1 Surface Integrals

1.1 Surface Integrals

1.1.1 Parameterized Surfaces

The surfaces can be described in a lot of ways. They can be consider as being graphics of real valued functions of two variables, this is can be considered as being sets of points (x, y, z) such that $z = f(x, y)$ and $(x, y) \in D$ where D is the domain of f. They can be defined as levels sets of valued real functions of three variables. But these definitions are not very good since a lot of surfaces can not be defined as graphic of functions of real valued functions of two variables. For instance, the spherical surface $x^2 + y^2 + z^2 = 1$ is not the graphic of a function of two variables. Therefore the first definition of surface is not very good. But the second definition is not also very good, since equations of the form $f(x,y,z)=0$ have solutions sets that are not surfaces. For instance, if we consider f such that $f(x,y,z)=x^2+y^2+z^2$ the level set of f of value 1 is the spherical surface centered at $(0,0,0)$ and radius 1, but the level set of f of value 0 is only the set $N_0(f)=\{(0,0,0)\}$. And, finally the definition of surface should allow to generalize the definition of surface of $\mathbb{R}^n$,when $n>3$. Therefore, we will present another definition of surface.

Definition 1.1.1. *{Parametrized Surface} Let D be a region of $\mathbb{R}^2$ such that $\overset{\circ}{D}\neq\emptyset$ that is a connected set of $\mathbb{R}^2$. A set S of points $\mathbb{R}^3$ is a parameterized surface of $\mathbb{R}^3$ if and only if $S=r(D)$ where $r:D\subset\mathbb{R}^2\mapsto\mathbb{R}^3$ is an injective continuous application on $\overset{\circ}{D}$, such that r^{-1} is an injective application on $r(\overset{\circ}{D})$. And we say that r is a parametrization of S.*

Definition 1.1.2. *{Regular parametrization of a parameterized surface } Let D be a set of $\mathbb{R}^2$ with $\overset{\circ}{D}\neq\emptyset$ and let S be a parameterized surface by the application r defined on D. One says that a parametrization r of S is a regular if and only if r is of class C^1 on $\overset{\circ}{D}$ and such that for any point $P=r(u,v)$ with $(u,v)\in\overset{\circ}{D}$ of S , $\frac{\partial r}{\partial u}(u,v)\times\frac{\partial r}{\partial v}(u,v)\neq 0$. And we say that S is a regular surface if $S=r(D)$ with r a regular parametrization of S and with D a set with non empty interior, or if $S=\cup_{i=1}^{n}S_i$ with*

$S_i = r(D_i)$ with D_i a connected set with non empty interior and r regular parametrization of S_i and $S_i \cap S_j = \emptyset$ and the tangent plane is defined on each point of S.

Observation 1.1.1. *The regular curves are curves that don't have corners. A regular surface is a surface such that at each of point of the surface is defined the tangent plane. In a simple language the regular surfaces are the surfaces that don't have edges and don't have corners.*

Definition 1.1.3. *We say that a surface S is a piecewise regular surface (or piecewise smooth surface) if and only if it can be divide on a finite number of regular surfaces, this is $S = \cup_{i=1}^{n} S_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$ and S_i for $i = 1, \cdots, n$ is a regular surface. We can say that planes, quadratic surfaces are regular surfaces. But cubes, prisms and parallelepiped are piecewise regular surfaces. In a easy way a surface is piecewise regular if it can be divided on pieces such that on each piece is a regular surface.*

I use these definition because I don't want to use a lot of mathematics.

Observation 1.1.2. *The condition $\frac{\partial r_i}{\partial u}(u_0, v_0) \times \frac{\partial r_i}{\partial v}(u_0, v_0) \neq 0$ for each $(u_0, v_0) \in D_i$ is called the regularity condition of r_i at the point (u_0, v_0). The points (x, y, z) of the surface S such that doesn't exist a parametrization r such that $r(u_0, v_0)) = (x, y, z)$ and $\frac{\partial r}{\partial u}(u_0, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0) \neq 0$ are* *called singular points of S. But we must have some care, a bad choice of a parametrization of a surface S can send us to the situation that at some points of the surface the conditions of the regularity can not be verified, even being the surface regular.*

Example 1.1.1. *The spherical surface S is a regular surface since $S = S_1 \cup S_2$ and $S_1 = r_1(D_1)$ and $S_2 = r_2(D_2)$ where r_1 and r_2 are such that $r_1(x,y) = (x,y,\sqrt{1-x^2-y^2})$ for all $(x,y) \in D_1$ with $D_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and we have $r_2(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2})$ that are regular parameterizations of S_1 and S_2 respectively, with $D_2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$, and $S_1 \cap S_2 = \emptyset$, and the tangent plane to each point of the surface S is well defined.*

Example 1.1.2. *Consider the surface S such that $S = r(\mathbb{R}^2)$ where r is defined by $r(u,v) = (x(u,v), y(u,v), z(u,v)) = (u,u,v)$ with $(u,v) \in \mathbb{R}^2$. Let $(x,y,z) \in S$. Let write $(x,y,z) = (u,u,v)$. Hence $y = x$ and $z \in \mathbb{R}$ and therefore S is formed by the points (x,y,z) such that $y = x$, this is the surface S is the vertical plane $y = x$.*

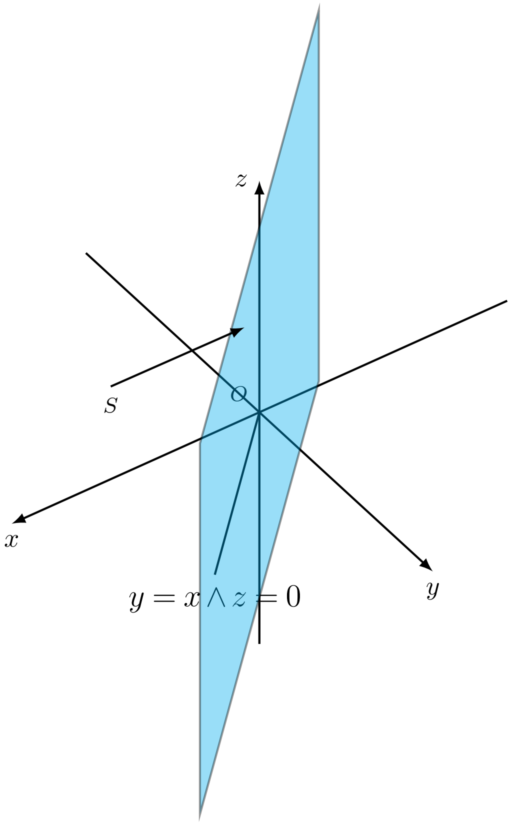


Figure 1.1: The surface S is the plane $y = x$

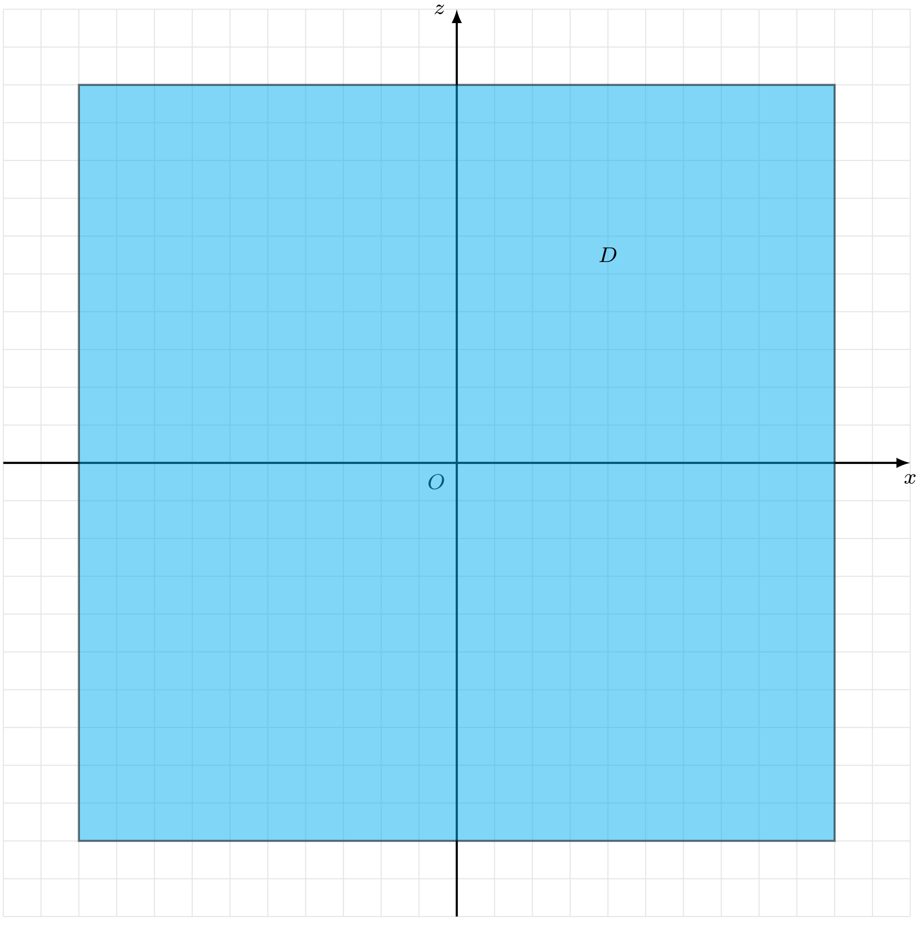


Figure 1.2: Domain of r

We must note that the application r that takes the set $D = \mathbb{R}^2$, the plane Oxz, on the plane $\mathbb{R}^3$ of equation $y = x$ is a regular parametrization of the plane. Evidently $S = r(D)$ and the set $D = \mathbb{R}^2$ is a connected set with nonempty interior. We must say that a connected set of $\mathbb{R}^2$ is connected if and only if is connected by arcs, this means if any two points of the set can be joined by a line inside the set. We must note that r such that

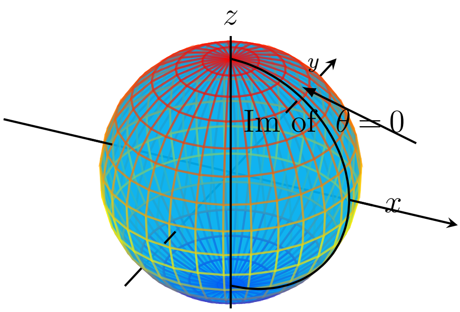
$r(u,v) = (u,u,v), \forall (u,v) \in \mathbb{R}^2$ *is a continuous injective function on* $\mathbb{R}^2$ *and that the inverse* r^{-1} *is such that* $r^{-1}(x,y,z) = (x,z), \forall (x,y,z) \in S$ *is continuous on* S *since* $r^{-1}(x,y,z) = (x,z)$ *is continuous on* $\mathbb{R}^3$.

Example 1.1.3. *Let* $D = [0,2\pi[\times]0,\pi[$ *and let consider the application* $r : D \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ *defined by* $r(\theta,\phi) = (a\cos(\theta)sen(\phi), asen(\theta)sen(\phi), a\cos(\phi))$ *with* $(\theta,\phi) \in D$.

Then

$$\begin{cases} x = a\cos(\theta)sen(\phi)) \\ y = asen(\theta)sen(\phi) \\ z = a\cos(\phi), 0 \leq \theta < 2\pi, 0 < \phi < \pi. \end{cases}$$

The parametric equations of the sphere $x^2+y^2+z^2 = 1$ *are the spherical coordinates with* $\rho = a$.



Then the parametrization r *such that*

$$r(\theta,\phi) = (a\cos(\theta)sen(\phi), asen(\theta)sen(\phi), a\cos(\phi)), (\theta,\phi) \in [0,2\pi[\times]0,\pi[$$

maps the region $D = [0, 2\pi[\times]0, \pi[$ *on the sphere of radius* a *centered at* $(0,0,0)$ *without the poles.*

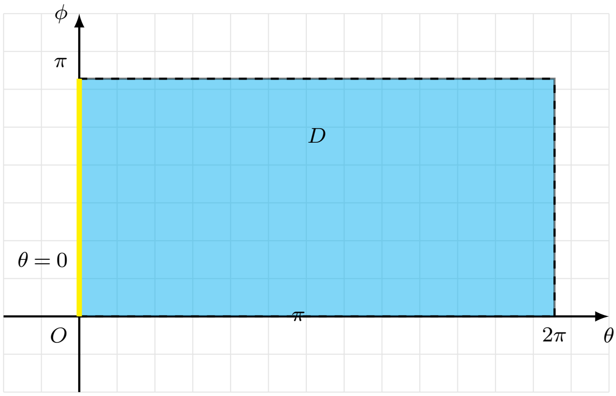


Figure 1.3: Domain of r

We most note also that the function r *is a regular parametrization of the sphere* S *without the poles, since* r *is a function of class* C^1 *on* $\overset{\circ}{D}$ *and since we have for* $(u,v) \in \overset{\circ}{D}$,

$$\frac{\partial \vec{r}}{\partial \theta}(\theta,\phi)\times\frac{\partial \vec{r}}{\partial \phi}(\theta,\phi)==\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -asen(\theta)sen(\phi) & a\cos(\theta)sen(\phi) & 0 \\ a\cos(\theta)\cos(\phi) & asen(\theta)\cos(\phi) & -asen(\phi)) \end{vmatrix}$$

$$= sen(\phi)\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -asen(\theta) & a\cos(\theta) & 0 \\ a\cos(\theta)\cos(\phi) & asen(\theta)\cos(\phi) & -asen(\phi)) \end{vmatrix}$$

$$= sen(\phi)\left(-a^2\cos(\theta)sen(\phi)\vec{i}-a^2sen(\theta)sen(\phi)\vec{j}\right)-$$

$$-a^2sen(\phi)\cos(\phi)\vec{k}.$$

From now on, we will use the following notation:

$$N(\theta,\phi)=\frac{\partial r}{\partial \theta}(\theta,\phi)\times\frac{\partial r}{\partial \phi}(\theta,\phi)$$

$$=-a^2sen(\phi)(\cos(\theta)sen(\phi),sen(\theta)sen(\phi),\cos(\phi)).$$

So, the parametrization r is a regular parametrization of the sphere without the poles.

Example 1.1.4. *Let consider the set $D=[0,2\pi[\times]-\infty,+\infty[$ and let r be the application that to the point $(\theta,z)\in D$ associates the point $r(\theta,z)=(a\cos(\theta),asen(\theta),z)$. Then the cylindrical surface S formed by the points $(x,y,z)\in\mathbb{R}^3$ such that $x^2+y^2=a^2$ verifies the equality $S=r(D)$.*

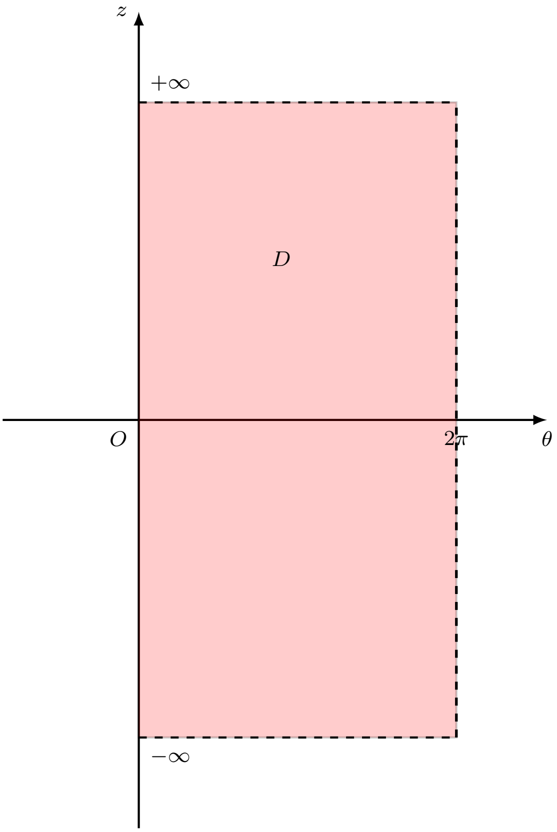


Figure 1.4: Domain of r

We present on figure 1.5 a graphical representation of the cylinder $x^2 + y^2 = a^2$.

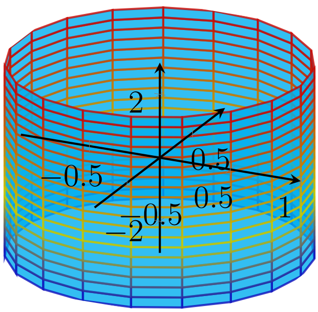


Figure 1.5: The cylinder $x^2 + y^2 = a^2$

We must note that r is an application of class C^1 on $\overset{\circ}{D}$.

And, we have $\frac{\partial r}{\partial \theta}(\theta,z) = (-asen(\theta), a\cos(\theta), 0)$ *and* $\frac{\partial r}{\partial \theta}(\theta,z) = (0,0,1)$.

And, therefore we write that

$$\frac{\partial \vec{r}}{\partial \theta}(\theta,z) \times \frac{\partial \vec{r}}{\partial z}(\theta,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -asen(\theta) & a\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= a\cos(\theta)\vec{i} + asen(\theta)\vec{j} + 0\vec{k}.$$

We conclude that

$$\frac{\partial r}{\partial \theta}(\theta,z) \times \frac{\partial r}{\partial \theta}(\theta,z) = (a\cos(\theta), asen(\theta), 0) \neq (0,0,0), \forall (\theta,z) \in \overset{\circ}{D}.$$

So, we can say that $S = r(D)$ is a regular surface.

Example 1.1.5. *Let $D \subset \mathbb{R}^2$ be an open connected set with non empty interior. Let $f : D \mapsto \mathbb{R}$ be a real valued function of two variables of class C^1 on D. Then $S = f(D)$ is a regular parameterized surface, since $S = r(D)$ with $r(x,y) = (x,y,f(x,y)), (x,y) \in D$ and since f is of class C^1 on D then r is a function of class on C^1 on D, and, we have $\frac{\partial r}{\partial x}(x,y) \times \frac{\partial r}{\partial y}(x,y) \neq 0$. Indeed, we have $\frac{\partial r}{\partial x}(x,y) = (1, 0, \frac{\partial f}{\partial x}(x,y))$ e*

$\frac{\partial r}{\partial y}(x,y) = (0,1,\frac{\partial f}{\partial y}(x,y))$, *hence*

$$\frac{\partial \vec{r}}{\partial x}(x,y) \times \frac{\partial \vec{r}}{\partial y}(x,y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(x,y) \\ 0 & 1 & \frac{\partial f}{\partial y}(x,y) \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 0 & \frac{\partial f}{\partial x}(x,y) \\ 1 & \frac{\partial f}{\partial y}(x,y) \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & \frac{\partial f}{\partial x}(x,y) \\ 0 & \frac{\partial f}{\partial y}(x,y) \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -\frac{\partial f}{\partial x}(x,y)\vec{i} - \frac{\partial f}{\partial y}(x,y)\vec{j} - 1\vec{k}.$$

So $\frac{\partial r}{\partial x}(x,y) \times \frac{\partial r}{\partial y}(x,y) \neq 0.$

Observation 1.1.3. *We must say that a lot of surfaces are nothing but graphics of real functions. For instance the surface S defined by the equation $x = y^2$ is the graphic of the function $f(y,z) = x(y,z) = y^2$. Therefore is natural to define the parametrization of S as being $r(y,z) = (y^2,y,z)$ and $D = \mathbb{R}^2$. For instance the paraboloid $z = x^2 + y^2$ is the graphic of the function f such that $f(x,y) = x^2 + y^2, \forall (x,y) \in \mathbb{R}^2$. Therefore it is natural to consider the parametrization r such that $r(x,y) = (x,y,x^2 + y^2), \forall (x,y) \in \mathbb{R}^2$ and we must consider the domain D of this parametrization as being $D = \mathbb{R}^2$. A lot of times the surfaces are nothing but parameterized regular surfaces when deleted from them the edges or the* *corner points. For instance the conic surface, S formed by the points $(x,y,z)\in\mathbb{R}^3$ such that $z=\sqrt{x^2+y^2}$ and $(x,y,z)\neq(0,0,0)$ is a parameterized regular surface, since considering the parametrization $r(x,y)=(x,y,\sqrt{x^2+y^2}),(x,y)\in\mathbb{R}^2\setminus(0,0)$ we obtain a regular parametrization of $S\setminus\{(0,0,0)\}$.*

But we must say that we could also obtain the parametrization of this surface by considering as parameters of the parametrization the coordinates r and θ of polar coordinates it is only necessary to make the composition of r with p where $r(x,y)=(x,y,\sqrt{x^2+y^2}),(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ and $p(r,\theta)=(r\cos(\theta),rsen(\theta)),(r,\theta)\in]0,+\infty[\times[0,\times2\pi[$ obtaining the parametrization $r_1=r\circ p$ with $D_1=]0,+\infty[\times[0,2\pi[$. And all, the parameterizations of the form $r(x,y)=(x,y,f(x,y)),(x,y)\in D$, or of the form $r(y,z)=(f(y,z),y,z),(y,z)\in D$ or of the form $r(x,z)=(x,f(x,z),z),(x,z)\in D$ can conduct us to other parameterizations more easy to work using the polares coordinates on the convenient planes if the domain D is a circle, or a circular sector circular,etc. In the calculation of a surface integral sometimes are used parameterizations that are nothing but the parametrization of the vector position of a point of the surface using cylindrical coordinates (for the case of cylindrical surfaces) ore using spherical coordinates for the spherical surfaces.

Now we will present the definition of coordinates curves of a surface S of $\mathbb{R}^3$.

1.1.2 Parameterized Coordinate Curve of a Surface at a Point of the Surface

Definition 1.1.4. *{Definition of coordinate curve of a surface}*

Let S be a regular parameterized surface such that $S = r(D)$ with r a regular parametrization of S defined by $r(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in D$ where D is connected set with non empty interior. Let $(u_0, v_0) \in \overset{\circ}{D}$. The curve C_v that is described by the position vector $r(u_0, v)$ when v takes values on an interval I_v such that $(u_0, v) \in D, \forall v \in I_v$ is called the v coordinate at (u_0, v_0) The curve C_u where u takes values on an interval I_u such that the position vector $r(u, v_0)$ belongs to $D, \forall u \in I_u$ is called the coordinate curve u at (u_0, v_0).

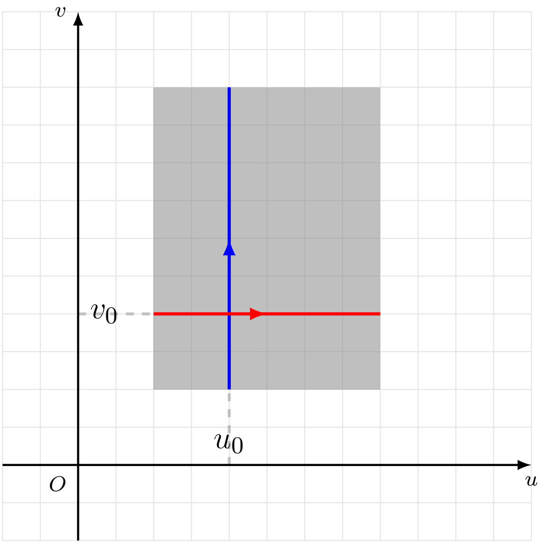
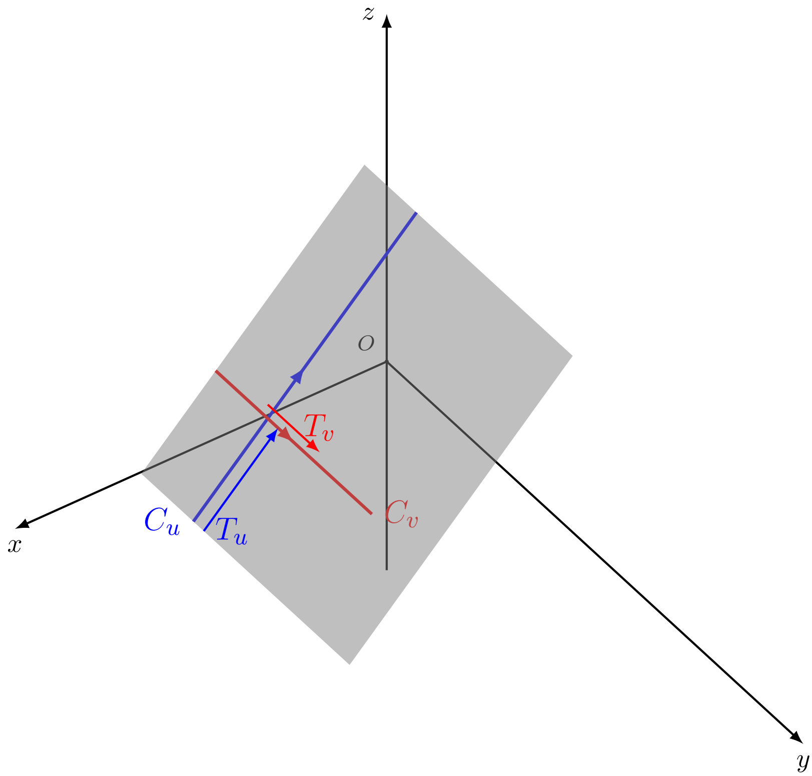


Figure 1.6: Coordinate Curves



Observation 1.1.4. *Let S be a regular parameterized surface such that*

$S = r(D)$ with $r(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in D$ where D is an open connected set with non empty interior. We will consider the following notation

$$T_u(u_0,v_0) = (\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0))$$

$$T_v(u_0,v_0) = (\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0))$$

with $T_v(u_0,v_0)$ the tangent vector to the curve C_v at the point $v = v_0$ parametrized by $r(u_0,v), v \in I_v$ and $T_u(u_0,v_0)$ the tangent vector C_u at the point $u = u_0$ parameterized by $r(u,v_0), u \in I_u$. Hence $T_u(u_0,v_0) = \frac{\partial r}{\partial u}(u_0,v_0)$ e $T_v(u_0,v_0) = \frac{\partial r}{\partial v}(u_0,v_0)$ are tangent vectors to the surface S at the point $r(u_0,v_0)$ and therefore the normal vector to S at the point $r(u_0,v_0)$ is

$$N(u_0,u_0) = \frac{\partial r}{\partial u}(u_0,v_0) \times \frac{\partial r}{\partial v}(u_0,v_0).$$

Sometimes throughout the text we will use also the notation $N(u_0,v_0) = T_u(u_0,v_0) \times T_v(u_0,v_0)$.