Open Journal of Statistics, 2023, 13, 941-954
https://www.scirp.org/journal/ojs
ISSN Online: 2161-7198
ISSN Print: 2161-718X

Combinations of Right Half-Plane Mappings and Vertical Strip Mappings Convex in the Vertical Direction

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How to cite this paper: Cai, Y.L., She, Z.R., Zeng, F.J. and Huang, L. (2023) Combinations of Right Half-Plane Mappings and Vertical Strip Mappings Convex in the Vertical Direction. *Open Journal of Statistics*, **13**, 941-954.
https://doi.org/10.4236/ojs.2023.136047

Received: November 29, 2023
Accepted: December 25, 2023
Published: December 28, 2023

Abstract

In this paper, we explore the linear combinations of right half-plane mappings and vertical strip mappings. We demonstrate that the combinations of these harmonic mappings are convex in the vertical direction provided they are locally univalent and sense-preserving. Furthermore, we extend this analysis to a more general case by setting specific conditions. Additionally, we take some common parameters such as $z, -z, e^{i\theta}z$ as the dilatation of these harmonic mappings, and prove the sufficient conditions that their combinations are locally univalent and convex in the vertical direction. Several examples are constructed by the Mathematica software to demonstrate our main results.

Keywords

Harmonic Mapping, Linear Combination, Cohn's Rules, Directional Convexity

1. Introduction

In-depth research into the properties of harmonic mappings is beneficial for addressing various problems encountered in the field of engineering. Generally, the linear combinations of harmonic mappings often fail to preserve the original characteristics. For instance, the linear combinations of two convex harmonic mappings may not necessarily be convex, in some cases, may not even be univalent. One can refer to the survey paper by Campbell in [1]. Therefore, a thorough investigation into the univalency and convexity properties of the combinations of some harmonic mappings becomes crucial.

A continuous complex-valued function $f = u + iv$ is said to be harmonic in the open unit disk $\mathbb{D}=\{z\in\mathbb{C},|z|<1\}$ if u and v are real-value harmonic function in $\mathbb{D}$. Such harmonic mappings can be written as $f=h+\overline{g}$ where h, g are both analytic in $\mathbb{D}$. The equation $\omega_f=g'/h'$ is called the dilatation of a harmonic mapping f. Lewy [2] has proved that the harmonic mapping f is locally univalent and sense-preserving in $\mathbb{D}$ if $|\omega_f(z)|<1$ for all $z\in\mathbb{D}$.

Let $S_{\mathcal{H}}$ denote the class of all locally univalent and sense-preserving harmonic mappings, and $S_{\mathcal{H}}^0$ be the subclass of $\mathcal{H}$ which normalized by the conditions $f(0)=f'_{\bar{z}}(0)=f_z(0)-1=0$.

A domain $\Omega\in\mathbb{C}$ is said to be convex in the direction φ ($0\le\varphi\le 2\pi$) if for all $\alpha\in\mathbb{C}$, the set $\Omega\cap\{\alpha+te^{i\varphi},t\in\mathbb{R}\}$ is either connected or empty. Specifically, a domain is convex in the vertical direction if for every line parallel to the imaginary axis has a connected intersection with Ω. A function $f(z)$ is called convex in the vertical direction if it maps $\mathbb{D}$ onto a domain Ω convex in the imaginary axis.

A common way to construct a new function is to take the linear combination of two functions with a real coefficient λ. Let $f_1=h_1+\overline{g}_1$ and $f_2=h_2+\overline{g}_2$ be two harmonic mappings, then $f_3=\lambda f_1+(1-\lambda)f_2$ ($0\le\lambda\le 1$) is called the linear combination of f_1 and f_2.

The following Lemma 1.1 called shear construction which was proposed by Clunie and Shile-Small in [3], can help us to verify the vertical convexity of harmonic mappings.

Lemma 1.1. A locally univalent and sense-preserving harmonic mapping $f=h+\overline{g}$ on $\mathbb{D}$ is univalent and maps $\mathbb{D}$ onto a domain convex in the direction of φ ($0\le\varphi\le 2\pi$) if and only if the analytic function $F=h-\mathrm{e}^{2i\varphi}g$ is univalent and maps $\mathbb{D}$ onto a domain convex in the direction of φ.

In particular, when the harmonic mapping maps the unit disk to convex along the horizontal direction, then the parameter $\gamma=0$, then it can be seen that both the analytic function $F=h-g$ and the harmonic mapping $f=h+\overline{g}$ are convex in the horizontal direction. Similarly, for the case that is convex in vertical direction $\gamma=k\pi+\pi/2$, then it can be obtained that the analytic function $F=h+g$ and the harmonic mapping $f=h+\overline{g}$ are both convex in the vertical direction. This lemma will help us construct harmonic mappings that are convex along some special directions, and extend them to arbitrary directions.

In order to judge the univalency of some harmonic mappings, we need to use the relevant Lemma 1.2 proposed by Rahman, Q. I. in [4], which is usually called Cohn's Rule. Converting the judgment of the univalency of the harmonic mappings into the problem of analyzing the distribution of the zeros of a polynomial function.

Lemma 1.2. (Cohn's Rule) Given a polynomial
$p(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$ of degree n, and let $p^*(z)$ satisfy the following equation

$$p^*(z)=z^n p\left(\frac{1}{\overline{z}}\right)=\overline{a}_n+\overline{a}_{n-1}z+\overline{a}_{n-2}z^2+\cdots+\overline{a}_0z^nA. \tag{1}$$

Assuming that $p(z)$ has r and s zeros located inside and on the unit disk $|z|=1$, respectively. If the coefficients in the polynomial satisfy $|a_0|<|a_n|$, then we can construct a function $p_1(z)$ as follows

$$p_1(z)=\frac{\overline{a}_n p(z)-a_0 p(z)}{z}. \tag{2}$$

Which is of degree $r-1$ and has $s-1$ number of zeros inside and on the unit disk $\mathbb{D}$.

The following result due to Hengarther [5] is useful in checking whether an analytic function is convex in the vertical direction.

Lemma 1.3. Suppose f is an analytic function and non-constant in $\mathbb{D}$. Then

$$\operatorname{Re}\left(F'\left(1-z^2\right)\right)\ge 0, z\in\mathbb{D}. \tag{3}$$

if and only if

1) It is univalent in $\mathbb{D}$;

2) It is convex in the vertical direction;

3) There exist sequences $\{z_n'\}$ and $\{z_n''\}$ converging to $z=1$ and $z=-1$, respectively, such that

$$\begin{aligned}\lim_{n\to\infty}\operatorname{Re}\{f(z_n')\}&=\sup_{|z|<1}\operatorname{Re}\{f(z)\},\\ \lim_{n\to\infty}\operatorname{Re}\{f(z_n'')\}&=\inf_{|z|<1}\operatorname{Re}\{f(z)\}.\end{aligned} \tag{4}$$

Linear combination of two functions is an important way to construct new harmonic mapping. However, some geometry properties may not exist after their combination [1]. In 2013, Wang, Z. G. [6] *et al.* derived several sufficient conditions of the linear combinations of harmonic univalent mappings to be univalent and convex in the horizontal direction. In 2016, Kumar, R. [7] *et al.* proposed a new family of univalent harmonic mappings which map the unit disk onto domains convex in the vertical direction, and he also identifies the conditions under which linear combinations of mappings from this family remain univalency and convex in the vertical direction. In 2018, Long, B. Y. [8] *et al.* considered the linear combination of two vertical strip mappings with various dilatation, and proved it is convex in the vertical direction. Zireh, A. [9] *et al.* proved the sufficient conditions for the linear combinations of two slanted half-plane harmonic mappings to be univalent and convex in an arbitrary direction of $-\gamma$. In recent years, Beig, S. [10] *et al.* have demonstrated that the linear combination of two different kinds of harmonic mappings is univalent and convex in a special direction γ, they have further generalized this result to more common cases by setting certain conditions.

In this paper, inspired by the research conducted in [6] [7] [8] [9] [10], we investigate the linear combinations $f_3=\lambda f_1+(1-\lambda)f_2$ $(0\le\lambda\le 1)$, where $f_1=h_1+\overline{g}_1$ represents right half-plane mappings and $f_2=h_2+\overline{g}_2$ represents vertical strip mappings. These harmonic mappings are sheared by the analytic functions respectively given by

$$F_1 = h_1 + g_1 = \frac{z}{1-z},$$
$$F_2 = h_2 + g_2 = \frac{1}{2i\sin\theta}\log\frac{1+z\mathrm{e}^{i\theta}}{1+z\mathrm{e}^{-i\theta}}. \tag{5}$$

In particular, we demonstrate that f_3 is univalent and convex in the vertical direction under certain conditions. Furthermore, by setting some coefficients such as $z, -z, \mathrm{e}^{i\theta}z$ be the dilatation of f_1 and f_2, we establish the sufficient conditions for their combination to be locally univalent and convex in the vertical direction.

2. Main Results

Theorem 2.1. For $j = 1,2$, let $h_j + \overline{g}_j \in S_{\mathcal{H}}$ and satisfy Equation (5). Then $f_3 = \lambda f_1 + (1-\lambda) f_2, (0 \le \lambda \le 1)$ is convex in the vertical direction for a real constant λ if it is locally univalent and sense-preserving.

Proof. Since

$$\begin{aligned} f_3 &= \lambda f_1 + (1-\lambda) f_2 \\ &= \lambda h_1 + (1-\lambda) h_2 + \overline{\lambda g_1 + (1-\lambda) g_2} \\ &= H + \overline{G}. \end{aligned} \tag{6}$$

We set

$$\begin{aligned} F &= H - \mathrm{e}^{2i\mu} G \\ &= \lambda h_1 + (1-\lambda) h_2 - \mathrm{e}^{2i\mu}\left(\lambda g_1 + (1-\lambda) g_2\right) \\ &= \lambda h_1 - \mathrm{e}^{2i\mu}\lambda g_1 + (1-\lambda) h_2 - \mathrm{e}^{2i\mu}(1-\lambda) g_2. \end{aligned} \tag{7}$$

By setting $\mu = \pi/2$ and differentiate the Equation (8), we get

$$\begin{aligned} F' &= H' + G' \\ &= \lambda h_1' + (1-\lambda) h_2' + \left(\lambda g_1' + (1-\lambda) g_2'\right) \\ &= \lambda\left(h_1' + g_1'\right) + (1-\lambda)\left(h_2' + g_2'\right). \end{aligned} \tag{8}$$

Substituting F' into $\operatorname{Re}\left(F'\left(1-z^2\right)\right)$, we can get

$$\begin{aligned} \operatorname{Re}\left(F'\left(1-z^2\right)\right) &= \lambda \operatorname{Re}\left(1-z^2\right)\left(h_1' + g_1'\right) + (1-\lambda)\operatorname{Re}\left(1-z^2\right)\left(h_2' + g_2'\right) \\ &= \lambda \operatorname{Re}\left(\frac{1+z}{1-z}\right) + (1-\lambda)\operatorname{Re}\left(\frac{1-z^2}{\left(1+z\mathrm{e}^{i\mu}\right)\left(1+z\mathrm{e}^{-i\mu}\right)}\right) \\ &= \lambda \operatorname{Re}(\varphi_1) + (1-\lambda)\operatorname{Re}(\varphi_2). \end{aligned} \tag{9}$$

It is obviously that $\operatorname{Re}(\varphi_1) > 0$ for all $z \in \mathbb{D}$, and we can easy to verify that $\varphi_2(0) = 1$, $\operatorname{Re}(\varphi_2(z)) = 0$ for $|z| = 1$. Therefore, by using the minimum principle for harmonic mappings with all $|z| \in \mathbb{D}$, we get $\operatorname{Re}\left(F'\left(1-z^2\right)\right) = \lambda \operatorname{Re}(\varphi_1) + (1-\lambda)\operatorname{Re}(\varphi_2) > 0$.

By applying Lemma 1.3, we can determine that analytic function $F = H - \mathrm{e}^{2i\mu} G$ is convex in the direction of $\mu = \pi/2$. According to Lemma 1.1, the harmonic mapping $f_3 = H + \overline{G}$ also convex in the vertical direction.

Theorem 2.2. For $j = 1,2$, let $f_j = h_j + \overline{g}_j \in S_{\mathcal{H}}$ and satisfy Equation (5).

Let $\omega_j = g_j'/h_j', (j=1,2)$ be the dilatation of these two harmonic mappings $f_j(j=1,2)$. Then the linear combination $f_3 = \lambda f_1 + (1-\lambda) f_2, (0 \le \lambda \le 1)$ is convex in the vertical direction for a real constant λ if one of the following conditions is met.

1) $\omega_1 = \omega_2$.

2) $\theta = \arccos\phi(z)$ and $\phi(z) = \dfrac{(1-\lambda)(1-z)^2(1+\omega_1) - \lambda(1+z)^2(1+\omega_2)}{\lambda z(1+\omega_2)}$.

Proof. In views of (6), we set $f_3 = \lambda f_1 + (1-\lambda) f_2 = H + \bar{G}$ where $H = \lambda h_1 + (1-\lambda) h_2$, $G = \lambda g_1 + (1-\lambda) g_2$. Let ω_3 be the dilatation of f_3, we get

$$\omega_3 = \frac{G'}{H'} = \frac{\lambda g_1' + (1-\lambda) g_2'}{\lambda h_1' + (1-\lambda) h_2'} = \frac{\lambda \omega_1 h_1' + (1-\lambda)\omega_2 h_2'}{\lambda h_1' + (1-\lambda) h_2'}. \tag{10}$$

Since $g_j' = \omega_j h_j', (1,2)$, the above equations give

$$\begin{aligned} h_1' &= \frac{1}{(1+\omega_1)(1-z)^2}, \\ h_2' &= \frac{1}{(1+\omega_2)\left(1+2z\cos\theta+z^2\right)}. \end{aligned} \tag{11}$$

Substituting h_1 and h_2 into ω_3, we get

$$\begin{aligned} \omega_3 &= \frac{\lambda\omega_1 \dfrac{1}{(1+\omega_1)(1-z)^2} + (1-\lambda)\omega_2 \dfrac{1}{(1+\omega_2)\left(1+2z\cos\theta+z^2\right)}}{\lambda \dfrac{1}{(1+\omega_1)(1-z)^2} + (1-\lambda) \dfrac{1}{(1+\omega_2)\left(1+2z\cos\theta+z^2\right)}} \\ &= \frac{\lambda\omega_1(1+\omega_2)\left(1+2z\cos\theta+z^2\right) + (1-\lambda)\omega_2(1+\omega_1)(1-z)^2}{\lambda(1+\omega_2)\left(1+2z\cos\theta+z^2\right) + (1-\lambda)(1+\omega_1)(1-z)^2} \end{aligned} \tag{12}$$

Let $\omega_1 = \omega_2$, we can simplify the Equation (12) to obtain $\omega_3 = \omega_1 = \omega_2$. This means that $|\omega_3| < 1$ and f_3 is locally univalent and sense-preserving. According to Theorem 2.1, we can conclude that f_3 is convex in the vertical direction. The proof of first condition is complete.

Next, we examine the second condition with $\theta = \arccos\phi(z)$, where $\phi(z)$ is defined by Equation (2) in Theorem 2.2. Substituting $\phi(z)$ into the equation $\theta = \arccos\phi(z)$, we have

$$\cos\theta = \frac{(1-\lambda)(1+\omega_1)(1-z)^2 - \lambda(1+\omega_2)\left(1+z^2\right)}{2\lambda z(1+\omega_2)}. \tag{13}$$

After simplify, we can get

$$1+2z\cos\theta+z^2 = \frac{(1-\lambda)(1-z)^2(1+\omega_1)}{\lambda(1+\omega_2)}. \tag{14}$$

Combining formulas (14) and (12), we can obtain

$$
\begin{aligned}
\omega_3 &= \frac{\lambda\omega_1(1+\omega_2)(1+2z\cos\theta+z^2)+(1-\lambda)\omega_2(1+\omega_1)(1-z)^2}{\lambda(1+\omega_2)(1+2z\cos\theta+z^2)+(1-\lambda)(1+\omega_1)(1-z)^2} \\
&= \frac{\omega_1+\omega_2}{2}.
\end{aligned}
\tag{15}
$$

Therefore, $|\omega_3| = |(\omega_1+\omega_2)/2| < 1$, which indicate that f_3 is locally univalent and sense-preserving. By applying Theorem 2.1, we know that f_3 is convex in the vertical direction. This concludes the proof of the second condition.

Theorem 2.3. For $j = 1,2$, let $f_j = h_j + \overline{g}_j \in S_{\mathcal{H}}$ and satisfy Equation (5). Let $\omega_1 = z$ and $\omega_2 = -z$ be the dilatation of f_1 and f_2 respectively. Then, for a real constant λ, ($0 \le \lambda \le 1$), the linear combination $f_3 = \lambda f_1 + (1-\lambda) f_2$ is convex in the vertical direction.

Proof. Let ω_3 be the dilatation of f_3 which satisfy the Equation (10). By setting $\omega_1 = z$, $\omega_2 = -z$, and substituting them into Equation (11), we obtain

$$
\begin{aligned}
h_1' &= \frac{1}{(1+\omega_1)(1-z)^2} = \frac{1}{(1+z)(1-z)^2}, \\
h_2' &= \frac{1}{(1+\omega_2)(1+2z\cos\theta+z^2)} = \frac{1}{(1-z)(1+2z\cos\theta+z^2)}.
\end{aligned}
\tag{16}
$$

Then, by substituting these into ω_3 from Equation (10), we can derive

$$
\begin{aligned}
\omega_3 &= \frac{\lambda z(1-z)(1+2z\cos\theta+z^2)+(1-\lambda)(-z)(1+z)(1-z)^2}{\lambda(1-z)(1+2z\cos\theta+z^2)+(1-\lambda)(1+z)(1-z)^2} \\
&= -z\frac{-z^3+z^2(-1+2\lambda\cos\theta)+(1-2\lambda\cos\theta)+(1-2\lambda)}{1+z(-1+2\lambda\cos\theta)+z^2(-1+2\lambda-2\lambda\cos\theta)+z^3(1-2\lambda)} \\
&= -z\frac{a_3z^3+a_2z^2+a_1z+a_0}{a_0z^3+a_1z^2+a_2z+a_3} \\
&= -z\frac{r(z)}{r^*(z)}
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
r(z) &= -z^3+z^2(-1+2\lambda\cos\theta)+(1-2\lambda\cos\theta)+(1-2\lambda), \\
r^*(z) &= 1+z(-1+2\lambda\cos\theta)+z^2(-1+2\lambda-2\lambda\cos\theta)+z^3(1-2\lambda).
\end{aligned}
\tag{18}
$$

And $r^*(z) = z^3\overline{r(1/\overline{z})}$, which satisfy Lemma 1.2. Therefore, let z_0 be a zero of $r(z)$ and $z_0 \ne 0$, implying $1/\overline{z}$ is a zero of $r^*(z)$. With this, we can re-write (17) as follow

$$
\omega_3(z) = \frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)}.
\tag{19}
$$

It is evident that $|a_0| = |1-2\lambda| < |a_3| = 1$ for $\lambda \in (0,1)$. Thus, we can apply Cohn's Rule to $r(z)$, and conclude that all zeros of $r_1(z)$ lie inside or on $|z| = 1$. Consequently, we have

$$
r_1(z) = \frac{a_3r(z)-a_0r^*z}{z} = a_2z^2+a_1z+a_0
\tag{20}
$$

where

$$\begin{aligned} a_2 &= 4\lambda(1-\lambda), \\ a_1 &= -1+2\lambda\cos\theta+(-1+2\lambda)(-1+2\lambda-2\lambda\cos\theta), \\ a_0 &= 4\lambda(\lambda-1)\cos\theta. \end{aligned} \tag{21}$$

By simple calculate, we get

$$|a_0| = |4\lambda(\lambda-1)\cos\theta| < |a_2| = |4\lambda(\lambda-1)|. \tag{22}$$

which satisfies the condition of Lemma 1.2. Therefore, we can apply Cohn's Rule again to reduce the quadratic polynomial to once and judge the size of the function root.

By setting $r_1^*(z) = z^2 r_1(1/\overline{z})$, substituting it into the Equation (2), we have

$$r_2(z) = \frac{a_2 r_1(z) - a_0 r_1^* z}{z} = a_1 z + a_0, \tag{23}$$

where

$$\begin{aligned} a_1 &= \lambda^2\left(16-16\cos^2\theta\right)(1-\lambda)^2, \\ a_0 &= -\lambda^2\left(16-16\cos^2\theta\right)(1-\lambda)^2. \end{aligned} \tag{24}$$

It is obviously that the coefficients given above satisfy $|a_1| = |a_0|$. Therefore, the unique root of $r_2(z)$ from Equation (23) lies on the unit circle. According to Lemma 1.2, we know that the dilatation of harmonic mapping f_3 satisfies $|\omega_3| < 1$. Moreover, by Theorem 2.1, we obtain that the linear combination of harmonic mapping $f_3 = \lambda f_1 + (1-\lambda) f_2$ is locally univalent and convex in the vertical direction. Thus, the proof of Theorem 2.3 is completed.

Theorem 2.4. For $j = 1,2$, let $f_j = h_j + \overline{g}_j \in S_{\mathcal{H}}$ and satisfy Equation (5). Considering some special dilatations of two harmonic mappings f_1 and f_2, by setting $\omega_1 = -z$ and $\omega_2 = z$ respectively. Then, for a real constant λ, the linear combination $f_3 = \lambda f_1 + (1-\lambda) f_2$ is convex in the vertical direction if the following inequation holds true

$$\left|\lambda^2(-8-4\cos\theta) + \lambda(8+2\cos\theta)\right| \le \left|4\lambda - 4\lambda^2\right|. \tag{24}$$

Proof. Let ω_3 be the dilatation of f_3 which satisfies the Equation (10). By setting $\omega_1 = -z$, $\omega_2 = z$, and substituting them into Equation (11), we obtain

$$\begin{aligned} h_1' &= \frac{1}{(1+\omega_1)(1-z)^2} = \frac{1}{(1-z)^3}, \\ h_2' &= \frac{1}{(1+\omega_2)\left(1+2z\cos\theta+z^2\right)} = \frac{1}{(1+z)\left(1+2z\cos\theta+z^2\right)}. \end{aligned} \tag{25}$$

Then, by substituting these into ω_3 from Equation (10), we can derive

$$\begin{aligned} \omega_3 &= \frac{\lambda(-z)(1+z)\left(1+2z\cos\theta+z^2\right)+(1-\lambda)z(1-z)^3}{\lambda(1+z)\left(1+2z\cos\theta+z^2\right)+(1-\lambda)(1-z)^3} \\ &= -z\frac{-z^3+z^2(3-4\lambda-t\cos\theta)+(-3+2\lambda-t\cos\theta)+(1-2\lambda)}{-1+z(3-4\lambda-\lambda\cos\theta)+(-3+2\lambda-\lambda\cos\theta)z^2+(1-2\lambda)z^3} \end{aligned}$$

$$\begin{aligned} &= -z\frac{a_3z^3 + a_2z^2 + a_1z + a_0}{a_0z^3 + a_1z^2 + a_2z + a_3} \\ &= -z\frac{r(z)}{r^*(z)}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} &r(z) = -z^3 + z^2(3 - 4\lambda - \lambda\cos\theta) + (-3 + 2\lambda - \lambda\cos\theta) + (1 - 2\lambda), \\ &r^*(z) = -1 + z(3 - 4\lambda - \lambda\cos\theta) + (-3 + 2\lambda - \lambda\cos\theta)z^2 + (1 - 2\lambda)z^3. \end{aligned} \tag{27}$$

Similarly, the above Equation (26) satisfies $r^*(z) = z^3 r(1/\overline{z})$ and the conditions of Lemma 1.2 mentioned earlier. Therefore, let z_0 be the zero of $r(z)$, and $1/\overline{z}$ be the zero of $r^*(z)$, and we express ω_3 in form of education (19) again. By simple calculations, we have

$$|a_0| = |-1 + 2\lambda| \le 1 = |a_3|, \tag{28}$$

which satisfies the condition of Lemma 1.2. Thus, we apply the Cohn's Rule and get

$$r_1(z) = \frac{a_3 r(z) - a_0 r^* z}{z} = a_2z^2 + a_1z + a_0, \tag{29}$$

where

$$\begin{aligned} a_2 &= 1 + (1 - 2\lambda)(-1 + 2\lambda), \\ a_1 &= -3 + 4\lambda + 2\lambda\cos\theta + (1 - 2\lambda)(3 - 2\lambda + 2\lambda\cos\theta), \\ a_0 &= (1 - 2\lambda)(-3 + 4\lambda + 2\lambda\cos\theta) + 3 - 2\lambda. \end{aligned} \tag{30}$$

Simplify the above equation, we have

$$\begin{aligned} &|a_2| = |1 + (1 - 2\lambda)(-1 + 2\lambda)| = \left|1 - (1 - 2\lambda)^2\right| = \left|4\lambda - 4\lambda^2\right|, \\ &|a_0| = \left|\lambda^2(-8 - 4\cos\theta) + \lambda(8 + 2\cos\theta)\right|. \end{aligned} \tag{31}$$

Since the condition $|a_0| \le |a_2|$ is met, we can use Cohn's Rule again and get

$$r_2(z) = \frac{a_2 r_1(z) - a_0 r_1^* z}{z} = a_1z + a_0. \tag{32}$$

where

$$\begin{aligned} a_1 &= \lambda^2\left(-48 - 64\cos\theta - 16\cos^2\theta\right)\left(1 - 2\lambda + \lambda^2\right), \\ a_0 &= \lambda^2\left(16 - 16\cos^2\theta\right)\left(1 + \lambda^2 - 2\lambda\right). \end{aligned} \tag{33}$$

For $\lambda \in (0,1), \theta \in (0, 2\pi)$, we have

$$\begin{aligned} |a_0| &= \left|\lambda^2\left(16 - 16\cos^2\theta\right)\left(1 + \lambda^2 - 2\lambda\right)\right| \\ &= \left|16\lambda^2\left(1 - 2\lambda + \lambda^2\right)(1 + \cos\theta)(1 - \cos\theta)\right| \\ &\le \left|16\lambda^2\left(1 - 2\lambda + \lambda^2\right)(1 + \cos\theta)(3 + \cos\theta)\right| \\ &= \left|\lambda^2\left(-48 - 64\cos\theta - 16\cos^2\theta\right)\left(1 - 2\lambda + \lambda^2\right)\right| \\ &= |a_1|. \end{aligned} \tag{34}$$

Thus, the unique zero of the above equation will lie inside or on the unit disk $|z|=1$ if the inequality (24) is met. According to Lemma 1.2 and Theorem 2.1, we know that harmonic mapping $f_3(z)$ is locally univalent and convex in the vertical direction. Thus, the proof of Theorem 2.4 is complete.

In recent years, Long B Y [8] *et al.* introduced a variable dilatation $e^{i\theta}$ for harmonic mappings. Therefore, following their setup, in the theorem below, we similarly make a transformation to the dilatation of harmonic mappings.

Theorem 2.5. For $j=1,2$, let $f_j=h_j+\overline{g}_j\in S_{\mathcal{H}}$ and satisfy Equation (5). Let $\omega_1=z$ and $\omega_2=e^{i\theta}z,\theta\in(0,2\pi)$ be the dilatations of two harmonic mappings f_1 and f_2 respectively. Then, for a real constant λ, the linear combination $f_3=\lambda f_1+(1-\lambda)f_2$ is convex in the vertical direction if

$$\begin{aligned}&|a_2|>|a_2^*|,\\&|a_1|>|a_1^*|.\end{aligned}\tag{35}$$

where

$$\begin{aligned}&a_2=1+\left(-1+\lambda-e^{-i\theta}\lambda\right)\left(1-\lambda+e^{i\theta}\lambda\right),\\&a_2^*=(-1+2\lambda)+2e^{-i\theta}\lambda\cos\theta+\left(-1+\lambda-e^{-i\theta}\lambda\right)\left(-1+e^{i\theta}\lambda+2\lambda\cos\theta\right).\end{aligned}\tag{36}$$

and

$$\begin{aligned}a_1=&\Big(e^{-2i\theta}\lambda^2\Big(\left(-1+e^{i\theta}\right)^4(-1+\lambda)^2+\Big(1+e^{i\theta}\left(-1+e^{i\theta}\right)(-1+\lambda)\\&+2\left(-1+e^{i\theta}\right)(-1+\lambda)\cos\theta\Big)\Big(1-\lambda-e^{i\theta}(-1+\lambda)(-1+2\cos\theta)\\&+e^{2i\theta}\left(-1+2(-1+\lambda)\cos\theta\right)\Big)\Big)\Big),\\a_1^*=&e^{-2i\theta}\lambda^2\Big(e^{i\theta}\Big(6-2e^{2i\theta}(-1+\lambda)^2+2e^{3i\theta}(-1+\lambda)^2-7\lambda+2\lambda^2\\&+e^{i\theta}\left(-5+(7-2\lambda)\lambda\right)\Big)+2\Big(1+e^{i\theta}\left(e^{i\theta}(4-3\lambda)+2(-1+\lambda)-\lambda\right)\cos\theta\\&+2\left(-1+e^{i\theta}\right)^2(-1+\lambda)^2\cos2\theta\Big)\Big).\end{aligned}\tag{37}$$

Proof. Let ω_3 be the dilatation of f_3 which satisfies the Equation (10). By setting $\omega_1=z$, $\omega_2=e^{i\theta}z$, substituting them into Equation (11), we have

$$\begin{aligned}\omega_3&=\frac{\lambda z\left(1+e^{i\theta}z\right)\left(1+2z\cos\theta+z^2\right)+(1-\lambda)e^{i\theta}z(1-z)^2(1+z)}{\lambda\left(1+e^{i\theta}z\right)\left(1+2z\cos\theta+z^2\right)+(1-\lambda)(1-z)^2(1+z)}\\&=e^{i\theta}z\frac{a_3z^3+a_2z^2+a_1z+a_0}{a_0z^3+a_1z^2+a_2z+a_3}\\&=e^{i\theta}z\frac{r_1(z)}{r_1^*(z)},\end{aligned}\tag{38}$$

where

$$\begin{aligned}&r_1(z)=z^3+z^2\left(-1+\frac{\lambda}{e^{i\theta}}+\lambda+2\lambda\cos\theta\right)+z\left(-1+2\lambda+\frac{2\lambda\cos\theta}{e^{i\theta}}\right)+\left(1+\frac{\lambda}{e^{i\theta}}-\lambda\right),\\&r_1^*(z)=z^3\left(1-\lambda+\lambda e^{i\theta}\right)+z^2\left(-1+2\lambda+2\lambda e^{i\theta}\cos\theta\right)-z\left(1-\lambda\left(1+e^{i\theta}+2\cos\theta\right)\right)-1.\end{aligned}\tag{39}$$

It can be easily calculated that the following inequality

$$|a_0| = \left|1 + \frac{\lambda}{e^{i\theta}} - \lambda\right| < |a_3| = 1. \tag{40}$$

consistently holds for $\lambda \in (0,1)$. Therefore, we can use the Cohn's Rule and get

$$r_2(z) = \frac{\bar{a}_3 r_1(z) - a_0 r_1^*(z)}{z} = a_2 z^2 + a_1 z + a_0, \tag{41}$$

where

$$\begin{aligned} a_2 &= 1 + \left(-1 + \lambda - e^{-i\theta}\lambda\right)\left(1 - \lambda + e^{i\theta}\lambda\right), \\ a_2^* &= (-1 + 2\lambda) + 2e^{-i\theta}\lambda\cos\theta + \left(-1 + \lambda - e^{-i\theta}\lambda\right)\left(-1 + e^{i\theta}\lambda + 2\lambda\cos\theta\right). \end{aligned} \tag{42}$$

Assuming the inequality $|a_2| > |a_2^*|$ holds, we can use the Cohn's Rule again and get

$$r_3(z) = \frac{\bar{a}_3 r_2(z) - a_0 r_2^*(z)}{z} \tag{43}$$

After simplification, we have

$$r_3(z) = a_1 z + a_0, \tag{44}$$

where

$$\begin{aligned} a_1 &= \Big(e^{-2i\theta}\lambda^2\Big(\left(-1 + e^{i\theta}\right)^4(-1+\lambda)^2 + \Big(1 + e^{i\theta}\left(-1 + e^{i\theta}\right)(-1+\lambda) \\ &\quad + 2\left(-1 + e^{i\theta}\right)(-1+\lambda)\cos\theta\Big)\Big(1 - \lambda - e^{i\theta}(-1+\lambda)(-1+2\cos\theta) \\ &\quad + e^{2i\theta}\left(-1 + 2(-1+\lambda)\cos\theta\right)\Big)\Big)\Big), \\ a_1^* &= e^{-2i\theta}\lambda^2\Big(e^{i\theta}\Big(6 - 2e^{2i\theta}(-1+\lambda)^2 + 2e^{3i\theta}(-1+\lambda)^2 - 7\lambda + 2\lambda^2 \\ &\quad + e^{i\theta}\left(-5 + (7 - 2\lambda)\lambda\right)\Big) + 2\Big(1 + e^{i\theta}\left(e^{i\theta}(4 - 3\lambda) + 2(-1+\lambda) - \lambda\right)\cos\theta \\ &\quad + 2\left(-1 + e^{i\theta}\right)^2(-1+\lambda)^2\cos 2\theta\Big)\Big). \end{aligned} \tag{45}$$

Since $|a_1| > |a_1^*|$, the zeros of $r_2(z)$ and $r_2(z)$ both lie inside or on the unit disk. Therefore, $|\omega_3| < 1$. By Theorem 2.1, we know that the harmonic mapping f_3 is locally univalent and convex in the vertical direction. Thus, all the proofs in this article have been completed.

3. Examples

In this section, we give several examples to illustrate our main results.

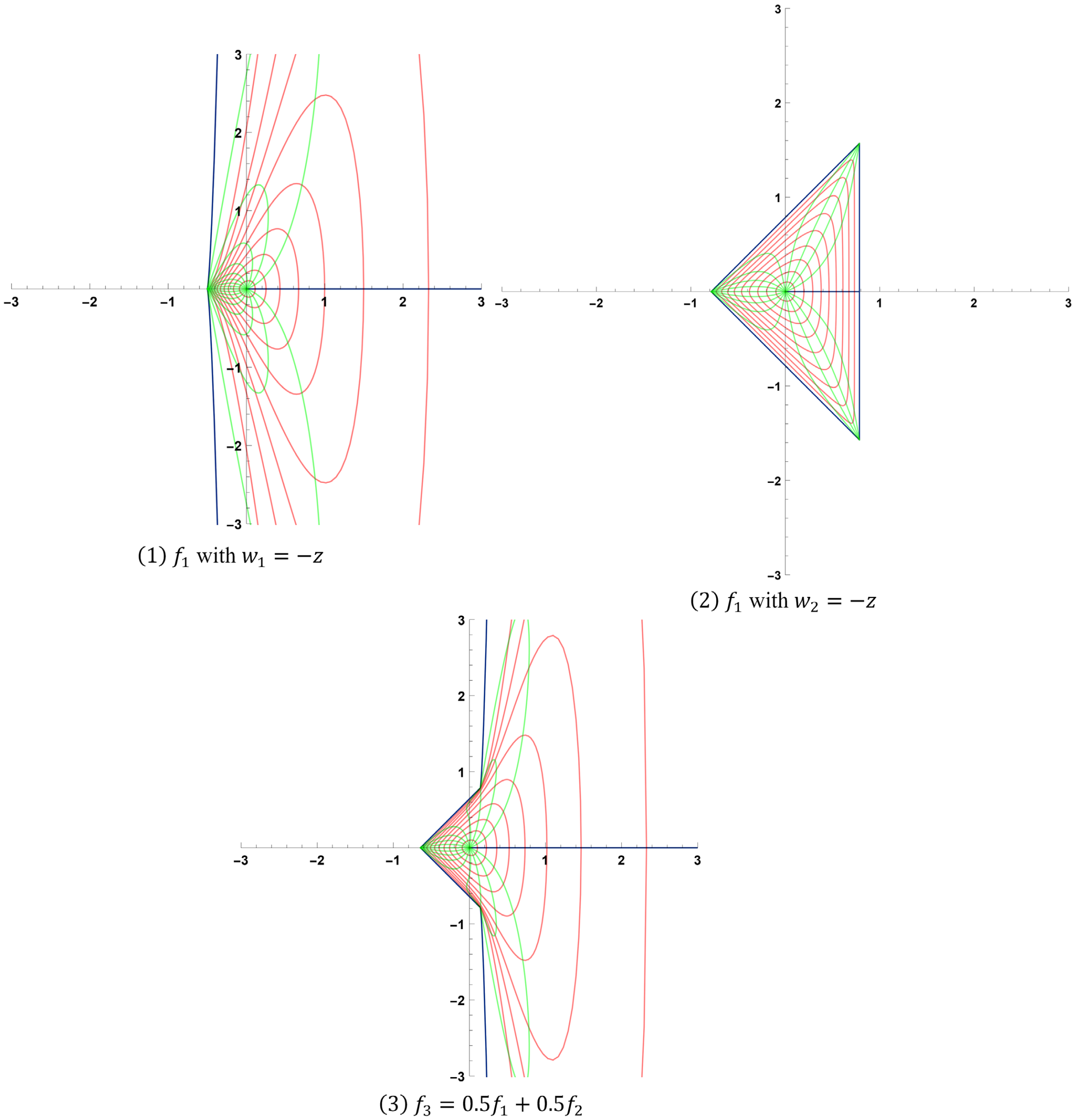
Example 1. Let $f_j = h_j + \bar{g}_j \in S_H, (j = 1,2)$ and satisfy (5), we consider the linear combination $f_3 = \lambda f_1 + (1-\lambda) f_2$ with $\omega_1 = \omega_2 = -z$, $\theta = \pi/2$ and $\lambda = 0.5$. Taking $\omega_1 = -z$, by shearing we obtain

$$\begin{aligned} h_1(z) &= \frac{z - \frac{z^2}{2}}{(1-z)^2} \\ g_1(z) &= -\frac{z^2}{2(1-z)^2} \end{aligned} \tag{46}$$

Also, taking $\omega_1 = -z$, $\theta = \pi/2$, we have

$$\begin{aligned} h_2 &= \frac{\arctan(z)}{2} + \frac{1}{4}\log\frac{1+z^2}{(1-z)^2}, \\ g_2 &= \frac{\arctan(z)}{2} - \frac{1}{4}\log\frac{1+z^2}{(1-z)^2}. \end{aligned} \tag{47}$$

Since $\omega_1 = \omega_2 = -z$ satisfy the condition of Theorem 2.1, the linear combination of harmonic mappings $f_3 = 0.5 f_1 + 0.5 f_2$ is convex in the vertical direction. The images of $\mathbb{D}$ under f_1, f_2 and f_3 are shown in **Figure 1**, respectively.

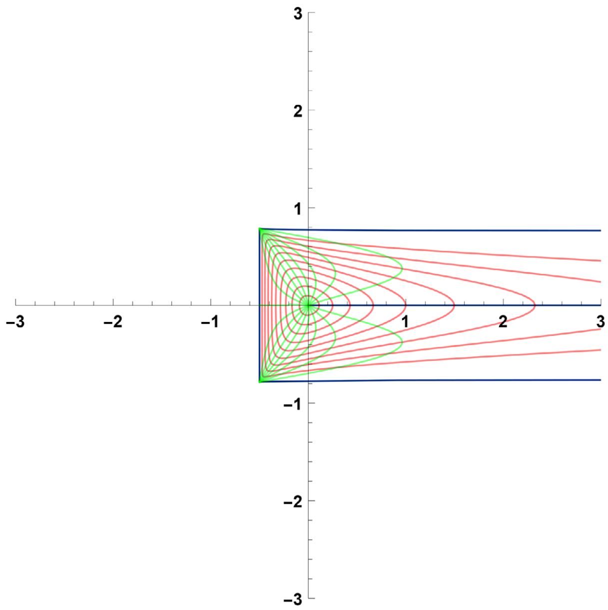


(1) f_1 with $w_1 = -z$

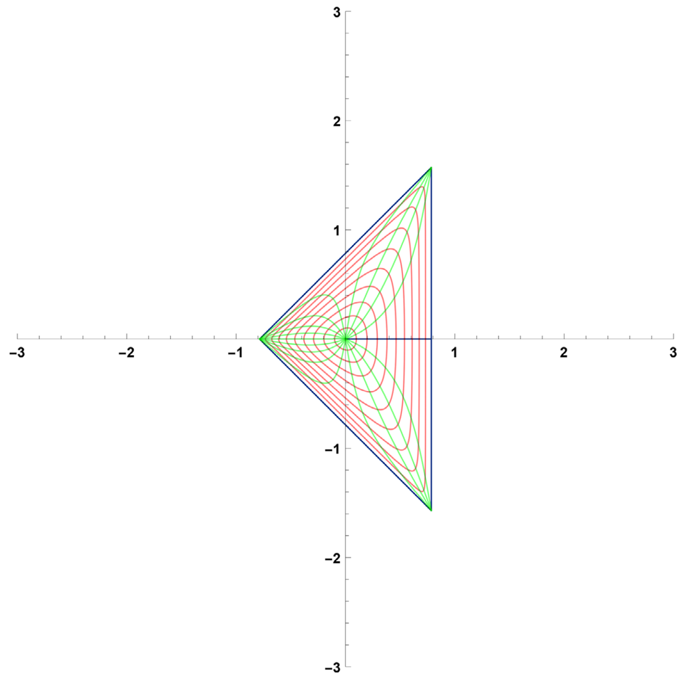
(2) f_1 with $w_2 = -z$

(3) $f_3 = 0.5 f_1 + 0.5 f_2$

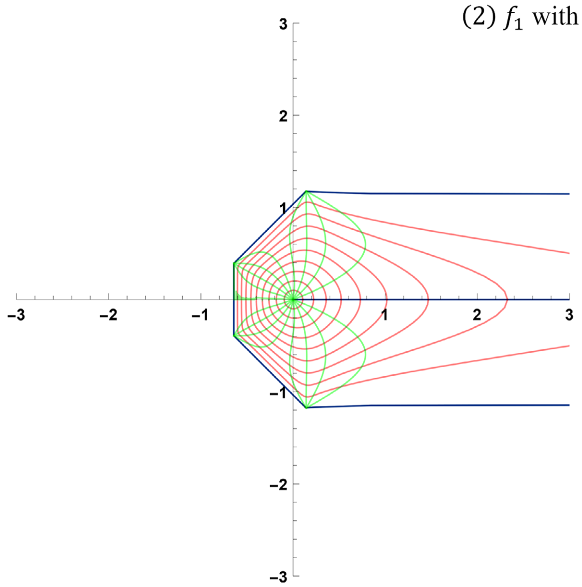
Figure 1. Images of $\mathbb{D}$ under f_1, f_2 and $f_3 = \lambda f_1 + (1-\lambda) f_2$ with $\lambda = 0.5$.



(1) f_1 with $w_1 = z$



(2) f_1 with $w_2 = -z$



(3) $f_3 = 0.5f_1 + 0.5f_2$

Figure 2. Images of $\mathbb{D}$ under f_1, f_2 and $f_3 = \lambda f_1 + (1-\lambda) f_2$ with $\lambda = 0.5$.

Example 2. Let f_1, f_2 be the harmonic mappings considered in Theorem 2.3 with $\omega_1 = z$, $\omega_2 = -z$, $\theta = \pi/2$ and $\lambda = 0.5$. Taking $\omega_1 = z$, by shearing we obtain

$$h_1 = \frac{1}{2(1-z)} + \frac{1}{4}\log\frac{1+z}{1-z},$$

$$g_1 = \frac{z}{2(1-z)} - \frac{1}{4}\log\frac{1+z}{1-z}.$$

Also, taking $\omega_1 = -z$, we have

$$h_2 = \frac{\arctan(z)}{2} + \frac{1}{4}\log\frac{1+z^2}{(1-z)^2},$$
$$g_2 = \frac{\arctan(z)}{2} - \frac{1}{4}\log\frac{1+z^2}{(1-z)^2}.$$

Let $f_1 = h_1 + \overline{g}_1, f_2 = h_2 + \overline{g}_2$ and $f_3 = 0.5f_1 + 0.5f_2$, the harmonic mapping f_3 is convex in the vertical direction. The images of $\mathbb{D}$ under f_1, f_2 and f_3 are shown in **Figure 2**, respectively.

4. Conclusions

In the main results, we demonstrate that the combinations of right half-plane mappings and vertical strip mappings are convex in the vertical direction if and only if they are locally univalent. Furthermore, we extend the above theorem to more general cases by imposing two conditions $\omega_1 = \omega_2$ and $\theta = \arccos\phi(z)$. By considering parameters $z, -z$ and $e^{i\theta}z$ as the dilatations of these harmonic mappings, respectively. We prove the sufficient conditions that their combinations are locally univalent and convex in the vertical direction.

From the above proofs in this paper, it is evident that the linear combinations of right half-plane mappings and vertical strip mappings are convex in the vertical direction only when specific conditions are met. In future studies, we can set the combination coefficient λ to be a complex number, consider the univalency and convexity properties of their combinations. This can extend the content presented by Liu Z. H. and Khurana, D. *et al.* in [11] [12]. Also, we can use the slanted half-plane harmonic mappings which proposed in [9], and prove the sufficient conditions that the combinations of slanted half-plane harmonic mappings and vertical-strip mappings are locally univalent and convex in some certain direction. This holds significant implications for the progress of research on harmonic mappings and minimal surfaces.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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