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Gumbel-Exponentiated Weibull {Logistic} Lifetime Distribution and Its Applications

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Abstract

A new generalized exponentiated Weibull model called Gumbel-exponentiated Weibull {Logistic} distribution is introduced and studied. The new distribution extends the exponentiated Weibull distribution with additional parameters and bimodal densities. Some new and earlier distributions formed the sub-models of the proposed distribution. The mathematical properties of the new distribution including expressions for the hazard function, survival function, moments, order statistics, mean deviation and absolute mean deviation from the mean, and entropy were derived. Monte Carlo simulation study was carried out to assess the finite sample behavior of the parameter estimates by maximum likelihood estimation approach. The superiority of the new generalized exponentiated Weibull distribution over some competing distributions was proved empirically using the fitted results from three real life datasets.

Keywords

Gumbel Distribution, Exponentiated Weibull, Logistic Distribution, Bimodal Densities, Hazard Function, Maximum Likelihood, Order Statistics, Quantile Function, Shannon Entropy, Simulation, T-R (Y) Family

1. Introduction

Modeling and analyzing lifetime data have received tremendous attention in recent years due to the changing nature of lifetime data structures. Data, which arise from several real-life problems, are assumed to follow specific probability distributions defined by a probability density function (pdf) and the shape of the hazard rate function (hrf) which could be monotonically increasing or decreasing, increasing-decreasing-increasing, decreasing-increasing-decreasing, and bathtub shaped or upside-down bathtub shaped. Several lifetime distributions have been used in literature to model these data sets with varying characteristics, including the Weibull distribution, exponential, gamma, Rayleigh, Lindley, Pareto distributions and their generalizations. It is worthy of note that most generalizations have followed several known methods for generating new families of distribution. Some of the continuous univariate generators which have received great deal of attention in recent decade include: method based on differential equations introduced by [1], method of generation based on transformations by [2], method based on quantiles proposed by [3] and [4], methods based on generating skewed distributions developed by [5], method based on adding parameters and generators proposed by [6], method based on compounding univariate continuous distributions and family of discrete distributions suggested by [7], methods based on generators proposed by [8], composite method by [9], and the transformed-transformer (T-X) approached introduced by [10] which was later modified as T-R{Y} by [11] and [12].

[6] introduced an extension of the Weibull distribution called the exponentiated Weibull distribution by adding additional shape parameter to this distribution and since then many works have been carried out by researchers to make the exponentiated weibull distribution more flexible. These efforts include: the Exponentiated Weibull family by [13], extended exponentiated Weibull lifetime distribution developed by [14], exponentiated Weibull-Weibull distribution studied by [15], while [16] suggested a new generalization of the exponentiated modified Weibull known as beta-exponentiated modified Weibull. For other modifications of the Weibull distribution, see [17] and [18]. The exponentiated Weibull distribution and its extensions have a great deal of usage and current generalization will be able to handle more intricate lifetime modeling situations.

This paper therefore focuses on deriving a new exponentiated Weibull lifetime distribution that will be capable of modeling both unimodal and bimodal datasets. Gumbel-Weibull, Gumbel-Rayleigh and Gumbel-exponential distributions form the special cases of the proposed distribution. The proposed distribution is flexible and can play important role in reliability analysis, because the hazard function can assume several shapes. The kurtosis of the Gumbel-Exponentiated Weibull distribution will achieve higher flexibility when compared with the baseline distribution.

The remainder of this paper is organized as follows: the cdf and pdf of the new generalized exponentiated Weibull distribution is derived in Section 2. The statistical and reliability properties of the proposed model are presented in Section 3. Estimation of parameters using the maximum likelihood approach is discussed in Section 4. Simulation of result is presented in Section 5. Section 6 presents the application of real-life dataset and conclusion is Section 7.

2. The Gumbel-Exponentiated Weibull {Logistic} Distribution (GEWLD)

Let T be a random variable following Gumbel distribution with the cumulative distribution and density function given by

$$Z_T(x) = \exp\left\{-\exp\left[-\left(\frac{x-s}{b}\right)\right]\right\}$$

and

$$z_T(x) = \frac{1}{b}\exp\left[-\left(\frac{x-s}{b}\right)\right]\exp\left\{-\exp\left[-\left(\frac{x-s}{b}\right)\right]\right\},$$
$$-\infty < x < \infty, b > 0, -\infty < s < \infty,$$

respectively and b is a scale parameter and s is a location parameter. Suppose R is an exponentiated Weibull distribution with cumulative distribution function (cdf) and probability density function (pdf) given by

$$Z_R(x) = \left[1-\exp\left\{-\left(\frac{x}{\lambda}\right)^{\alpha}\right\}\right]^{\beta},$$

pdf is given by,

$$z_R(x) = \frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta-1}, x > 0, \alpha, \beta, \lambda > 0$$

where λ is a scale parameter and α, β are the shape parameters (see [6]).

Let the density function $z_Y(x)$ and quantile function $\vartheta_Y(p)$ of a Y random variable following a standard logistic distribution be given respectively by

$$z_Y(x) = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}, -\infty < x < \infty, \quad \vartheta_Y(p) = \log\left(\frac{p}{1-p}\right), 0 < p < 1.$$

We introduce the 5-parameter GEWLD in this section, by adopting the T-R{Y} family of distributions generation approach introduced by [11]. Here we define the cdf of the T-R{Y} as

$$\begin{aligned} Z_X(x) &= \int_a^{\vartheta_Y\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right)} z_T(t)\,dt \\ &= P\left[T \le \vartheta_Y\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right)\right] \\ &= Z_T\left(\vartheta_Y\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right)\right), x \in \Re \end{aligned} \quad (1)$$

The corresponding pdf associated with (1) is

$$\begin{aligned} z_X(x) = z_R(x) \times \vartheta_Y'\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right) \\ \times z_T\left(\vartheta_Y\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right)\right) \end{aligned}$$

$$= z_R(x) \times \frac{z_T\left(\vartheta_Y\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right)\right)}{z_Y\left(\vartheta_Y\left(\left\{\left[1-\exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right)\right]^{\beta}\right\}\right)\right)}, x \in \Re \quad (2)$$

where

$$\vartheta_Y'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\vartheta_Y(x)$$

where, $Z_T(x)$, $Z_R(x)$ and $Z_Y(x)$ are the cdfs of Gumbel, exponentiated Weibull and standard Logistic random variables with their corresponding cdfs $z_T(x)$, $z_R(x)$ and $z_Y(x)$ and the quantile function of Y random variable $\vartheta_Y(p)$, $0<p<1$, respectively.

Suppose $\xi=(\alpha,\beta,b,s,\lambda)$ are the parameters of the proposed model, where $x>0$, $\alpha>0$, $\theta>0$, $b>0$, $\lambda>0$, $-\infty<s<\infty$. Applying (1) and (2) above, then the cdf of the Gumbel-exponentiated Weibull {Logistic} distribution (GEWLD) is obtained as

$$Z_{GEWL}(x;\xi) = \exp\left\{-\mathrm{e}^{\frac{s}{b}}\left(\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right]^{-1}-1\right)^{-\frac{1}{b}}\right\} \quad (3)$$

Let $X \sim GEWL(\xi)$ be a random variable with cdf (3), then the pdf of X is

$$z_{GEWLD}(x;\xi) = \frac{\alpha\beta\mathrm{e}^{\frac{s}{b}}}{\lambda b}\left(\frac{x}{\lambda}\right)^{\alpha-1}\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta-1}$$
$$\times\left(1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}\times\left[\left(1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\frac{1}{b}-1} \quad (4)$$
$$\times\exp\left\{-\mathrm{e}^{\frac{s}{b}}\left(\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right]^{-1}-1\right)^{-\frac{1}{b}}\right\}$$

2.1. Shape of the Density

The study of the first and second derivative of the cdf (3) helps us to understand the main features of the density shape. The process to obtain the first derivative of $\ln\{z(x)\}$ is provided below.

Let $k=1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}$. Then $k<1$, $-\ln(1-k)=\left(\frac{x}{\lambda}\right)^{\alpha}$ and $\lambda\left[-\ln(1-k)\right]^{\frac{1}{\alpha}}=x$. Rewrite (4) into a function in k. For $k<1$, we have,

$$z\left(\lambda\left[-\ln(1-k)\right]^{\frac{1}{\alpha}}\right)$$
$$=\frac{\alpha\beta a}{\lambda b}\left[-\ln(1-k)\right]^{1-\frac{1}{\alpha}}(1-k)k^{\beta-1}(1-k)^{-2}\left\{\left(1-k^{\beta}\right)^{-1}-1\right\}^{-(b+1)}$$
$$\times\exp\left\{-a\left(\left(1-k^{\beta}\right)^{-1}-1\right)^{-\frac{1}{b}}\right\}$$

Taking logarithm of both sides of the equation, we have

$$\ln\left(k\left(\lambda\left[-\ln(1-k)\right]^{\frac{1}{\alpha}}\right)\right)$$
$$=\ln\left(\frac{\alpha\beta a}{\lambda b}\right)+\left(1-\frac{1}{\alpha}\right)\ln\left[-\ln(1-k)\right]-\ln k+(\beta-1)\ln k-2\ln\left(1-k^{\beta}\right)$$
$$-\left(\frac{1}{b}-1\right)\ln\left\{\left(1-k^{\beta}\right)^{-1}-1\right\}-a\left(\left(1-k^{\beta}\right)^{-1}-1\right)^{-\frac{1}{b}}$$

Taking derivatives, we obtain

$$\frac{\partial\left(\ln\left(z\left(\lambda\left[-\ln(1-z)\right]^{\frac{1}{\alpha}}\right)\right)\right)}{\partial k}$$
$$=\frac{\alpha-1}{\alpha}\frac{1}{(1-k)\ln(1-k)}-\frac{1}{k}+\frac{\beta-1}{k}+\frac{2\beta k^{\beta-1}}{1-k^{\beta}}+\frac{b-1}{b}\frac{\beta k^{\beta-1}\left(1-k^{\beta}\right)^{-2}}{\left(1-k^{\beta}\right)^{-1}-1}$$
$$+\frac{a\beta}{b}k^{\beta-1}\left(1-k^{\beta}\right)^{-2}\left[\left(1-k^{\beta}\right)^{-1}-1\right]^{-\left(1+\frac{1}{b}\right)}$$

The first derivative of $\ln\left(z(x)\right)$ of the Gumbel-Exponentiated Weibull {Logistic} distribution after replacing k by $1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}$ accordingly, is:

$$=-\frac{\alpha+1}{\alpha}\mathrm{e}^{\left(\frac{x}{\lambda}\right)^{\alpha}}\left(\frac{x}{\lambda}\right)^{\alpha}-\beta\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{-1}+2\beta\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta-1}\left(1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}$$
$$-\frac{1+b}{b}\beta\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta-1}\left(1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}\left[\left(1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-1}$$
$$+\frac{a\beta}{b}\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta-1}\left[1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right]^{-2}\left[\left(1-\left(1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\left(\frac{1}{b}+1\right)}$$

The $\frac{\partial\ln\left(z(x)\right)}{\partial x}$ does not exist when $x=0$. If there are modes for GEWLD it will satisfy $\frac{\partial\ln\left(z(x)\right)}{\partial x}=0$.

Let, $\lambda=1$, since it is a scale parameter and does not affect the shape of the density, then

$$
\begin{aligned}
&-\frac{\alpha+1}{\alpha}x^{\alpha}\mathrm{e}^{-x^{\alpha}}+\beta\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{-1}+2\beta\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta-1}\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-1}\\
&-\frac{1+b}{b}\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta-1}\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-2}\left[\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-1}\\
&+\frac{a\beta}{b}\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta-1}\left[1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right]^{-2}\left[\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\left(\frac{1}{b}+1\right)}=0
\end{aligned}
\tag{5}
$$

Multiplying the first part of the Equation (5) by $-\alpha\left(1-\mathrm{e}^{-x^{\alpha}}\right)$, we obtain

$$
\begin{aligned}
&b(\alpha-1)x^{-\alpha}\mathrm{e}^{-x^{\alpha}}\left(1-\mathrm{e}^{-x^{\alpha}}\right)-2\beta\alpha b\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-1}\\
&+\alpha^{2}\beta(1+b)\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-2}\left[\left(1-\left(1-\mathrm{e}^{-x^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-1}=\alpha\beta b
\end{aligned}
\tag{6}
$$

So, the modes of $z(x)$ are the roots of Equation (6). Another possible root of the density is $b(\alpha-1)$ and there may be other roots. If we have $x=x_0$ as a root of (6) then it corresponds to the local maximum of $\partial(z(x))/\partial x>0$ for all $x<x_0$ and $\partial(z(x))/\partial x<0$ for all $x>x_0$. It corresponds to local maximum of $\partial(z(x))/\partial x<0$ for all $x<x_0$ and $\partial(z(x))/\partial x>0$ for $x>x_0$. The solution tallies to the point of inflexion if either $\partial(z(x))/\partial x<0$ for all $x\neq x_0$ or $\partial(z(x))/\partial x>0$ for all $x\neq x_0$.

The plots in **Figure 1** give (a) bimodal density shapes with fixed values of $\lambda=1,\beta=1.2$ and varying values of *a*, *b* and *s*, (b) left-skewed density shapes

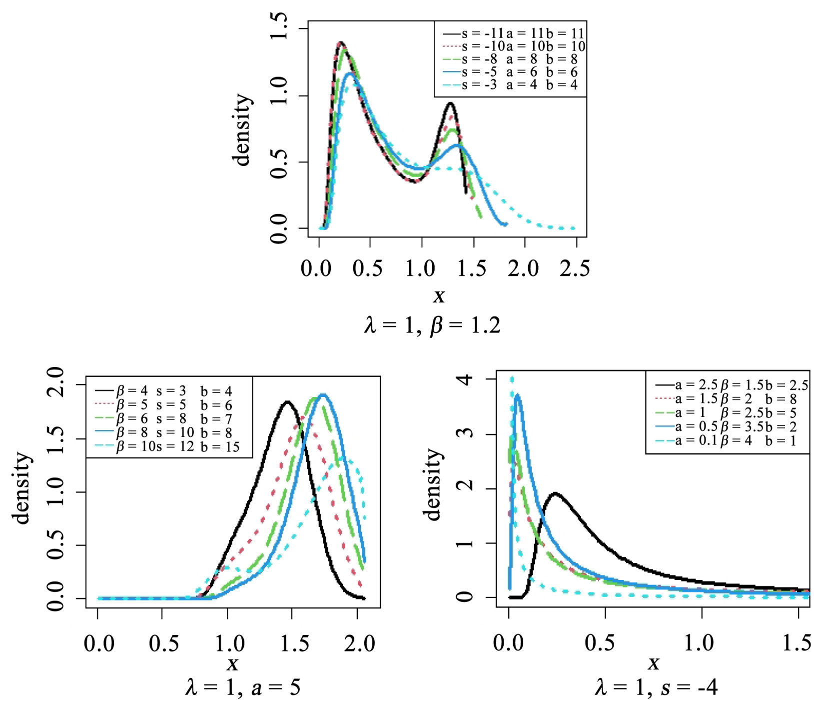


Figure 1. Plots of the GEWLD pdf for $\lambda=1,\beta=1.2$ and some values of *a*, *b* and *s*.

with fixed values of $\lambda = 1, a = 5$ and varying values of β, b, and s, (c) reversed-J, unimodal right-skewed density shapes with fixed values of $\lambda = 1, s = -4$ and varying values of a, β and b.

The Survival function which is used in engineering to model time-to-event surviving beyond time x in reliability analysis is presented in Equation (6)

$$S(x) = 1 - \exp\left\{ -\mathrm{e}^{\frac{s}{b}} \left(\left[1 - \left(1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \right)^{\beta} \right]^{-1} - 1 \right)^{-\frac{1}{b}} \right\}, \tag{7}$$

It is easy to verify that the hazard rate function (hrf) is of GEWLD is given by

$$hrf(x) = \frac{\alpha\beta\mathrm{e}^{\frac{s}{b}}}{\lambda b} \left(\frac{x}{\lambda}\right)^{\alpha-1} \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \left(1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta-1} \times \left(1 - \left(1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}$$

$$\times \frac{\left[\left(1 - \left(1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \right)^{\beta} \right)^{-1} - 1 \right]^{-\frac{1}{b}-1} \times \exp\left[-\mathrm{e}^{\frac{s}{b}} \left(1 - \left(1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \right)^{\beta} \right)^{-1} - 1 \right]^{-\frac{1}{b}}}{1 - \exp\left\{ -\mathrm{e}^{\frac{s}{b}} \left(\left[1 - \left(1 - \mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \right)^{\beta} \right]^{-1} - 1 \right)^{-\frac{1}{b}} \right\}}, \tag{8}$$

From the graph of the hazard rate function (hrf) plots in **Figure 2**. below, the GEWLD displays decreasing, upside-down bathtub shape, bathtub shape and increasing hazard rate with time at different values of the parameters.

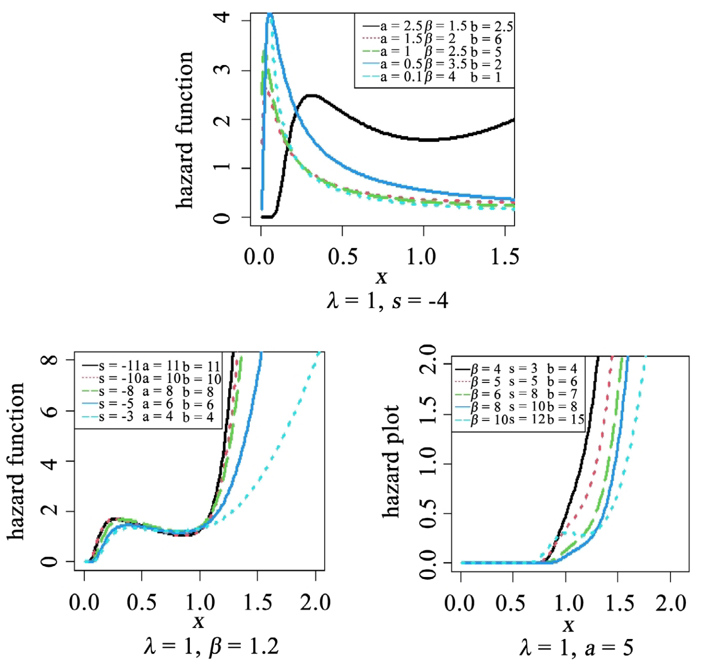


Figure 2. Hazard rate function plots of GEWLD for selected parameter value.

2.2. Submodels of the GEWL Distribution

The following results in the proposed distribution provide some immediate characterization of the Gumbel-Exponentiated Weibull {Logistic} distribution (GEWLD).

1) If the shape parameter $\beta = 1$, the GEWLD reduces to the four parameter Gumbel-Weibull (GW) distribution introduced by [19] with cdf and pdf

$$Z_{GW}(x) = \exp\left\{ -e^{\frac{s}{b}} \left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1 \right)^{-\frac{1}{b}} \right\}$$

$$z_{GW}(x) = \frac{\alpha e^{\frac{s}{b}}}{b\lambda} \left(\frac{x}{\lambda}\right)^{\alpha-1} \left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1 \right)^{-\left(1+\frac{1}{b}\right)} \exp\left\{ -e^{\frac{s}{b}} \left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1 \right)^{-\frac{1}{b}} + \left(\frac{x}{\lambda}\right)^{\alpha} \right\}$$

with $0 < x < \infty, \alpha, \lambda, b > 0$ and $-\infty < s < \infty$, and α, b are the shape parameters, λ and s are the scale parameters.

2) If the parameter $\alpha = \beta = 1$, then the GEWLD reduces to the Gumbel exponential distribution with cdf given as

$$Z_{GE}(x) = \exp\left\{ -e^{\frac{s}{b}} \left(e^{\frac{x}{\lambda}} - 1 \right)^{-\frac{1}{b}} \right\}$$

$$z_{GE}(x) = \frac{e^{\frac{s}{b}}}{\lambda b} \left(e^{\frac{x}{\lambda}} - 1 \right)^{-\left(\frac{1}{b}+1\right)} \exp\left[-e^{\frac{s}{b}} \left(e^{\frac{x}{\lambda}} - 1 \right)^{-\frac{1}{b}} + \frac{x}{\lambda} \right]$$

where $0 < x < \infty, b, \lambda > 0$ and $-\infty < s < \infty$ and λ, s are the scale parameters and b is the shape parameter.

3) If $\alpha = 2, \beta = 1$, then the GEWLD reduces to Gumbel-Rayleigh distribution with cdf and pdf given respectively as

$$Z_{GR}(x) = \exp\left\{ e^{\frac{s}{b}} \left(1 - e^{\left(\frac{x}{\lambda}\right)^{2}} \right)^{-\frac{1}{b}} \right\}$$

and

$$z_{GR}(x) = \frac{2xe^{\frac{s}{b}}}{b\lambda^2} \left\{ \left[1 - e^{\left(\frac{x}{\lambda}\right)^{2}} \right]^{-\left(\frac{1+b}{b}\right)} e^{\left\{ e^{\frac{s}{b}} \left\{ 1 - e^{\left(\frac{x}{\lambda}\right)^{2}} \right\}^{-\frac{1}{b}} + \left(\frac{x}{\lambda}\right)^{2} \right\}} \right\}$$

where $0 < x < \infty, b, \lambda > 0$ and $-\infty < s < \infty$, b is the shape parameter, b and s are the scale parameters.

3. Some structural Properties

3.1. Expansions for the Cumulative and Density Function of the GEWLD

Very important mixture expression of the cumulative distribution function (cdf) and probability density function pdf of the GEWLD are presented here. By using some series expansions with obtain the cdf of GEWL distribution which is given by

$$Z(x)=\varnothing_g \mathrm{e}^{\left(\frac{x}{\lambda}\right)^{\alpha k(m+1)}} \tag{9}$$

where

$$\varnothing_g=\sum_{i=0}^{\infty}\frac{(-1)^i a^i}{i!}\left[\sum_{j,k,l,m=0}^{\infty}(-1)^{j+k+l+m}\binom{-1}{j}\binom{\beta j}{k}\binom{-\left(\frac{i}{b}+1\right)}{l}\binom{-\left(\frac{n}{b}+l\right)}{m}\right]^{-\frac{i}{b}}$$

we also obtain the expression for the pdf of GEWL distribution as a linear combination of Gumbel and exponentiated Weibull densities as

$$z(x)=\varnothing_f x^{\alpha-1}\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha[1-q(1+t)+v+\beta w]}}, \tag{10}$$

where

$$\varnothing_f=\frac{\alpha\beta a}{\lambda^\alpha b}\sum_{n,p,q,h,t,v,w,\tau=0}^{\infty}(-1)^{n+p+q+h+t+v+w+\tau}\frac{a^n}{n!}\binom{\beta-1}{v}\binom{-2}{w}\binom{p}{\tau}$$
$$\times\binom{-\frac{1}{b}(n+b+1)+1}{h}\binom{-\frac{1}{b}(n+b+1)+bh}{t}.$$

3.2. Moments

The rth ordinary moment of $X\sim GEWLD(x)$ can be expressed using (10) as

$$\mu_r'=E\left(X^r\right)=\frac{\varnothing_f}{\alpha(1-q(1+t)+v+\beta w)}\Gamma\left(\frac{r+\alpha}{\alpha(1-q(1+t)+v+\beta w)}\right)\lambda^{r+\alpha}$$

For any positive integer n, the first four moments of GEWLD are

$$E(X)=\frac{\varnothing_f}{\alpha(1-q(1+t)+v+\beta w)}\Gamma\left(\frac{1+\alpha}{\alpha(1-q(1+t)+v+\beta w)}\right)\lambda^{1+\alpha}$$

$$E\left(X^2\right)=\frac{\varnothing_f}{\alpha(1-q(1+t)+v+\beta w)}\Gamma\left(\frac{2+\alpha}{\alpha(1-q(1+t)+v+\beta w)}\right)\lambda^{2+\alpha}$$

$$E\left(X^3\right)=\frac{\varnothing_f}{\alpha(1-q(1+t)+v+\beta w)}\Gamma\left(\frac{3+\alpha}{\alpha(1-q(1+t)+v+\beta w)}\right)\lambda^{3+\alpha}$$

$$E\left(X^4\right)=\frac{\varnothing_f}{\alpha(1-q(1+t)+v+\beta w)}\Gamma\left(\frac{4+\alpha}{\alpha(1-q(1+t)+v+\beta w)}\right)\lambda^{4+\alpha}$$

3.3. Incomplete Moments

The incomplete moment of a distribution is very useful and answers many questions in medicine, economics, reliability analysis, demography and insurance when dealing with the Bonferroni and Lorenz curves.

The *s*th incomplete moment $\varphi_{(s)}(t)=\delta(s,t)$, is defined as

$$\varphi_s(t)=\int_0^t x^s z(x)\mathrm{d}x=\int_0^t x^s \varnothing_f x^{\alpha-1}\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha(1-q(1+t)+v+\beta w)}}\mathrm{d}x$$

Set $z=\alpha(1-q(1+t)+v+\beta w)$,

Then

$$\varphi_s(t)=\varnothing_f\int_0^t x^{\alpha+s-1}\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^z}\mathrm{d}x=\frac{\lambda^{\alpha+s}}{Z}\varnothing_f\delta\left(\frac{\alpha}{Z}+\frac{s}{Z},\lambda^{-Z}t^Z\right)\lambda^{-s-\alpha}Z^{-1} \tag{11}$$

where $\delta(s,t)=\int_0^t x^{s-1}\mathrm{e}^{-x}\mathrm{d}x$, is the lower incomplete Gamma function. Besides, the *s*th conditional moments of *X*, say $\omega_s(t)$, is defined as

$$E\left(X^s/X>t\right)=\omega_s(t)=\int_t^\infty x^s z(s)\mathrm{d}x$$

Hence by (3.42), we obtain

$$\omega_s(x)=\frac{\lambda^{\alpha+s}}{z}\varnothing_f\Gamma\left(\frac{\alpha}{z}+\frac{s}{z},\lambda^{-z}t^z\right)\lambda^{-s-\alpha}Z^{-1}, \tag{12}$$

where $\Gamma(s,t)=\int_t^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x$, is the upper incomplete gamma function.

3.4. Generating Function

The moment generating function for GEWL distribution is given by

$$M_X(t)=E\left(\mathrm{e}^{tx}\right)$$

$$M_X(t)=E\left[\sum_{r=0}^{\infty}\frac{t^r x^r}{r!}\right]=\sum_{n=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)$$

$$M_X(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\varnothing_f}{\alpha(1-q(1+t)+v+\beta w)}\Gamma\left(\frac{r+\alpha}{\alpha(1-q(1+t)+v+\beta w)}\right)\lambda^{r+\alpha} \tag{13}$$

3.5. Shannon Entropy

The Shannon Entropy as a measure of randomness or uncertainty [11] has provided important solution to several studies such as anomalous diffusion, DNA sequencing, daily temperature randomness, study of content of information signals, etc. According to [18], the Shannon's entropy for the GEWLD is will be written as

$$\mathbb{E}\left[-\ln z(X)\right]=\mathbb{E}\left\{-\ln\left[z_R(X)\frac{z_T\left(Q_Y\left(z_R(X)\right)\right)}{z_Y\left(Q_Y\left(z_R(X)\right)\right)}\right]\right\}=\eta_X$$

$$\eta_X=\eta_T+\mathbb{E}\left[\ln z_Y(T)\right]-\mathbb{E}\left[\ln z_R(X)\right]$$

But T is a Gumbel, Y is logistic, and R is Exponentiated Weibull distribution random variables respectively, hence η_T in equation is the Shannon entropy of the Gumbel distribution given by $\ln\theta+\zeta+1$, where ζ is the Euler Mascheroni constant. ζ is approximately equal to 0.5772.

But

$$\ln\left(z_R\left(X\right)\right)=\ln\left(\frac{\alpha\beta}{\lambda}\right)+\left(\alpha-1\right)\ln\left(\frac{X}{\lambda}\right)-\left(\frac{X}{\lambda}\right)^{\alpha}+\left(\beta-1\right)\ln\left[1-\exp\left(-\left(\frac{X}{\lambda}\right)^{\alpha}\right)\right]$$

$$\mathbb{E}\left[\ln Z_R\left(X\right)\right]=\ln\alpha+\ln\beta-\ln\lambda+\left(\alpha-1\right)\mathbb{E}\left[\ln X\right]-\left(\alpha-1\right)\ln\lambda$$
$$-\lambda^{-\alpha}\mathbb{E}\left[X^{\alpha}\right]+\left(\beta-1\right)\ln\left[1-\mathrm{e}^{-\left(\frac{X}{\lambda}\right)^{\alpha}}\right]$$

Recall that $z_Y\left(x\right)=\frac{\mathrm{e}^{-x}}{\left(1-\mathrm{e}^{-x}\right)^{2}}$ which implies that $z_Y\left(T\right)=\frac{\mathrm{e}^{-T}}{\left(1-\mathrm{e}^{-T}\right)^{2}}$

Therefore $\ln z_Y\left(T\right)=-T-2\ln\left(1+\mathrm{e}^{-T}\right)$

$$\mathbb{E}\left[\ln z_Y\left(T\right)\right]=-\mathbb{E}\left(T\right)-2\mathbb{E}\left[\ln\left(1+\mathrm{e}^{-T}\right)\right]=-\mu_T-2\mathbb{E}\left[\ln\left(1+\mathrm{e}^{-T}\right)\right]$$

Substituting $\mathbb{E}\left[\ln Z_R\left(X\right)\right]$ and $\mathbb{E}\left[\ln z_Y\left(T\right)\right]$ in η_X we have

$$\eta_X=\eta_T-\mu_T-2\mathbb{E}\left[\ln\left(1+\mathrm{e}^{-T}\right)\right]-\ln\alpha-\ln\beta+\ln\lambda-\left(\alpha-1\right)\mathbb{E}\left[\ln X\right]$$
$$+\left(\alpha-1\right)\ln\lambda+\lambda^{-\alpha}\mathbb{E}\left[X^{\alpha}\right]-\left(\beta-1\right)\ln\left[1-\mathrm{e}^{-\left(\frac{X}{\lambda}\right)^{\alpha}}\right]$$

$$\eta_X=\eta_T-\mu_T-\lambda^{-\alpha}\mu'_{\alpha}-\ln\alpha-\ln\beta+\ln\lambda-\left(\alpha-1\right)\mathbb{E}\left[\ln X\right]$$
$$+\left(\alpha-1\right)\ln\lambda-2\mathbb{E}\left[\ln\left(1+\mathrm{e}^{-T}\right)\right]-\left(\beta-1\right)\ln\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{1}{i}\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^{\alpha i}} \tag{14}$$

3.6. Quantiles

The p^{th} quantile x_p of the Gumbel-Exponentiated Weibull{Logistic} distribution is obtained by direct reversal of the cdf given in (3)

$$x_p=-\lambda\ln\left[1-\left(1-\left[\left[-\mathrm{e}^{-\frac{s}{b}}\ln\left(p\right)\right]^{-b}+1\right]^{-1}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}},0<p<1 \tag{15}$$

The first three quantiles of GEWLD can be obtained by using x_p above for $p=\frac{1}{4},\frac{1}{2}$ and $\frac{3}{4}$. Generally, to generate a random variate "x" from GEWL from the uniform random number "u" we can use the formula

$$x=-\lambda\ln\left[1-\left(1-\left[\left[-\mathrm{e}^{-\frac{s}{b}}\ln\left(u\right)\right]^{-b}+1\right]^{-1}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\alpha}},0<u<1 \tag{16}$$

3.7. Order Statistics

Let $X_1, X_2, \cdots, X_m$ be a random sample GEWL distribution, Suppose $X_{k:m}$ denote the k^{th} order statistic. The probability density function of the $X_{k:m}$ can be expressed as

$$z_{k:m}(x) = \frac{1}{B(k, m-k+1)} z(x)\left[Z(x)\right]^{k-1}\left[1-Z(x)\right]^{m-k}$$

$$= \sum_{i=1}^{m-k} \frac{(-1)^i}{(k+1)B(k, m-k+1)} \binom{m-k}{i} (k+1) z(x)\left[Z(x)\right]^{k+i-1}$$

$$z_{k:m}(x) = \sum_{i=1}^{m-k} \varnothing_{i,k,m} z(x)\left[Z(x)\right]^{k+i-1} \quad (17)$$

where $\varnothing_{i,k,m} = \frac{(-1)^i}{(k+1)B(k, m-k+1)} \binom{m-k}{i}$

Given the general expression of the cdf (9) and pdf (10) of the GEWL distribution, we get the pdf of the k^{th} order statistic for the GEWL distribution as

$$z_{k:m} = \sum_{i=1}^{m-k} \sum_{j=0}^{\infty} \sum_{w=0}^{\infty} \varnothing_{i,k,m} \varnothing_f \left[\varnothing_g\right]^{k+i-1} x^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^{[z-v(k+i-1)]}}$$

The r^{th} moment of the k^{th} order statistic of the Gumbel-exponentiated Weibull {Logistic} distribution is given by

$$E\left(X_{k:m}^r\right) = \int_0^{\infty} x_{k:m}^r z_{k:m}(x)\,dx$$

$$= \sum_{i=1}^{m-k} \sum_{j=0}^{\infty} \sum_{w=0}^{\infty} x_{k:m}^r \varnothing_{i,k,m} \varnothing_f \left[\varnothing_g\right]^{k+i-1} x^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^{[z-v(k+i-1)]}}$$

Let $S = vk + vi - v$

$$= \sum_{i=1}^{m-k} \sum_{j=0}^{\infty} \sum_{w=0}^{\infty} \varnothing_{i,k,m} \varnothing_f \left[\varnothing_g\right]^{k+i-1} \int_0^{\infty} x^{r+\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^{[z-s]}}\,dx$$

$$E\left(X_{k:m}^r\right) = \frac{1}{B(k, m-k+1)} \sum_{i=1}^{m-k} \sum_{j=0}^{\infty} \sum_{w=0}^{\infty} \varnothing_f \left[\varnothing_g\right]^{r+\alpha-1} \frac{\lambda^{r+\alpha}}{z-s} \Gamma\left(\frac{r+\alpha}{z-s}\right) \quad (18)$$

4. Estimation

The model parameters of the GEWL distribution can be estimated using the method of maximum likelihood. Suppose $x = (x_1, x_2, \cdots, x_n)'$ is a random independent sample size n from the GEWLD distribution with parameter vector $B = (\alpha, \beta, b,, s, \lambda)'$, then log-likelihood function is expressed as

$$L = n\log\alpha + n\log\beta + \frac{ns}{b} - n\log\lambda - n\log b + (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right)$$

$$-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha} + (\beta-1)\sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right] - 2\sum_{i=1}^{n}\log\left[1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right]$$

$$+\left(-\frac{1}{b}-1\right)\sum_{i=1}^{n}\log\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]$$
$$-e^{\frac{s}{b}}\sum_{i=1}^{n}\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\frac{1}{b}} \tag{19}$$

Solving the log-likelihood function analytically is complex though it can be maximized numerically by using global iterative optimization methods available with software like R and Mathematica. Taking partial derivatives of the log-likelihood function with respect to the parameters we obtain the follows:

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right)-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)$$
$$+(\beta-1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}}$$
$$+2\sum_{i=1}^{n}\frac{\beta\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta-1}e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)}{1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}}$$
$$-\left(\frac{1}{b}+1\right)\sum_{i=1}^{n}\frac{\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}\beta\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta-1}e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)}{\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1} \tag{20}$$
$$+e^{\frac{s}{b}}\sum_{i=1}^{n}-\frac{1}{b}\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\frac{1}{b}-1}\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}$$
$$\times\beta\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta-1}e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right]+2\sum_{i=1}^{n}\frac{\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\log\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)}{1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}}$$

$$+\left(-\frac{1}{b}-1\right)\sum_{i=1}^{n}\frac{\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\log\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)}{\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]}$$

$$-e^{\frac{s}{b}}\sum_{i=1}^{n}-\frac{1}{b}\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\frac{1}{b}-1} \tag{21}$$

$$\times\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\log\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)$$

$$\frac{\partial L}{\partial b}=-\frac{ns}{b}-\frac{n}{b}+\frac{1}{b^2}\sum_{i=1}^{n}\log\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]$$

$$-\left[e^{\frac{s}{b}}\left(\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right)\right]^{\frac{1}{b}}\times\log\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]\frac{1}{b^2} \tag{22}$$

$$-\left[\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\frac{1}{b}}\frac{s}{b^2}e^{\frac{s}{b}}$$

$$\frac{\partial L}{\partial s}=\frac{n}{b}-\frac{e^{\frac{s}{b}}}{b}\sum_{i=1}^{n}\left[\left(\left(1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right)\right]^{-\frac{1}{b}} \tag{23}$$

$$\frac{\partial L}{\partial \lambda}=-\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{\frac{x_i}{\lambda^2}}{\frac{x_i}{\lambda}}+\sum_{i=1}^{n}\alpha\left(\frac{x_i}{\lambda}\right)^{\alpha-1}\frac{x_i}{\lambda^2}$$

$$+(\beta-1)\sum_{i=1}^{n}\frac{\alpha\left(\frac{x_i}{\lambda}\right)^{\alpha-1}\frac{x_i}{\lambda^2}e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}}$$

$$-2\sum_{i=1}^{n}\frac{\alpha\beta\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta-1}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}\frac{x_i}{\lambda^2}e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}}{1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}}$$

$$
\begin{aligned}
&+\left(-\frac{1}{b}-1\right)\sum_{i=1}^{n}\frac{\alpha\beta\left(1-\left(1-\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-2}\left(1-\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta-1}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}\frac{x_i}{\lambda^2}\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}}{\left(1-\left(1-\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1} \\
&-\mathrm{e}^{\frac{s}{b}}\sum_{i=1}^{n}\left(-\frac{1}{b}\right)\alpha\beta\left[\left(1-\left(1-\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta}\right)^{-1}-1\right]^{-\frac{1}{b}-1} \\
&\times\left(1-\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}\right)^{\beta-1}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}\frac{x_i}{\lambda^2}\mathrm{e}^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}}
\end{aligned}
\tag{24}
$$

Solving the equations $\cup(B)=\left(\frac{\partial L}{\partial\alpha},\frac{\partial L}{\partial\beta},\frac{\partial L}{\partial b},\frac{\partial L}{\partial s},\frac{\partial L}{\partial\lambda}\right)'=0$ simultaneously gives the maximum likelihood estimate (MLE) $\hat{B}=\left(\hat{\alpha},\hat{\beta},\hat{b},\hat{s},\hat{\lambda}\right)'$ of $B=\left(\alpha,\beta,b,s,\lambda\right)'$.

Inverting the Fisher information matrix $I_{i,j}(B)$ which can be gotten using the second partial derivatives of the log-likelihood function w. r. t each parameter $I_{i,j}(B)=\left[\frac{\partial^2 L}{\partial B_i B_j}\right]$, the asymptotic variance-covariance matrix of the MLE parameters is obtained.

As $n\to\infty$, $\sqrt{n}\left(\hat{B}-B\right)\xrightarrow{d}N_5\left(0,I^{-1}(B)\right)$.

5. Simulation Study

In this section, assessment of the performance of the MLEs of the GEWLD parameters given in (20), (21), (22), (23) and (24) based on simulation study with respect to sample size n is carried out. The simulation is repeated for $N=5000$ times each with sample size $n=25,70,150,400$ and 700 and arbitrarily, the fixed choice of parameter values is $\alpha=4.5$, $\beta=2.5$, $b=5$, $s=4.5$, $\lambda=1.5$. The evaluation of estimates is based on mean estimates (ME), average bias (AVB), roots mean square error (RMSE), average width (AW) and coverage probability (CP) using the R-software and result presented in Table 1. The results in Table 1 indicate that the estimates are stable and the values of the mean estimates approached true values as the sample size increases. Moreover, Table 1 indicates that the values of the RMSE, AVB and AW decreased as the sample size increases and that the coverage probabilities (CP) of the confidence intervals are quite close to the nominal level of 95%.

6. Applications

In this section, empirical illustrations of the GEWLD using real-life datasets are

Table 1. Results of Monte Carlo simulations for the GEWLD M, AVB, RMSE, AW and CP.

Parameter	Sample size	ME	AVB	RMSE	AW	CP
α	$n = 25$	6.4366	3.1474	4.4091	45.0321	0.90
	$n = 70$	5.4334	3.1283	4.3894	27.3043	0.95
	$n = 150$	4.8143	3.1101	4.3557	24.1081	0.89
	$n = 400$	4.7196	3.0914	4.2943	20.2772	0.98
	$n = 700$	4.6520	3.0900	4.2516	16.3491	0.99
β	$n = 25$	1.8258	0.9325	6.1357	415.5115	1
	$n = 70$	1.9827	0.6310	5.7255	367.4212	1
	$n = 150$	2.7325	1.5508	5.2362	283.9237	0.90
	$n = 400$	2.6018	0.5637	3.3525	198.2219	0.80
	$n = 700$	2.5112	0.8967	2.4461	160.6621	0.96
λ	$n = 25$	0.4669	2.1004	3.0742	62.9456	1
	$n = 70$	0.7264	1.8786	3.0088	58.2126	1
	$n = 150$	1.9816	2.3099	2.7861	53.1782	1
	$n = 400$	1.5625	2.4349	2.1500	47.3689	1
	$n = 700$	1.5012	1.8523	2.0033	29.3421	0.99
s	$n = 25$	6.2326	2.4216	1.5002	27.1733	1
	$n = 70$	6.0524	2.2452	1.4271	22.2195	1
	$n = 150$	4.9031	2.3584	1.3216	20.2290	1
	$n = 400$	4.8416	2.4337	1.2112	15.5751	1
	$n = 700$	4.5710	2.2686	1.1823	11.2931	1
b	$n = 25$	7.2290	3.3082	8.1438	92.2425	0.89
	$n = 70$	7.0568	3.2192	6.6862	50.4314	0.90
	$n = 150$	6.0326	2.1519	4.0988	34.2228	0.80
	$n = 400$	5.2810	2.9359	2.4096	21.3001	0.90
	$n = 700$	5.0028	1.7254	1.1207	17.2481	1

analyzed. All the datasets used are based on complete observations in different lifetime situations showing the usefulness of the GEWLD. The goodness-of-fit criterion, Akaike information criterion, Kolmogorov-Smirnov (K-S) statistic and p-value are used to compare the GEWLD with other competing models.

The first dataset represents the Breaking Stress of Carbon Fibers of 50 mm Length (GPa). The data was obtained from [20]. The dataset is unimodal and is approximately symmetric (Skewness = −0.1285 and excess kurtosis = 0.1261). Six distributions are used to fit the dataset as shown in **Table 2** namely: the proposed Gumbel-exponentiated Weibull {logistic} distribution (GEWLD), the Weibull distribution (WD) [21], the Gumbel distribution (GD) [22], the exponentiated Weibull distribution (EWD) [6], the beta exponential distribution

Table 2. Maximum likelihood fit of the breaking stress of carbon Fibers of 50 mm Length (GPa).

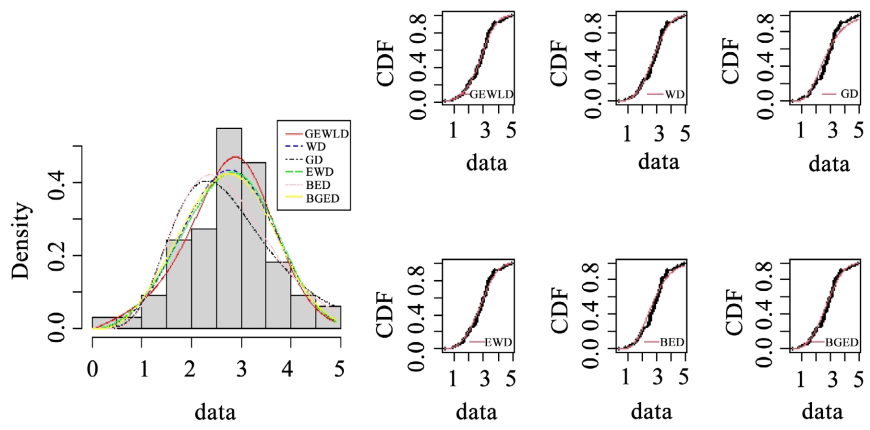
Distribution	GEWLD	WD	GD	EWD	BED	BGED
Parameter estimates	$\hat{\alpha}=2.673(0.654)$ $\hat{\beta}=0.320(0.834)$ $\hat{s}=4.093(2.284)$ $\hat{b}=2.903(2.727)$ $\hat{\lambda}=1.659(0.881)$	$\hat{\alpha}=3.441(0.330)$ $\hat{\lambda}=3.062(0.114)$	$\hat{s}=2.310(0.119)$ $\hat{b}=0.911(0.079)$	$\hat{\alpha}=3.910(1.067)$ $\hat{\beta}=0.800(0.352)$ $\hat{\lambda}=3.230(0.345)$	$\hat{a}=7.512(1.291)$ $\hat{b}=21.131(45.301)$ $\hat{\lambda}=0.112(0.213)$	$\hat{a}=0.405(0.290)$ $\hat{b}=24.129(26.647)$ $\hat{\alpha}=10.670(8.769)$ $\hat{\lambda}=2.813(1.573)$
Log-Lik	−84.96	−86.07	−92.40	−85.94	−91.22	−86.43
AIC	179.91	176.14	188.79	177.89	188.44	180.85
K-S	0.0704	0.0823	0.1352	0.0809	0.1334	0.0936
p-value	0.8757	0.7314	0.1627	0.7498	0.1745	0.5773

(standard error of estimate in parenthesis)

(BED) [23] and the beta generalized exponential distribution (BGED) [24]. Results in **Table 2** clearly show that the proposed GEWLD not only provided a good fit to the dataset but also outperformed the other distribution in fitting the data set given that its p-value of the K-S statistic is highest and very close to unity. Again, the p-value of the K-S statistic for all the fitted distributions is greater than the nominal 0.05 level of significance indicating that all the distributions fitted the data considerably well and that the proposed GEWLD presented the best fit. From **Figure 3**, the pdfs and cdfs plots suggest good adjustments to the dataset and that GEWLD fits almost symmetric and unimodal dataset very well.

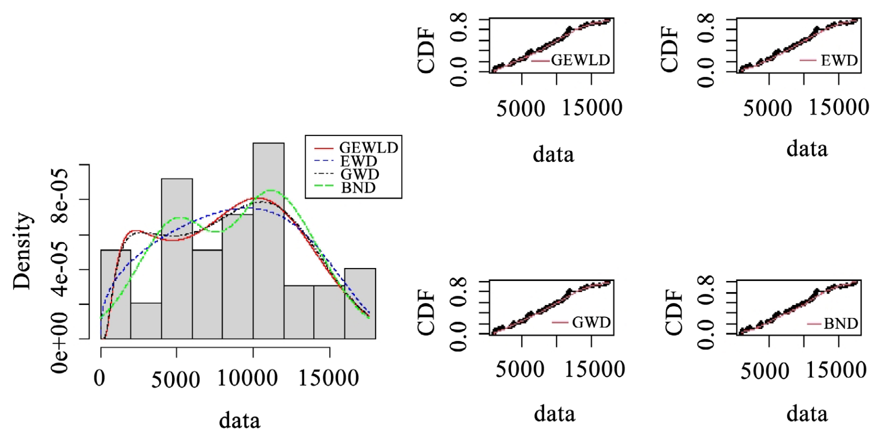
The data set represents Kevlar 49/epoxy strands failure times data (pressure at 70%). The dataset was used in the work of [19] to compare the fit of the Gumbel-Weibull distribution (GWD) with that of the EWD and the beta-normal distribution (BND) [25]. The data is multimodal, platykurtic, and approximately symmetric. (Skewness = 0.0938, excess kurtosis = −0.9154). Four distributions are used to fit the dataset namely: the proposed GEWLD, the EWD, the GWD and the BND. The results of the maximum likelihood fit of all the distributions are contained in **Table 3**. Results in **Table 3** clearly show that the proposed GEWLD not only provided a good fit to the bimodal dataset but also outperformed the other distribution in fitting the data set given that its p-value of the K-S statistic is highest and very close to unity. Again, the p-value of the K-S statistic for all the fitted distributions is greater than the nominal 0.05 level of significance indicating that all the distributions fitted the data considerably well and that the proposed GEWLD presented the best fit. The estimated pdfs and cdfs are presented in **Figure 4** and the figure indicated GEWLD provides a better than the competing models.

The third dataset represents Kevlar 49/epoxy strands failure times data (pressure at 90%). The dataset was collected by [26] and obtained from [27]. The dataset is highly skewed to the right with a reverse-J shape. The data is unimodal and leptokurtic (Skewness = 2.9573, excess kurtosis = 13.3798) (**Figure 5**).



(a) (b)

Figure 3. Fitted pdfs (3a) and cdfs (3b) of GEWLD and other completive models for dataset 1.



(a) (b)

Figure 4. Fitted pdfs (4a) and cdfs (4b) for the bimodal Kevlar 49/epoxy strands failure times data (pressure at 70%).

Table 3. Maximum likelihood fit of the Kevlar 49/epoxy strands failure times data (pressure at 70%).

Distribution	GEWLD	EWD	GWD	BND
Parameter estimates	$\hat{\alpha}=2.4400(0.4209)$ $\hat{\beta}=1.4643(0.8018)$ $\hat{s}=1.8012(1.0821)$ $\hat{b}=5.1439(2.4147)$ $\hat{\lambda}=5006.5(3.0016)$	$\hat{\alpha}=5.2208(2.9744)$ $\hat{\beta}=0.2646(0.1880)$ $\hat{\lambda}=10000.5(1995.8)$	$\hat{\alpha}=2.6741(0.3582)$ $\hat{\epsilon}=1.1546(0.7454)$ $\hat{\sigma}=4.1036(1.0343)$ $\hat{\lambda}=6116.9(246.54)$	$\hat{a}=0.1150(0.1489)$ $\hat{b}=0.0806(0.1068)$ $\hat{\mu}=7796.1(1390.6)$ $\hat{\sigma}=1087.1(794.9)$
Log-Lik	−478.5	−479.0	−478.51	−480.4
AIC	967.0	964.1	965.0	968.8
K-S	0.0695	0.0825	0.0742	0.0832
p-value	0.9587	0.8651	0.9316	0.8590

(standard error of estimate in parenthesis).

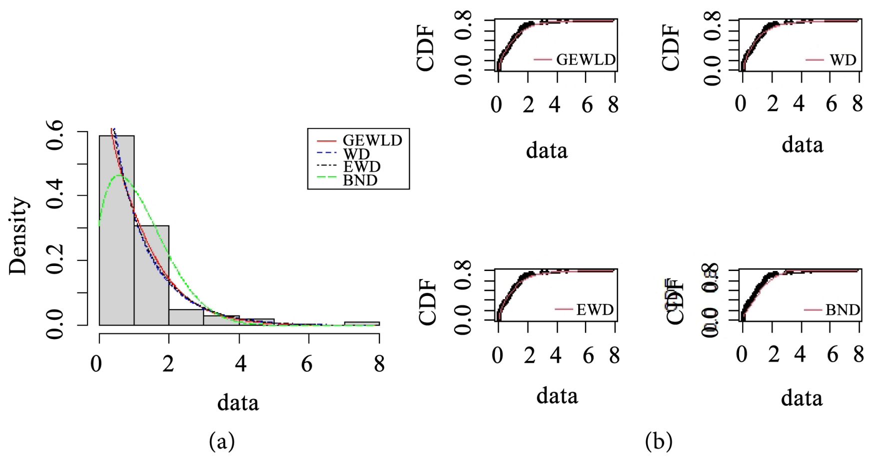


Figure 5. Fitted pdfs (5a) and cdfs (5b) for the right-skewed and heavy-tailed Kevlar 49/epoxy strands failure times data (pressure at 90%).

7. Conclusion

This work has introduced a new generalized exponentiated Weibull univariate continuous distribution called the Gumbel-exponentiated Weibull {Logistic} distribution (GEWLD). The probability density function showed that the distribution is capable of modeling bimodal lifetime dataset. Several mathematical properties of the distribution were derived. Certain characterization results were also presented in the work. Using the method of maximum likelihood estimation method, the estimates of the model parameters were obtained. The new model provides adequate fits when compared with other models competing very well in terms of AICs but best when the smallest K-S statistics and highest P-values are considered.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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