A Kind of Half-Discrete Hardy-Hilbert-Type Inequalities Involving Several Applications

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Contents

Chapter 1. Introduction ········ 3
1.1 Background of the Analytic Inequalities ········ 4
1.2 Important Periods of Hilbert-Type Inequalities ········ 9
1.3 The Organization of This Book ········ 11
Chapter 2. Half-Discrete Mulholland-Type Inequalities with a Internal Variable ········ 12
2.1 Some Lemmas ········ 13
2.2 Main Results ········ 16
2.3 Operator Expressions and Some Particular Cases ········ 20
2.4 The Reverses ········ 27
Chapter 3. Half-Discrete Hilbert-Type Inequalities Involving One Derivative Function of Higher-Order ········ 42
3.1 Some Lemmas ········ 42
3.2 Main Results ········ 45
3.3 Equivalent Forms and Operator Expressions ········ 49
3.4 The Reverses ········ 52
Chapter 4. Half-Discrete Hilbert-Type Inequalities Involving One Multiple Upper Limit Function ········ 62
4.1 Some Lemmas ········ 62
4.2 Main Results ········ 65
4.3 Equivalent Forms and Operator Expressions ········ 69
4.4 The Reverses ········ 72
Chapter 5. Some Reverse Half-Discrete Hilbert-Type Inequalities with One Partial Sums ········ 84
5.1 Some Lemmas ········ 84
5.2 The Reverse Inequality Involving One Multiple Upper Limit Function ········ 87
5.3 The Reverse Inequalities Involving One Derivative Function of Higher-Order ········ 98
5.4 Equivalent Forms and Some Particular Inequalities ········ 109
Chapter 6. Some Reverse Half-Discrete Hilbert-Type Inequalities with Two Internal Variables and One Partial Sums ········ 117

6.1 Some Lemmas ··· 118
6.2 The Reverse Inequalities Involving One Upper Limit Function ··· 121
6.3 The Reverse Inequalities Involving One Derivative Function ··· 131
Chapter 7. Half-Discrete Hilbert-Type Inequalities with Two Internal Variables ··· 141
7.1 Some Lemmas ··· 141
7.2 The Inequalities Involving One Upper Limit Function ··· 143
7.3 The Inequalities Involving One Derivative Function ··· 161
References ··· 179

Preface

Hilbert-type inequalities including Hilbert's inequalities (built-in 1908), Hardy-Hilbert -type inequalities (built-in 1934) and Yang-Hilbert-type inequalities (built-in 1998) played an important role in analysis and its applications, which are mainly divided into three classes of integral, discrete and half-discrete. In recent thirty years, there are many advances in research on Hilbert-type inequalities and applications, especially in Yang-Hilbert-type inequalities.

In this book, applying the weight functions, the idea of introduced parameters and the techniques of real analysis and functional analysis, we provide a new kind of half-discrete Hilbert-type inequalities named in Mulholland-type inequality. Then, we consider its several applications involving the derivative function of higher-order or the multiple upper limit function. Some new reverses with the partial sums are obtained. We also consider some half-discrete Hardy-Hilbert's inequalities with two internal variables involving one derivative function or one upper limit function in the last chapter. The lemmas and theorems provide an extensive account of these kinds of half-discrete inequalities and operators.

There are seven chapters in this book. In Chapter 1, we introduce some recent developments of Hilbert-type integral, discrete and half-discrete inequalities. In Chapter 2, by using the weight functions and the techniques of real analysis, a new half-discrete Mulholland-type inequality with the nonhomogeneous kernel is given, and the case of the homogeneous kernel is deduced. The equivalent forms and some equivalent statements of the best possible constant factors related to several parameters are obtained. We also consider the operator expressions as well as some reverses. In Chapters 3-4, two kinds of applications involving one derivative function of higher-order or one multiple upper limit function are obtained. In Chapters 5-6, we consider some new reverses with the partial sums. In Chapter 7, some new Hardy-Hilbert's inequalities with two internal variables involving the derivative function or the upper limit function are given. So, we finish the topic involving a kind of half-discrete Hilbert-type inequalities and several applications.

We hope that this monograph will prove to be useful especially to graduate students of mathematics, physics, and engineering sciences.

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> As long as a branch of knowledge offers an abundance of problems, it is full of vitality.

David Hilbert

> ... we have always found with most inequalities, that we have a little new to add.
>
> ... in a subject (inequalities) like this, which has applications in every part of mathematics but never been developed systematically.

G.H. Hardy

1 Introduction

This chapter provides an overview of the historical context and theoretical foundation of analytic inequalities, specifically focusing on Hilbert-type inequalities. In section 1.1, we delve into the background and origins of analytic inequalities. Then, in section 1.2, we introduce the important periods of Hilbert-type inequalities, each with their unique properties and applications. Finally, in section 1.3, we outline the overall structure and organization of this book.

1.1 Background of the Analytic Inequalities

Analytic inequality theory is a highly significant mathematical theory that deals with inequalities established or proven using analytical techniques and methods, such as calculus and mathematical analysis. These inequalities are of fundamental importance in various mathematical disciplines and offer valuable insights into the connections between mathematical objects. Research in this field has been disseminated globally and with the abundance of recent research findings, the theoretical content of analytic inequality theory is continuously evolving and enhancing. The status of analytic inequality theory in the mathematical community is becoming increasingly esteemed, and its results find applications in every realm of mathematics.

In modern times, the study of analytic inequalities arose in Europe. There exist numerous analytic inequalities in mathematics, including but not limited to Cauchy inequality, Minkowski inequality, and Hilbert-type inequality. Moreover, the Hilbert-type inequality can be classified into three different types, namely Hilbert-type integral inequality, half-discrete Hilbert-type inequality, and Hilbert-type discrete inequality. All these inequalities are mainly basic in the real analysis theory, such as the Fubini Theorem, Lebesgue term-by-term integration Theorem, Levi's Theorem, and so on. The following sections will introduce these three typical inequalities.

1.1.1 Cauchy's Inequality

The Cauchy inequality, which is also referred to as the Cauchy-Schwarz inequality is a fundamental inequality in mathematics that relates to inner products or dot products of vectors. It is named after the French mathematician Augustin-Louis Cauchy.

In an inner product space, supposing two vectors u, and v, the Cauchy inequality states [1]:

$$|u \cdot v| \leq \|u\| * \|v\|, \tag{1.1}$$

where $u \cdot v$ represents the inner product of u and v. The norm of a vector u is denoted by $\|u\|$, while the norm of a vector v is represented by $\|v\|$. In simpler terms, the magnitude of the dot product between two vectors is always less than or equal to the product of their magnitudes. Geometrically, this means that the cosine of the angle bound is between -1 and 1 in two vectors.

The Cauchy inequality can also be expressed in vector form using vectors u and v as follows:

$$(u \cdot v)^2 \leq (u \cdot u) * (v \cdot v). \tag{1.2}$$

This form demonstrates that the square of the dot product is always less than or equal to the product of the squares of the individual norms.

Cauchy inequality can also be discussed as follows:

$$\sum_{i=1}^{n} a_i b_i \leq \left[\left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) \right]^{\frac{1}{2}}. \tag{1.3}$$

In the cases equality holds if and only if $b_1 = b_2 = \cdots = b_n = 0$ or

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Cauchy inequality has numerous applications in various branches of mathematics, including linear algebra, functional analysis, probability theory, and inequalities theory. It is a powerful tool for establishing bounds and deriving other important results.

1.1.2 Minkowski's Inequality

Minkowski inequalities are a set of mathematical inequalities named after the German mathematician Hermann Minkowski. They provide bounds on the norms of vector sums in a vector space, particularly in Euclidean spaces.

Let's consider a vector space V with a norm $\|\cdot\|$. For any vectors u and v in vector space v, Minkowski's inequalities state [1]:

(1). The triangle inequality form: $\|u+v\| \leq \|u\| + \|v\|$. This inequality expresses that the norm of two vectors' summation is always less than or equal to the summation of their norms. It is analogous to triangle inequality in geometry, where the summation of the lengths of any two sides will always be greater than or equal to the third side length of a triangle.

(2). The reverse triangle inequality form: $\|u+v\| \geq \|u\| + \|v\|$. This inequality expresses that the norm of two vectors' summation is always greater than or equal to the summation of their norms. It provides a lower bound for the norm of the difference Minkowski inequalities are fundamental properties of vector norms and play a crucial role in many areas of mathematics, including functional analysis, optimization, and signal processing. They provide a way to measure the size or length of vectors and establish relationships between vector quantities.

In summary, the Cauchy inequality and Minkowski inequality are specific inequalities that relate to inner products and norms in vector spaces, while Hilbert-type inequalities are more general inequalities that extend and generalize these concepts to various mathematical settings. It is also can be discussed as the following:

Sppose that the numbers $u_{jk}(j = 1, \cdots, m; k = 1, \cdots, n)$ are nonnegative. If $p > 1$

$$\left[\sum_{k=1}^{n}\left(\sum_{j=1}^{m} u_{jk}\right)^{p}\right]^{\frac{1}{p}} \leq \sum_{j=1}^{m}\left(\sum_{k=1}^{n} u_{jk}^{p}\right)^{\frac{1}{p}}; \tag{1.4}$$

if $0 < p < 1$ then

$$\left[\sum_{k=1}^{n}\left(\sum_{j=1}^{m} u_{jk}\right)^{p}\right]^{\frac{1}{p}} \geq \sum_{j=1}^{m}\left(\sum_{k=1}^{n} u_{jk}^{p}\right)^{\frac{1}{p}}. \tag{1.5}$$

In the cases equality holds if and only if the numbers in sets $(u_{11,\cdots,}u_{1n}), \cdots, (u_{m1,\cdots,}u_{mn})$ are proportional.

The Cauchy inequality and Minkowski inequality can be seen as specific cases or building blocks within the broader framework of Hilbert-type inequalities. Therefore, the research focuses on the Mullholland-type inequality, which belongs to the class of Hilbert-type inequality.

1.1.3 Hilbert-Type Inequalities and Operator Expressions

Since its establishment in 1908 by the esteemed German mathematician D. Hilbert, Hilbert-type inequality has undergone a century of rigorous development under the collective efforts of mathematicians. The result of this persistent endeavor is the emergence of systematic integral and discrete Hilbert inequality theories. The Hilbert-type inequality, which bears the name of its founder, serves as a fundamental mathematical tool that has found wide-ranging applications in various areas of mathematical research. The latest advancements in Hilbert-type inequality involve the incorporation of kernels, constructing weight functions, and introducing multi-parameters. These new developments have led to various generalizations and extensions of Hilbert-type inequalities, which involve the utilization of special functions such as the beta function, gamma function [2], Bernoulli's function [3], and the hypergeometric function [4] with many applications. Above all, Hilbert-type inequality becomes an essential branch of modern mathematics [3].

(a) Hilbert-type discrete inequalities

Throughout the 20th century, the significance of both discrete and integral inequalities in mathematics have been substantial, with a wide range of practical applications in different areas of applied and pure mathematics. D. Hilbert [5], a German mathematician, published the well-known Hilbert discrete inequality in 1908:

For $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, the best possible constant factor is π as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}} \tag{1.6}$$

Hilbert-type discrete inequality refers to a class of inequalities that involve discrete variables and resemble the structure or properties of the classical Hilbert inequality.

In 2012, the advancements of Hilbert-type discrete and integral inequalities with kernels were described [2]. These inequalities typically involve summations or sequences of discrete variables and may incorporate weights, exponents, and additional conditions. Several dynamic Hilbert-type inequalities with the application of time scales were showcased in previous works [6]-[7]. The intricate nature of Hilbert-type discrete inequalities posses a challenge in their study. The research outcomes have limited relevance to Hilbert-type integral inequalities.

(b) Hilbert-type integral inequalities

The Hilbert-type integral inequality was formed [8]:

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty f^2(y)dy \right)^{\frac{1}{2}}. \tag{1.7}$$

The best possible constant factor is π. The Hilbert-type integral inequality is a category of inequalities that expands upon the classical Hilbert inequality by incorporating integrals. These inequalities involve various functions and integrals, and they can also include additional parameters to further enhance their complexity.

In contrast to Hilbert-type discrete inequalities, Hilbert-type integral inequalities have proven to be a more approachable subject of study. As a result, scholars have achieved significant academic progress through the exploration of integral inequalities. These advancements have contributed greatly to both the theoretical and practical understanding of inequalities, thereby establishing a firm basis for the study of half-discrete Hilbert-type inequalities.

(c) Half-discrete Hilbert-type inequalities

In the classical Hilbert-type inequality, all the variables involved are continuous. However, in many practical situations, it is more realistic to consider a combination of discrete and continuous variables. The half-discrete Hilbert-type inequality addresses this by allowing some variables to be discrete while others remain continuous. The half-discrete Hilbert-type inequality, expressed as [9],

$$\sum_{m=1}^{\infty} n^{p-2} \left(\int_0^\infty K(nx)f(x)dx \right)^p < \phi^p \left(\frac{1}{q} \right) \int_0^\infty f^p(x)dx, \tag{1.8}$$

represents a significant formulation within the field. The half-discrete Hilbert-type inequality theory has presented several inequalities for the nonhomogeneous kernel. However, conclusive proof for the optimal constant factor is still lacking. Theorem 315 in [9] holds particular importance in advancing research in this area, ushering in a fresh perspective on half-discrete Hilbert-type inequality.

In 2005, the half-discrete Hilbert inequality involving nonhomogeneous kernels was considered by introducing interval variables, followed by the proving of the best possible constant factor [10]. This particular area of study has captured the attention of scholars due to its potential implications, and wide-ranging applications in mathematical fields. Therefore, the investigation and analysis of half-discrete Hilbert-type inequalities have become a prominent area of interest and research in the academic community.

1.1.4 The Operation Expressions of Hilbert-Type Inequalities

With the ongoing expansion of theoretical research on inequality, scholars have progressively directed their focus toward exploring its practical applications. This shift has yielded a wealth of fruitful outcomes, utilizing diverse application types to gain deeper insights and tackle the multifaceted challenges associated with inequality. Among these applications, the operator expression serves as a fundamental link between inequality and real analysis.

Over the past few decades, the fundamental theory about half-discrete Hilbert-type inequalities and operator expressions has yet to develop into a fully mature and comprehensive system. The operator expressions of Hilbert inequalities can be traced back to the 1920s. In 1923, the Hilbert operator for the first time and abstractly described the Hilbert-type inequality by using the inner product operation of the operator and the norm relation [11]. It was obtained as follows:

If $L^2(0,\infty)$ is a real function space, we Hilbert's integral operator as $\overline{T}: L^2(0,\infty) \to L^2(0,\infty)$, there exist a $h = \overline{T}f \in L^2(0,\infty)$, satisfying

$$\left(\overline{T}f\right)(y) = h(y) := \int_0^\infty \frac{f(x)}{x+y} dx, y \in (0,\infty).$$

Hence for any $g \in L^2(0,\infty)$, we define the inner product of $\overline{T}f$ and g as follows:

$$(\overline{T}f, g) = \int_0^\infty \int_0^\infty \frac{1}{x+y} f(x)g(y)dxdy.$$

Then (1.7) may be rewritten as

$$(\overline{T}f, g) < \pi \|f\|_2 \|g\|_2 . \tag{1.9}$$

We have $\left\|\overline{T}\right\| = \pi$. This result establishes a connection between the Hilbert's inequality and the operator norm.

Since the 1960s, operator theory has developed more significantly in matrix theory [12]-[13], differential equations [14]-[15], statistics [16], and many other mathematical branches.

Therefore, quantum mechanics [17], physics [17], and other fields have also been widely used. Operator inequality has become an essential part of operator theory. In 2002, the spectrum of self-adjoint operators was proposed [18]. Kewei Zhang determined the half-positive qualitative of -1 homogeneous accounting operators and then established the strengthened inner product inequality theory. Subsequent applications of the method led to enhancements in Hilbert-type inequalities (1.6) and (1.7) individually. As a result, the fusion of the Hilbert operator and half-discrete Hilbert-type inequality remarkably elevated the theoretical caliber of the latter. This advancement not only strengthened the practical utilization of Hilbert-type inequality but also served as a pivotal juncture for the development of half-discrete Hilbert-type inequalities.

Furthermore, a novel half-discrete Hilbert-type inequality, which encompasses the integration with a variable upper limit and partial sums, was derived in [19]. This groundbreaking inequality introduces a fresh perspective and expands the scope of the half-discrete Hilbert-type inequality. In light of the novelty and complexity of this inequality, researchers have started to investigate various applications, such as multiple upper limits integral and partial sums [19], higher-order derivative functions [20]-[21], and other related areas. As a result, these research directions have gained increasing attention and are expected to become a burgeoning research field in the future.

1.2 Important Periods of Hilbert-Type Inequalities

The development of Hilbert-type inequalities involves the study of generalizations and extensions of this inequality to various settings, including higher dimensions, integral inequalities, discrete inequalities, and inequalities involving other mathematical functions. Researchers have been investigating the conditions under which such inequalities hold, deriving new inequalities, and exploring applications in diverse branches of mathematics, including analysis, optimization, and mathematical physics [22]. Three important periods in the history of Hilbert-type inequality are summarized as follows:

The first period is from 1908 to 1934. In 1908, David Hilbert [5] initially established double series Hilbert-type inequality in his lectures on integral equations (1.6), although he did not provide an exact determination of the constant involved. A few years after Hilbert's demonstration, Issai Schur [8] presented a novel proof in (1.7), establishing that (1.6) remains valid with the optimal sharp constant of π. This revelation by Schur unveiled the true nature of the constant factor associated with inequality. This inequality does not have any parameter variables and belongs to -1 homogeneous special kernel inequality.

In 1925, the famous British mathematician Hardy and his research team introduced two conjugate indices (p,q), discussed the situation of $p>1, \frac{1}{p}+\frac{1}{q}=1$. The Hilbert inequality (1.6) is generalized as follows, for $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geq 0$ $(m,n \in \mathbf{N}=\{1,2,\cdots\})$, $0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, the following Hardy-Hilbert inequality [23] was obtained:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1.10}$$

with the best possible constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$. This work introduced a pair of conjugate indices, which is the extension of Hilbert-type inequality.

In 1934, the research results of more than 100 published papers in Hardy's monograph [9]. They replaced the kernel $\frac{1}{m+n}$ of inequality (1.10) with a homogeneous kernel of -1 homogeneous $k(m,n)$. Hardy used the best possible constant factor of generalized integral $k_p=\int_0^{\infty} k(u,1)u^{-\frac{1}{q}}du$ to replace $\frac{\pi}{\sin(\frac{\pi}{p})}$. He adeptly introduced inequality (1.10) and explored its equivalent formulation. Furthermore, Hardy et al. proposed the associated general-1 homogeneous kernel integral inequality, which encompasses the general equation (1.7) and encompasses various equivalent expressions and multiple generalized forms. The Hardy-type integral inequality incorporates the integration with a variable upper limit and highlights the optimal constant factor. The theory of inequalities associated with the -1 homogeneous kernel is referred to as the Hardy-Hilbert inequality theory.

However, many mathematicians found it challenging to generalize Hardy's theory. From the 1930s to the 1990s, these 60 years became the 'blank period' in the research of Hilbert-type inequality theory, and there was no substantial progress.

The second period is from 1991 to 2015. It is the golden age of the development of Hilbert-type inequality theory. In 1991, China's famous mathematician professor Lizhi Xu et al. [24]-[25] published two papers in core journals, which improved the Hilbert inequality type and Hardy-Hilbert inequality uncovered research. The establishment of the enhancement Hilbert inequality (1.6) by using the weight coefficient method and introducing the following weighted inequality was initiated:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \le \left(\sum_{n=1}^{\infty}\omega(n)a_n^2\sum_{n=1}^{\infty}\omega(n)b_n^2\right)^{\frac{1}{2}}. \tag{1.11}$$

The reinforcement form of the equation was obtained, and then mathematicians focused on the best possible constant factor in the strengthened inequality. After that, Bicheng Yang and Mingzhe Gao [26]-[27] enhanced Xu's weight coefficient through optimization techniques and employed the refined Euler-Maclaurin summation formula for improved calculations. The following strengthened version of inequality (1.11) was given:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} &< \left[\sum_{m=1}^{\infty}\left(\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{1-\gamma}{m^{\frac{1}{p}}}\right)a_m^p\right]^{\frac{1}{p}}\\ &\times\left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{1-\gamma}{n^{\frac{1}{q}}}\right)b_n^q\right]^{\frac{1}{q}}. \end{aligned}\tag{1.12}$$

By optimizing Xu's weight coefficient method, researchers have expanded the systematic study of Hilbert-type inequality after nearly twenty years of unremitting efforts. In 1998, Bicheng Yang [28]-[29] introduced independent parameters and the Beta function which improved the method of weight coefficient and obtained the following generalization of inequality (1.7):

For $\lambda > 0, 0 < \int_0^{\infty} x^{1-\lambda} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} y^{1-\lambda} g^2(y)dy < \infty$, then

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}\frac{f(x)f(y)}{(x+y)^{\lambda}}dxdy\\ &< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^{\infty}x^{1-\lambda}f^2(x)dx\int_0^{\infty}y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}},\end{aligned}\tag{1.13}$$

where, the $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is the best possible constant factor.

During this period, a notable development emerged with the introduction of independent parameters $\lambda > 0$, accompanied by these conjugate indices (p,q) and (r,s) in [30]. This introduction of independent parameters and conjugate indices represented a significant breakthrough in the field as follows:

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}\frac{f(x)f(y)}{x^{\lambda}+y^{\lambda}}dxdy\\ &< \frac{\pi}{\lambda\sin(\frac{\pi\lambda}{r})}\left[\int_0^{\infty}x^{p(1-\frac{\lambda}{r})}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^{\infty}y^{q(1-\frac{\lambda}{r})}g^q(x)dy\right]^{\frac{1}{p}}.\end{aligned}\tag{1.14}$$

A generalization of the Hilbert-type inequality, incorporating the best possible constant factors, was established. Additionally, the application of operator theory was introduced to express the inequality, describe the norm, and apply it to specific cases. This incorporation of operator theory enhanced the understanding and utilization of inequality in various scenarios. Building upon the aforementioned research, a significant advancement occurred after 1998 with the introduction of the Yang-Hilbert inequality, which encompassed Hilbert-type inequalities for the homogeneous kernel of general real numbers. This breakthrough expanded the scope and applicability of the Hilbert-type inequalities. From the early 1990s to the present, the structural characteristics of Yang-Hilbert inequalities have been explored in depth. The research methods involve weight function, summability theory, real analysis, and functional analysis [31]-[41]. In the Yang-Hilbert inequality theory of homogeneous kernel, a real number with 12 basic classes is constructed. Its general and multiple applications are extended, especially in the Riemann-Zeta function [42]-[51]. It provides a theoretical basis for Riemann's conjecture.

The third period is from 2016 to the present. The year 2016 witnessed the publication of the sufficient and necessary conditions, along with equivalent statements, concerning the optimal constant factor associated with the parameters in the inequality [52]. This research work establishes a scientific assurance for understanding the interplay between the best possible constant factors and the parameters involved. Therefore, a new research field was created (cf. [53]- [63]). During this period, the theory of Hilbert-type inequality was greatly developed, and a large number of research results emerged. The theoretical system of Hilbert-type inequalities was gradually established. Starting in 2020, the applications of the Hilbert-type inequality have expanded to encompass scenarios on multiple upper-limit function and the higher-order derivative function [64]-[78]. The latest research field expanded the application range of inequalities.

During the aforementioned research endeavors, a novel research domain known as the half-discrete Hilbert-type inequality emerged, signifying an extension of the Hilbert-type integral inequality and Hilbert-type discrete inequality. Numerous mathematicians have made notable advancements in the discovery and verification of various new half-discrete Hilbert-type inequalities. These inequalities have found practical applications in diverse fields, including mathematical physics, pure mathematics, and applied mathematics [1].

1.3 The Organization of This Book

In Chapter 2, a new half-discrete Mulholland-type inequality is presented. This new inequality involves a non-homogeneous kernel represented by $h(v(x)lnn)$, and the best possible constant factor is determined using weight functions and techniques from real analysis. Additionally, the chapter explores the reverse half-discrete Mulholland-type inequality, which also features the same kernel. Several specific cases are examined in detail.

In Chapter 3, the application of half-discrete Hilbert-type inequality involving one higher-order derivative function and the best possible constant factor is obtained, by using the weight functions and the technique of real analysis. The chapter also explores the equivalent statements of the best possible constant factor related to some parameters. It further presents the equivalent forms and operator expressions. Lastly, the chapter delves into the reverse of the half-discrete Hilbert-type inequality involving a higher-order derivative function, along with the equivalent forms and other types of reverses.

In Chapter 4, a new half-discrete Hilbert-type inequality is introduced. This inequality involves one multiple upper limit function and has a more general kernel compared to previous studies. Furthermore, the equivalent statement and operator expressions of the inequality are also examined. The special cases involving the beta function are discussed in detail. Lastly, the reverse half-discrete Hilbert-type inequality, which also involves one multiple upper limit function, is explored. The equivalent forms and other types of reversals are also investigated.

In Chapter 5, we derive a set of new reverse half-discrete Hilbert-type inequalities. These inequalities involve one partial sum and either a multiple upper limit function or a higher-order derivative function. We employ weight functions, the mid-value theorem, and techniques of real analysis to establish these results. Additionally, we investigated the equivalent expressions for the optimal constant factors about various parameters. Furthermore, as applications, the equivalent forms and some particular inequalities are provided.

In Chapter 6, some new reverse half-discrete Hilbert-type inequalities with two internal variables and one partial sums involving one upper limit function or one derivative function are obtained. The equivalent statements of the best possible constant factors related to several parameters are considered. As applications, some particular inequalities are provided.

In Chapter 7, some new half-discrete Hilbert-type inequalities with two internal variables involving one upper limit function or one derivative function are obtained, by using the weight functions, the mid-value theorem, and the techniques of real analysis. The equivalent statements of the best possible constant factors related to several parameters, the equivalent forms, and the operator expressions are considered. The reverses are also obtained and some particular inequalities are provided.

2 Half-Discrete Mulholland-Type Inequalities with a Internal Variable

In this chapter, a new half-discrete Mulholland-type inequality with the non-homogeneous kernel as $h(v(x)lnn)$ and the best possible constant factor is obtained, by using the weight functions and the techniques of real analysis. The equivalent forms, the operator expressions are considered. As corollaries, we deduce some new equivalent inequalities with the homogeneous kernel as $k_\lambda(v(x), lnn)$. Some new particular inequalities are obtained. Additionally, some reverses are considered.