The Classical Spin or Intrinsic Angular Momentum

1.1. Introduction

It is possible that in classical mechanics the concept of spin, in the sense of intrinsic angular momentum, has not been given the importance that, due to its unique character, it perhaps deserves. Nevertheless, there is more than sufficient evidence backing up the importance of taking spin into account, not only in quantum mechanics but also in classical mechanics. Its unique physical nature distinguishes it from what is known as orbital angular momentum, although both are represented as pseudovectors and have, from a mathematical perspective, many things in common.

If we study the hydrogen atom using quantum mechanics and without taking spin into account, the energy levels are represented by three quantum numbers: n, m and l. These energy levels would be, as we know, degenerate. And what breaks this degeneracy is the z-axis component of the electron spin, which could be either $\left(+1/2\right)\hbar$ or $\left(-1/2\right)\hbar$. So, in order to define the entire state, it is known that one must specify n, l, m_l and m_s, where m_l and m_s are respectively the projections of the orbital angular momentum, L, and the spin angular momentum, S, along the z-axis. Admittedly, the concept of orbital angular momentum for the electron has been recognized and put into practice since the beginning of quantum mechanics. This would be, for example, ever since it was determined that the hydrogen atom problem could not be solved using the concept of orbital angular momentum alone. At this time, it became necessary for Goudsmit and Uhlenbeck, basing their work on that of Wolfgang Pauli, to propose the presence of spin as the only explanation for the unusual experimental results obtained by Stern and Gerlach. This meant, however, that the presence of intrinsic quantum angular momentum (spin) does not come out of quantum mechanics. A hypothesis must be presented.

Schrödinger's equation alone is incapable of predicting the intrinsic angular momentum for the spin of an electron and is therefore incapable of predicting the results of experiments, like the one made by Stern and Gerlach. It is important to emphasize that, since the first advances in wave mechanics, the wave function, Ψ has been considered an indivisible entity. However, in trying to explain the Stern-Gerlach experiment using wave mechanics, this strength of the Ψ wave function disappeared

In order to explain the Stern-Gerlach experiment, it was necessary to propose two discrete quantum mechanical components of the same wave function, which were defined as such in order to explain the two well-defined bands that appeared during the experiment. An explanation for this unexpected fact was only first found when, in 1928, the English theoretical physicist Paul Adrien Maurice Dirac came up with the idea to combine quantum mechanics with the special relativity theory in order to obtain a relativistic wave equation. From that point on, elementary particles with $\frac{1}{2}$ spin, like the electron, can be described in a manner consistent with quantum mechanical principles and with the special relativity theory.

It became almost obligatory, from a mathematical point of view, for the wave function, Ψ, to lose its indivisible status and become a wave function that, for an electron with intrinsic angular momentum, is made up of two parts symbolically representable as a column vector:

$$\Psi = \begin{pmatrix} \phi_{1/2}(\boldsymbol{r},t) \\ \phi_{-1/2}(\boldsymbol{r},t) \end{pmatrix} \tag{1.1}$$

In actuality, a completely relativistic description of an electron, or any other particle of spin $\frac{1}{2}$ requires the use of a 4-element *vector matrix*, called a —four-spinor—, where two of these elements represent the antiparticle (the positron, in the case of the electron). However, throughout our analysis, the energies in play make the presence of antiparticles irrelevant, which is why 2-element column vectors, called —bispinors—, are used in this situation.

And so, it is necessary to introduce the relativistic wave equation in order to describe elementary particles with $\frac{1}{2}$ spin like, for example, the electron. This means to say that spin requires that special relativity be incorporated into quantum mechanics in order for it to be explained.

Once more we should emphasize that if, from a mathematical point of view, the orbital angular momentum and the intrinsic angular momentum (spin) are both pseudo-vectors, then, from a physical point of view, only one of them, the spin, is the object of a different type of interaction when torque is applied. Thus, since spin is an immutable characteristic of a particle, the orbital angular momentum for elementary particles, which generally is quantized, is not such a characteristic. Even more notable, as you will see in the following chapters, the spin of the particle can be interacted with in order to generate torques and make the particle nutate and precess. Nevertheless, the precession of the orbital angular momentum is a consequence of the precession of the trajectory plane of the spin (and of the particle). The interactions are never with the orbital angular momentum. They are always due to the torques interacting with the particle's spin. The magnitude of the spin vector remains constant and, as previously mentioned, the movements, aside from displacement, that can be described relative to an inertial reference frame, are: **nutation and precession.**

Let's define the movement of **nutation** as: that motion that the spin of the particle as a whole undergoes when following a circular path.[1] We may define the movement of **precession** as: that motion which the spin undergoes when the circumference, around which it nutates, rotates around one of its diameters. These definitions will be illustrated more clearly and in more detail in the following chapters.

Given that it has been necessary to resort to special relativity in order to explain the transcendental nature of spin in quantum mechanics, a contemporary theory with classical origins; we ask ourselves if it might make sense to extend the concept of spin, understood to be intrinsic angular momentum, into classical mechanics. This requires unifying those basic concepts that should be taken into account when developing expressions in both fields: the classical and the quantum.

With this idea in mind, we ask ourselves the following questions:

1) Does it make any sense to talk about intrinsic angular momentum (spin) in the

[1]Although a circular path has been used in the definition of nutation, this is not the generic case. However, it is the most common.