## Chapter 1. The Scalar Laplace PDE

### 1.1. SKEF Functions

A method of Decomposition in Invariant Structures (DIS) is introduced in Section 1.1 by solving the scalar Laplace Partial Differential Equation (PDE) through a Stationary Kinematic Euler (SKE) structure with complex parameters. The DIS method generalizes the Euler method of solving Ordinary Differential Equations (ODEs) through an exponential function with complex parameters, the method of separation of variables, and the method of series solutions. It is also shown in Section 1.2 that the exponential function itself is an invariant hyperbolic structure.

### 1.1.1. Mathematical Formulation

Find Stationary Kinematic Euler-Fourier (SKEF) functions as partial solutions to a global Laplace PDE

$$
\begin{equation*}
\frac{\partial^{2} h}{\partial x^{2}}+\frac{\partial^{2} h}{\partial z^{2}}=0 \tag{1.1}
\end{equation*}
$$

for a scalar variable $h_{n}(x, z, t)$, depending on two spatial variables $(x, z)$ and time $t$ through the SKE structure

$$
\begin{equation*}
h=H \exp (K x X+K z Z) \tag{1.2}
\end{equation*}
$$

where $H$ is a real coefficient,

$$
\begin{equation*}
[K x=L x+i M x, K z=L z+i M z] \tag{1.3}
\end{equation*}
$$

are complex parameters in the $x$ - and $z$-directions, $i$ is the imaginary unit, $[L x, L z]$ are real parts of $[K x, K z],[M x, M z]$ are imaginary parts of $[K x, K z]$,

$$
\begin{array}{ll}
{\left[X=x-x_{0}-V x\left(t-t_{0}\right),\right.} & \left.Z=z-z_{0}-V z\left(t-t_{0}\right)\right], \\
{[X=x-V x t+S x,} & Z=z-V z t+S z] \tag{1.5}
\end{array}
$$

are propagation variables in the mathematical (1.4) and computational (1.5) forms, $\left[x_{0}, z_{0}, t_{0}\right]$ are reference values of the Cartesian coordinates and time, $[V x, V z]$ are components of the propagation velocity of a harmonic wave, and

$$
\begin{equation*}
\left[S x=-x_{0}+V x t_{0}, \quad S z=-z_{0}+V z t_{0}\right] \tag{1.6}
\end{equation*}
$$

are spatiotemporal shifts in the $x$ - and $z$-directions, correspondingly. Use the superposition principle to find a fundamental SKEF structure of type $a$.

When $(x, z, t)=\left[x_{0}, z_{0}, t_{0}\right]$, then $(X, Z)=[0,0]$. These relations mean that the origin $[0,0]$ of the reference frame $(X, Z)$, moving with the wave velocity $[V x, V z]$, is located at the reference point $\left[x_{0}, z_{0}\right]$ of the laboratory frame $(x, z)$ at the reference moment $t=t_{0}$. If $(x, z, t)=[0,0,0]$, then $(X, Z)=[S x, S z]$. The latter relationships signify that the origin $[0,0]$ of the laboratory frame $(x, z)$ is located at the initial point $[S x, S z]$ of the moving frame
$(X, Z)$ at the initial moment $t=0$. Thus, the SKE structure (1.2)-(1.6) describes a two-dimensional (2-D) propagation of the harmonic wave with stationary parameters in two dimensions.

### 1.1.2. Reduction of the Laplace PDE to Characteristic AEs

We readily find first derivatives of the SKE structure $h$ with respect to $(x, z, t)$ as

$$
\begin{equation*}
\frac{\partial h}{\partial x}=K x h, \frac{\partial h}{\partial z}=K z h, \frac{\partial h}{\partial t}=-(K x V x+K z V z) h . \tag{1.7}
\end{equation*}
$$

Since spatial and temporal derivatives vary only in structural coefficients of $h$, the SKE structure $h$ is structurally invariant with respect to spatial and temporal differentiation of any order.

Second derivatives of $h$ in with respect to $x$ and $z$ become

$$
\begin{equation*}
\frac{\partial^{2} h}{\partial x^{2}}=K x^{2} h, \frac{\partial^{2} h}{\partial z^{2}}=K z^{2} h . \tag{1.8}
\end{equation*}
$$

Substitution of spatial derivatives (1.8) in (1.1) and collection of like terms reduce the scalar Laplace PDE to the Laplace Algebraic Equation (AE)

$$
\begin{equation*}
\left(K x^{2}+K z^{2}\right) h=0 . \tag{1.9}
\end{equation*}
$$

For the Laplace AE to be satisfied for all independent variables and structural parameters $[x, z, t, L x, L z, M x, M z, V x, V z, S x, S z]$, a structural coefficient of $h$ should vanish:

$$
\begin{equation*}
K x^{2}+K z^{2}=0 . \tag{1.10}
\end{equation*}
$$

Expressing structural parameters $[K x, K z]$ through the real and imaginary parts by (1.3), vanishing the real and imaginary parts, and multiplying the imaginary part by $1 / 2$ transform the characteristic AE in complex variables (1.10) to a characteristic system of AEs in real variables. The first AE describes length invariance of vectors $[L x, L z]$ and $[M x, M z]$ by

$$
\begin{equation*}
L x^{2}+L z^{2}-\left(M x^{2}+M z^{2}\right)=0 . \tag{1.11}
\end{equation*}
$$

The second AE

$$
\begin{equation*}
L x M x+L z M z=0 \tag{1.12}
\end{equation*}
$$

sets for these vectors the orthogonality condition.
To summarize, the DIS method enables to reduce the Laplace PDE through the Laplace AE and the characteristic AE in complex variables to the characteristic system in real variables [ $L x, L z$ ] and $[M x, M z]$.

### 1.1.3. Structural Parameters

Primarily, we separate variables of (1.11)-(1.12) using a polar coordinate system. Set polar presentations of $[L x, L z]$ and $[M x, M z]$ with the same radial distance $R z$ and various polar angles $\alpha$ and A as follows:

$$
\begin{align*}
& {[L x=R z \cos (\mathrm{~A}), \quad L z=R z \sin (\mathrm{~A})],}  \tag{1.13}\\
& {[M x=R z \cos (\alpha), M z=R z \sin (\alpha)] .} \tag{1.14}
\end{align*}
$$

Set substitutions, which are produced by the Pythagorean identities, by

$$
\begin{equation*}
\cos ^{2}(\mathrm{~A})=1-\sin ^{2}(\mathrm{~A}), \cos ^{2}(\alpha)=1-\sin ^{2}(\alpha) . \tag{1.15}
\end{equation*}
$$

Substitution of (1.13)-(1.15) in (1.11) confirms that the first characteristic AE is satisfied.

Secondly, we substitute (1.13)-(1.14) in the second characteristic AE, vanish a coefficient of $R z^{2}$, and combine terms by the trigonometric identity

$$
\begin{equation*}
\cos (\mathrm{A}) \cos (\alpha)+\sin (\mathrm{A}) \sin (\alpha)=\cos (\mathrm{A}-\alpha) \tag{1.16}
\end{equation*}
$$

to yield a trigonometric equation for polar angles

$$
\begin{equation*}
\cos (\mathrm{A}-\alpha)=0 \tag{1.17}
\end{equation*}
$$

A general solution of (1.17) is

$$
\begin{equation*}
\mathrm{A}=\alpha+\frac{\pi}{2}+\pi j \tag{1.18}
\end{equation*}
$$

where an integer $j=0, \pm 1, \pm 2, \ldots$.
Finally, substituting relationship (1.18) into (1.13) and evaluating the trigonometric functions give the following solution for the real parts $[L x, L z]$ :

$$
\begin{equation*}
\left[L x=-(-1)^{j} R z \sin (\alpha), L z=(-1)^{j} R z \cos (\alpha)\right] \tag{1.19}
\end{equation*}
$$

while the imaginary parts $[M x, M z]$, which are independent parameters, are defined by (1.14).

### 1.1.4. SKEF Functions

The Euler formula for expansion of an exponential function of a complex argument reads:

$$
\begin{equation*}
\exp (a+i b)=\exp (a)[\cos (b)+i \sin (b)] \tag{1.20}
\end{equation*}
$$

where $a$ and $b$ are real and imaginary parts, respectively. Substitution of (1.3), (1.19), (1.14) in (1.2) and use of (1.20) represent

$$
\begin{equation*}
h=H(f+i g) \tag{1.21}
\end{equation*}
$$

as a superposition of SKEF functions $[f, g](x, z, t)$ with structural coefficients $[H, i H]$, where $[f, g]$ are computed as follows:

$$
\begin{align*}
& f=e^{-(-1)^{j} R z[\sin (\alpha) X-\cos (\alpha) Z]} \cos (R z[\cos (\alpha) X+\sin (\alpha) Z]),  \tag{1.22}\\
& g=e^{-(-1)^{j} R Z[\sin (\alpha) X-\cos (\alpha) Z]} \sin (R z[\cos (\alpha) X+\sin (\alpha) Z]) . \tag{1.23}
\end{align*}
$$

In the general case, the polar angle $\alpha$ varies from a propagation angle $\phi$, which is defined through the propagation velocities $[V x, V z]$ by

$$
\begin{equation*}
\cos (\phi)=\frac{V x}{\sqrt{V x^{2}+V z^{2}}}, \sin (\phi)=\frac{V z}{\sqrt{V x^{2}+V z^{2}}} . \tag{1.24}
\end{equation*}
$$

We assume that the reference frame $(X, Z)$ moves along the $x$-axis. The polar axis then coincides with the $x$-axis of the Cartesian coordinate system, i.e. $\alpha=0, V z=0, S z=0, \phi=0$. The SKEF functions (1.22)-(1.23) then are reduced to

$$
\begin{equation*}
f=e^{(-1)^{j} R z z} \cos (R z X), g=e^{(-1)^{j} R z z} \sin (R z X) . \tag{1.25}
\end{equation*}
$$

For $j=1$ and $j=0$, the SKEF functions (1.25) match the eigenfunctions computed by the method of separation of variables [3]. They model a one-dimensional (1-D) propagation of a 2-D harmonic wave, which decays in the $z$-direction. Hence, the DIS method produces more general eigenfunctions (1.22)-(1.23) than the method of separation of variables.

### 1.1.5. The Fundamental SKEF Structure of Type a

We should satisfy an upper condition on vanishing at infinity,

$$
\begin{equation*}
\left.h\right|_{z=+\infty}=0, \tag{1.26}
\end{equation*}
$$

for an upper solution of the Laplace PDE (1.1) in an upper domain (see Figure 1.1)

$$
\begin{equation*}
U=\{x \in(-\infty,+\infty), z \in(0,+\infty)\} \tag{1.27}
\end{equation*}
$$

and a lower condition on vanishing at infinity,

$$
\begin{equation*}
\left.h\right|_{z=-\infty}=0, \tag{1.28}
\end{equation*}
$$

for a lower solution of (1.1) in a lower domain

$$
\begin{equation*}
L=\{x \in(-\infty,+\infty), z \in(-\infty, 0)\} . \tag{1.29}
\end{equation*}
$$

In (1.25), the sign index $j=1$ and $j=0$ for upper and lower harmonic waves, respectively.
Two laboratory frames for the upper and lower harmonic waves in $U$ and $L$ are shown in Figure 1.1. The harmonic internal waves propagate along the $x$-axis and decay along the $z$-axis. A generation area of the internal waves is shaded. There are different realizations of the generation area in various applications. For instance, the generation area may be implemented by a charged strip in electromagnetism, a source of heat in heat transfer, an area of surface waves in fluid dynamics, etc. Surface forcing of excitation, propagation, and interference of the internal waves via the generation area is modelled by the Dirichlet and Neumann conditions on the lower boundary $z=0$ of the upper domain $U$ and the upper boundary $z=0$ of the lower domain $L$. Problems with these boundary conditions are considered in Sections 1.4 and $\mathbf{1 . 5}$.

Using the superposition principle [3], we set up a global solution $h(x, z, t)$ of the global Laplace $\operatorname{PDE}(1.1)$ in $U$ and $L$ as a superposition of local solutions $h_{n}(x, z, t)$ by

$$
\begin{equation*}
h=\sum_{n=1}^{N} h_{n} \tag{1.30}
\end{equation*}
$$

where $N$ is a number of internal waves. The local solutions then obey a local Laplace PDE

$$
\begin{equation*}
\frac{\partial^{2} h_{n}}{\partial x^{2}}+\frac{\partial^{2} h_{n}}{\partial z^{2}}=0, \Delta h_{n}=0, \Delta=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial z^{2}}, \tag{1.31}
\end{equation*}
$$

where $\Delta$ is the Laplacian.
Define Stationary Kinematic Fourier (SKF) functions

$$
\begin{equation*}
c x_{n}=\cos \left(R z_{n} X_{n}\right), s x_{n}=\sin \left(R z_{n} X_{n}\right), \tag{1.32}
\end{equation*}
$$

and a Stationary Kinematic Euler (SKE) function $e z_{n}$ in real variables for $n=1,2, \ldots, N$ by

$$
\begin{equation*}
e z_{n}=\exp \left((-1)^{j} R z_{n} z\right), e z_{n}=\exp \left(Z_{n}\right), \tag{1.33}
\end{equation*}
$$

where the propagation variable $X_{n}$ and the decay variable $Z_{n}$ are

$$
\begin{equation*}
\left[X_{n}=x-V x_{n} t+S x_{n}, Z_{n}=(-1)^{j} R z_{n} z\right] . \tag{1.34}
\end{equation*}
$$

The expression of $e z_{n}$ through $z$ is required to take into account the conditions at infinity. The expression of $e Z_{n}$ in terms of $Z_{n}$ is used to process compatibility conditions both in $U$ and $L$ simultaneously.


Figure 1.1. Configuration of the upper domain $U$ and the lower domain $L$ of generation, propagation, and interference of internal waves in two dimensions.

We then construct a local solution $h_{n}$ both in the upper and lower domains using a fundamental SKEF structure of type $a$, which is defined as a superposition of the SKEF functions

$$
\begin{equation*}
\left[f_{n}=e z_{n} c x_{n}, g_{n}=e z_{n} s x_{n}\right] \tag{1.35}
\end{equation*}
$$

with structural coefficients $\left[F h_{n}, G h_{n}\right.$ ] as follows:

$$
\begin{equation*}
h_{n}=a h_{n}=\left(F h_{n} c x_{n}+G h_{n} s x_{n}\right) e z_{n} . \tag{1.36}
\end{equation*}
$$

The fundamental SKEF structure $a h_{n}$ satisfies the conditions on vanishing at infinity (1.26) and (1.28) since

$$
\begin{equation*}
\left.a h_{n}\right|_{z=-(-1) j_{\infty}}=0 . \tag{1.37}
\end{equation*}
$$

For waves propagating along the $y$-axis and decaying along the $z$-axis, the propagation coordinate $Y_{n}$, the SKF functions $\left[c y_{n}, s y_{n}\right.$ ], and the SKEF functions $\left[f_{n}, q_{n}\right]$ are

$$
\begin{gather*}
Y_{n}=y-V y_{n} t+S y_{n},  \tag{1.38}\\
c y_{n}=\cos \left(R z_{n} Y_{n}\right), s y_{n}=\sin \left(R z_{n} Y_{n}\right),  \tag{1.39}\\
{\left[f_{n}=e z_{n} c y_{n}, q_{n}=e z_{n} s y_{n}\right],} \tag{1.40}
\end{gather*}
$$

where the decay coordinate $Z_{n}$ is given by (1.34) and the SKE function by (1.33).
For waves propagating along the $x$-axis and decaying along the $y$-axis, the SKE function $e y_{n}$ and the decay variable $Y_{n}$ become

$$
\begin{equation*}
e y_{n}=\exp \left((-1)^{j} R y_{n} y\right), Y_{n}=(-1)^{j} R y_{n} y, e y_{n}=\exp \left(Y_{n}\right), \tag{1.41}
\end{equation*}
$$

while the propagation variable $X_{n}$ and the SKF functions $\left[c x_{n}, s x_{n}\right]$ are defined by (1.34) and (1.32), respectively, and the SKEF functions [ $f_{n}, h_{n}$ ] are

$$
\begin{equation*}
\left[f_{n}=e y_{n} c x_{n}, h_{n}=e y_{n} s x_{n}\right] . \tag{1.42}
\end{equation*}
$$

In notation of the SKF functions, a first symbol from the list $[c, s]$ stands for the trigonometric functions [cos, $\sin$ ], respectively, and a second symbol from the list $[x, y]$ signifies the direction of wave propagation. In notation of the SKE functions, the first symbol $e$ always denotes the exponential function exp and a second symbol from the list $[y, z]$ means the direction of wave decay.

The same approach is used for notation of structural parameters. A first symbol stands for the name of a structural parameter, for instance, $V$ for velocity, $S$ for shift, and $R$ for scale (radius). A second symbol from the list $[x, y, z]$ means the direction. Consequently, names of structural parameters are capitalized, like $V x_{n}, V y_{n}, S x_{n}, S y_{n}, R x_{n}, R y_{n}, R z_{n}$, but names of structural functions are written in lower case, similar to $c x_{n}, c y_{n}, s x_{n}, s y_{n}, e z_{n}, e y_{n}$.

### 1.2. Fundamental Scalar Solutions

The SKEF functions derived in Section 1.1 are used in Section 1.2 to construct a complete set of the fundamental SKEF structures, which constitute an orthogonal SKEF structural basis to the scalar Laplace PDE in two dimensions. Differential, integral, and algebraic properties of the SKEF structural basis are considered, as well.

### 1.2.1. Mathematical Formulation

For a harmonic variable $h_{n}(x, z, t)$, which is expandable in the invariant SKEF structures, show that there are two independent fundamental SKEF structures

$$
\begin{align*}
& a h_{n}=\left(F h_{n} c x_{n}+G h_{n} s x_{n}\right) e z_{n},  \tag{1.43}\\
& b h_{n}=\left(G h_{n} c x_{n}-F h_{n} s x_{n}\right) e z_{n}, \tag{1.44}
\end{align*}
$$

which constitute the orthogonal SKEF structural basis to the scalar Laplace PDE (1.31), where [ $F h_{n}, G h_{n}$ ] are fundamental SKEF coefficients, $\left[c x_{n}, s x_{n}, e z_{n}\right]$ are the SKF and SKE functions (1.32)-(1.34).

In notation of the fundamental SKEF coefficients, a first capitalized symbol from the list $[F, G]$ refers to the name of the SKF function (1.35) from the list $[f, g]$, correspondingly. A second symbol stands for the name of a harmonic variable, for instance, $h$ for a scalar field, $[u, w]$ for components of a vector field $\boldsymbol{u}$. In notation of the fundamental SKEF structures, a first symbol corresponds to the structure type (1.43)-(1.44) from the list $[a, b]$ and a second symbol stands for the name of a harmonic variable. To summarize, names of structural coefficients are capitalized, but names of harmonic variables and fundamental SKEF structures are written in lower case.

When $\left[F h_{n}, G h_{n}\right]$ are given numerically, the structural invariance allows for construction of any of two fundamental SKEF structures: $a h_{n}$ or $b h_{n}$, see examples and exercises provided in Section 1.6.

### 1.2.2. Derivatives of Structural Functions

In the current subsection, we calculate spatial and temporal derivatives of the first and second orders of the SKE, SKF, and SKEF functions.

Taking first and second spatial derivatives of the SKF functions with respect to $x$ yields

$$
\begin{array}{cl}
\frac{\partial c x_{n}}{\partial x}=-R z_{n} s x_{n}, & \frac{\partial s x_{n}}{\partial x}=+R z_{n} c x_{n}, \\
\frac{\partial^{2} c x_{n}}{\partial x^{2}}=-R z_{n}^{2} c x_{n}, & \frac{\partial^{2} s x_{n}}{\partial x^{2}}=-R z_{n}^{2} s x_{n} . \tag{1.46}
\end{array}
$$

The first derivatives are structurally covariant since they change the SKF functions [ $c x_{n}, s x_{n}$ ] to cofunctions $\left[s x_{n}, c x_{n}\right.$ ] and the second derivatives are structurally invariant as they do not

