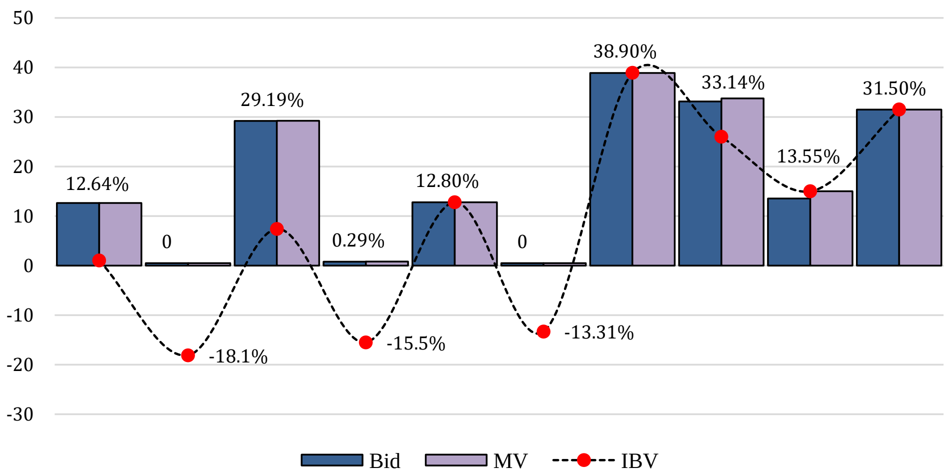

Figure 9. Brazilian portfolio returns (Bid versus MV) from 2010 to 2019. Source: prepared by the authors.

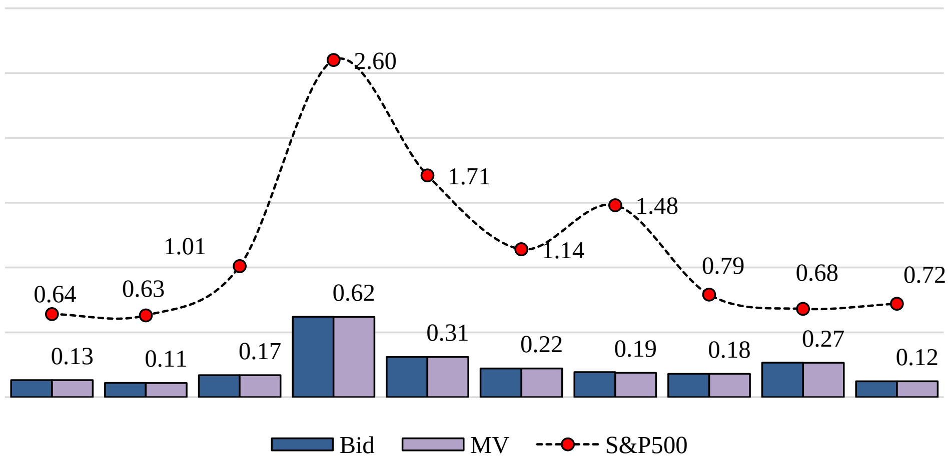

Figure 10. US portfolio risks (Bid versus MV-percentage) from 2005 to 2014. Source: prepared by the authors.

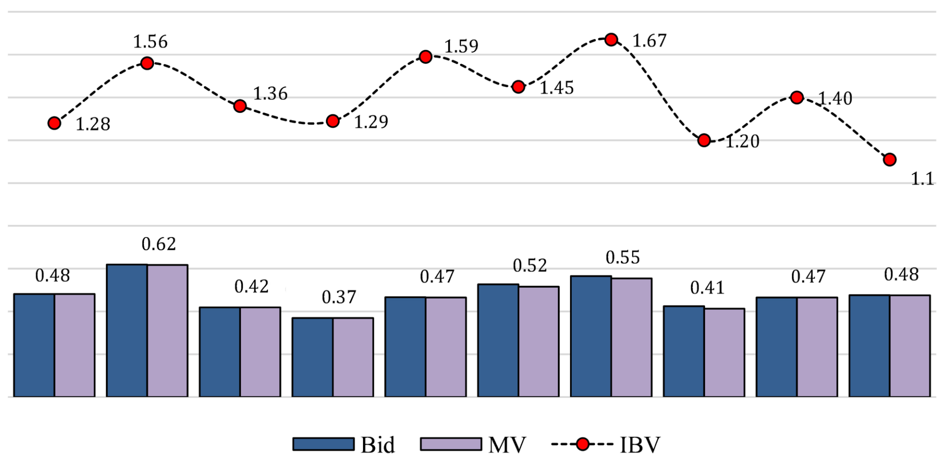

Figure 11. Brazilian portfolio risks (Bid versus MV-percentage) from 2010 to 2019. Source: prepared by the authors.

Table 1. Statistical summary.

Statistical procedure	Objective	α	Result
Hypothesis test B3	The difference between the Bid-MV and MV returns rates is equal to zero (0)	1%	There was evidence of $\mu_{\text{Bid-MV}} = \mu_{\text{MV}}$
Hypothesis test B3	The difference between the Bid-MV and MV standard deviations (risk) is equal to 0.3%	1%	There was evidence of $\sigma_{\text{Bid-MV}} = \sigma_{\text{MV}}$
Hypothesis test NYSE	The difference between the Bid-MV and MV returns rates is equal to zero	1%	There was evidence of $\mu_{\text{Bid-MV}} = \mu_{\text{MV}}$
Hypothesis test NYSE	The difference between the Bid-MV and MV standard deviation (risk) is equal to zero (0)	1%	There was evidence of $\sigma_{\text{Bid-MV}} = \sigma_{\text{MV}}$

Source: prepared by the authors.

The main advantage of the proposed method is that the liquidity-risk-return is simultaneously balanced during the optimization process. This process can be linked to the investor's profile, as shown in Figure 12. First, the risk-aversity of the investor is incorporated into the analysis to assess the range of their liquidity sensitivity. The traditional risk-return analysis is then carried out to achieve the best allocation. Risk-averse investors can choose a zero (0.00) proportion of low liquidity stocks (LL), while moderate or aggressive investors can accept a greater proportion. The portfolio, therefore, could be formed respecting the risk-return rules once the investor's liquidity profile has been defined. The low liquidity proportion can be adjusted to avoid allocation failure. On the whole, the proposed method has a range of liquidity alternatives that incorporate the MV results.

Institutional investors such as investment banks, brokers, and pension funds have usually used the investor's profile as an investment recommendation strategy. The method proposed here, however, offers a more realistic allocation because it considers the liquidity aspect based on the Bid-ask spread. Particularly in emerging markets such as Brazil, the liquidity approach during investment allocation is crucial. For example, Table 2 shows the stratification of the average daily trading volume of the most traded shares in 2019 in Brazilian currency (Reais). As can be seen, 33.1% of the shares had daily trades of less than R$1 million (~US$200,000). A significant proportion of the shares were of no business interest to institutional investors. In other words, these shares are difficult to buy and/or sell. Therefore, if there is no liquidity balancing, optimization could result in unfeasible portfolios being formed.

In summary, Figures 6-11, Table 1 and Table 2 show that the proposed method formed portfolios with similar return-risk performance compared to the MV model but with a more realistic liquidity approach in terms of the bid-ask spread and daily trading volume.

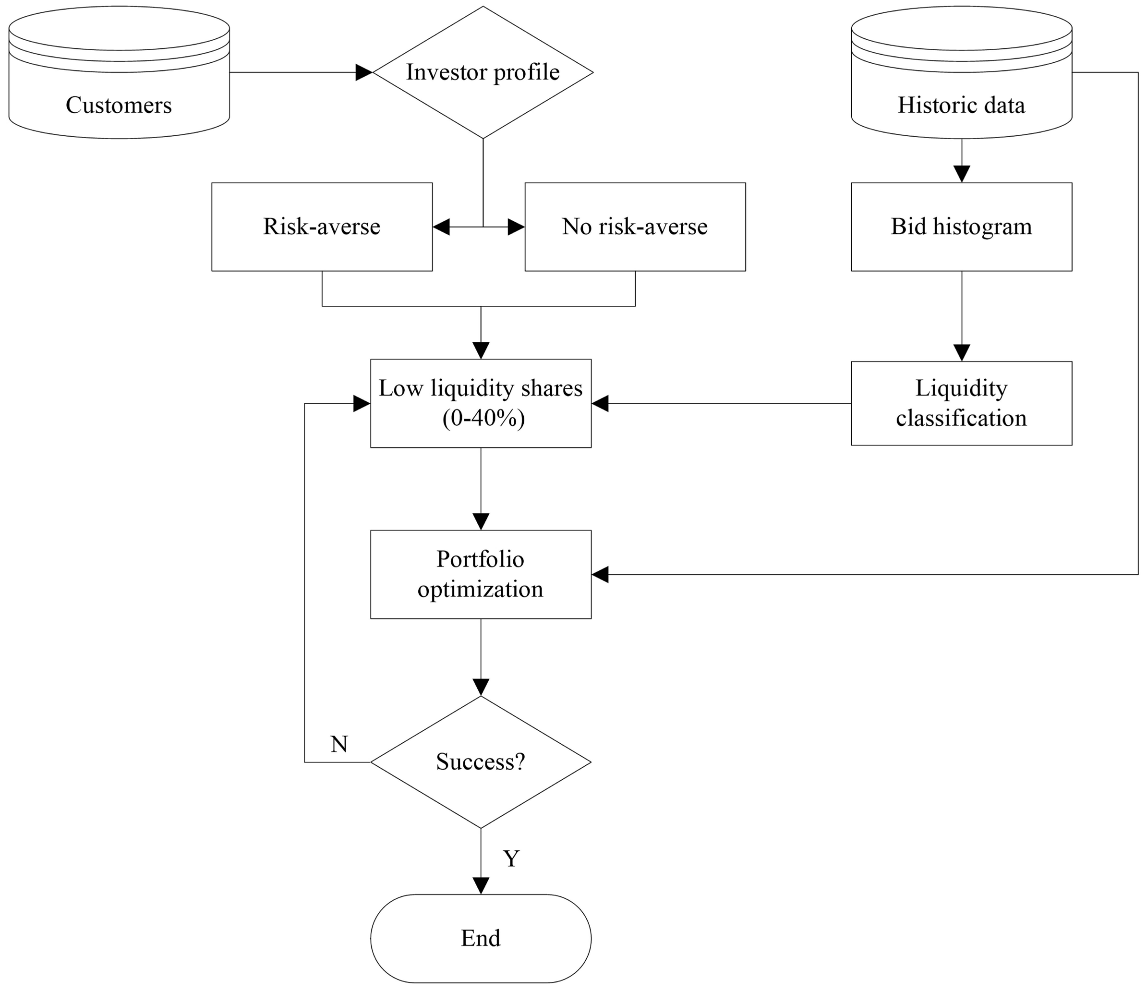

Figure 12. Brazilian portfolio risks (Bid versus MV). Source: prepared by the authors.

Table 2. Liquidity stratification of the most traded Brazilian stocks in 2019.

Financial Volume R$ (×1000)	Stocks	Percentage (%)
0 - 500	24	12.6
500 - 1000	39	20.5
1000 - 2000	34	17.9
2000 - 5000	34	17.9
5000 - 10000	59	31.1
Total	190	100

Source: prepared by the authors.

5. Conclusion and Suggested Future Work

This study developed a two-stage method on combining a liquidity measure (bid-ask spread) with the mean-variance model using public data take from the American (NYSE) and Brazilian (B3) stock exchange market for the assessment.

The method we employed uses the bid-ask spread measure to classify stock liquidity. The performance achieved was positively significant in all of the periods we analyzed. In both markets, the average Bid-MV portfolios had similar return-risk results with superior liquidity (10.30% on average) when compared to the MV model because the proportion of low-liquidity shares can be adjusted (e.g., equal to zero). Furthermore, in the MV model, stocks with low liquidity or no liquidity at all can participate in the portfolio composition. This study also showed that the proposed method includes the MV portfolio and incorporates the investor's profile from a liquidity-return-risk relationship perspective, which is particularly relevant for institutional investors.

A natural line of research would be to assess the performance of this method using other portfolio optimization models, by balancing the participation of low liquidity assets to improve the expected returns. Another exciting research line would be to combine the popular Bid with other proxies in order to achieve more accurate liquidity results.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix: Basic Statistics of the Data Used in This Study

Table A1. Descriptive statistics of the Brazilian stock exchange (B3) data.

Year	[1]IBV Yearly Return (%)	[2]Traded Days	[2]Analyzed Stocks	[2]Avg Price	[2]Std	[2]Avg Bid	[2]Std
2010	1.04	247	191	22.67	64	0.033	0.08
2011	−18.10	249	185	24.82	105	0.034	0.08
2012	7.40	246	182	18.34	31	0.032	0.07
2013	−15.50	248	187	16.87	22	0.037	0.09
2014	12.78	248	189	15.59	21	0.037	0.10
2015	−13.31	246	172	14.09	22	0.039	0.10
2016	38.90	249	167	15.25	27	0.035	0.09
2017	26.00	246	176	18.86	32	0.025	0.07
2018	15.00	244	195	19.91	25	0.025	0.07
2019	31.50	248	199	22.65	27	0.022	0.06
Average	8.57	247	184	18.90	38	0.032	0.08

[1]Ibovespa yearly return—YAHOO FINANCE; [2]B3.

Table A2. Descriptive statistics of the US stock market (NYSE) data.

Year	[1]S&P500 Yearly Return (%)	[2]Traded Days	[2]Analyzed Stocks	[2]Avg Price	[2]Std	[2]Avg Bid	[2]Std
2005	3.84	250	1454	29.07	42	0.054	0.07
2006	11.78	251	1446	32.29	42	0.054	0.07
2007	4.37	250	1441	36.27	47	0.066	0.09
2008	−38.40	252	1400	29.50	35	0.097	0.17
2009	23.45	251	1399	21.60	21	0.057	0.08
2010	12.78	251	1442	29.01	46	0.051	0.16
2011	0.00	251	1417	29.97	27	0.027	0.04
2012	13.41	249	1416	31.07	29	0.029	0.04
2013	29.60	251	1413	36.93	36	0.025	0.04
2014	11.39	251	1415	41.25	41	0.022	0.04
Average	7.22	251	1424	31.69	37	0.048	0.08

[1]YAHOO FINANCE; [2]NYSE-Bloomberg.

Table A3. Data sample of the Brazilian stock exchange (B3).

Ticker	Average price	Closing price	BBP	BAP	Traded stocks	[1]Financial Volume (R$)
AALR3	13.33	13.25	13.2	13.25	264,200	3,523,962
ABCB4	16.78	17.12	17.05	17.12	571,700	95,983,00

Continued

ABEV3	15.95	16.15	16.13	16.15	18,692,900	298,151,755
ADHM3	1.49	1.5	1.43	1.54	1800	2698
AFLT3	5.43	5.4	4.6	5.4	500	2716
AGRO3	15.86	15.95	15.9	15.95	30,000	475,985
ALPA3	16.51	16.8	16.8	17.67	5500	90,848
ALPA4	17.29	17.5	17.48	17.5	432,100	7,474,510
ALSC3	19.37	19.65	19.65	19.69	621,800	12,047,465
ALUP3	7.15	7.31	7.02	7.32	3800	27,177
ALUP4	5.77	5.8	5.53	5.8	6600	38,137
AMAR3	5.44	5.54	5.54	5.55	1,023,700	5,572,007
ANIM3	17.84	18.01	18.01	18.09	241,400	4,306,977

[1]Brazilian currency (Reais).