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Boundary Control Problems for 2 × 2 Cooperative Hyperbolic Systems with Infinite Order Operators

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Abstract

In this study, boundary control problems with Neumann conditions for 2 × 2 cooperative hyperbolic systems involving infinite order operators are considered. The existence and uniqueness of the states of these systems are proved, and the formulation of the control problem for different observation functions is discussed.

Keywords

Cooperative, Infinite Order, Boundary Control, Neumann Conditions, Observation Function, Hyperbolic Systems

1. Introduction

The earliest theory of optimal control was introduced by Lions [1].

Majority of the research in this field has focused on discussing the optimal control problem by using several operator types (such as elliptic, parabolic, or hyperbolic operators) [2]-[11], and by varying the nature of control (such as distributed control [6] [11] [12] [13] and boundary control [3] [5] [8]).

References [14] [15] were among the first studies that presented the control problems of systems including infinite order operators. These problems were then extended in different ways, such as for higher system degrees [16] [17], and for parabolic and hyperbolic systems [14] [17] [18] [19] [20] [21].

Based on the theories proposed by Lions [1] and Dubinskii [22] [23] [24], the distributed control problem with Dirichlet conditions for 2 × 2 non-cooperative hyperbolic systems involving infinite order operators was discussed in a previous study [13]; in this study, we extend this problem to cooperative hyperbolic systems of the boundary type with Neumann conditions for different observation functions.

The system can be defined as

$$\begin{cases} \frac{\partial^2 y_1}{\partial t^2} + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2|\alpha|} y_1(x) = a y_1(x) + b y_2(x) + f_1 & \text{in } Q, \\ \frac{\partial^2 y_2}{\partial t^2} + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2|\alpha|} y_2(x) = c y_1(x) + d y_2(x) + f_2 & \text{in } Q, \\ y_1, y_2 \to 0, \quad |x| \to \infty, \\ \left.\frac{\partial y_1}{\partial \upsilon}\right|_\Sigma = g_1, \quad \left.\frac{\partial y_2}{\partial \upsilon}\right|_\Sigma = g_2, \\ y_1(x,0) = y_{1,0}(x), \quad y_2(x,0) = y_{2,0}(x) & \text{in } \Omega, \\ \frac{\partial y_1(x,0)}{\partial t} = y_{1,1}(x), \quad \frac{\partial y_2(x,0)}{\partial t} = y_{2,1}(x) & \text{in } \Omega, \end{cases} \quad (1)$$

where a, b, c and d are constant such that $b, c > 0$.

(This implies that the system (1) is cooperative.)

$$y_1, y_2 \in L_2(Q), \quad \frac{\partial y_1}{\partial t}, \frac{\partial y_2}{\partial t} \in L_2(Q), \quad (2)$$

and $Q = \Omega \times]0,T[$ with the boundary as $\Sigma = \Gamma \times]0,T[$.

The rest of this paper is organized into four sections. Section 2 presents the Sobolev spaces of infinite order, which we refer to later in the paper. In Section 3, the state of the cooperative system with Neumann conditions is discussed. In Section 4, the nascency and sufficient conditions for optimal boundary control are derived. Finally, in Section 5, the formulation of the control problem for boundary observation function is studied.

2. Necessary Spaces

The Sobolev spaces of infinite order operators, which are used in this study, have already been presented in Reference [13]. We list them briefly below:

- $H^\infty(\Omega) = H^\infty\{a_\alpha, 2\}(\Omega) = \left\{\phi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha \phi\right\|_2^2 \le \infty\right\}$,
- The formal conjugate space to the space $H^\infty\{a_\alpha, 2\}(\Omega)$ is defined as

$$H^{-\infty}(\Omega) = H^{-\infty}\{a_\alpha, 2\}(\Omega) = \left\{\psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \psi_\alpha(x)\right\},$$

where $\psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha \psi_\alpha\right\|_2^2 < \infty$.

Then, we have the following chain:

- $H^\infty(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}(\Omega)$,
- $L^2(Q) = L^2(0,T,L^2(\Omega))$ denotes the space of measurable functions

  $t \to \phi(t), t \in]0,T[$, such that $\|\phi\|_{L^2(Q)} = \left(\int_0^T \|\phi(t)\|_2^2 \, dt\right)^{\frac{1}{2}} \le \infty$,

$$(f,g) = \int_0^T (f(t), g(t))_{L^2(\Omega)} \, dt.$$

$L^2(Q)$ is a Hilbert space.

- In a similar manner as that of $L^2(Q)$, we obtain the constructed space $L^2\left(0,T,H^{\infty}(\Omega)\right)=L^2\left(H^{\infty}(Q)\right)$, and the following chains:
- $L^2\left(H^{\infty}(Q)\right)\subseteq L^2(Q)\subseteq L^2\left(H^{-\infty}(Q)\right)$,
- $\left(L^2\left(H^{\infty}(Q)\right)\right)^2\subseteq\left(L^2(Q)\right)^2\subseteq\left(L^2\left(H^{-\infty}(Q)\right)\right)^2$.

Finally,

- $W(0,T)=\left\{f\in L^2\left(H^{\infty}(Q)\right):\frac{\mathrm{d}f}{\mathrm{d}t}\in L^2\left(H^{-\infty}(Q)\right)\right\}$,

with the norm

$$\|f(t)\|_{W(0,T)}=\left(\int_{(0,T)}\|f(t)\|^2_{H^{\infty}\{a_\alpha,2\}(\Omega)}\mathrm{d}t+\int_{(0,T)}\left\|\frac{\mathrm{d}f}{\mathrm{d}t}\right\|^2_{H^{-\infty}\{a_\alpha,2\}(\Omega)}\mathrm{d}t\right)^{1/2},$$

which is also a Hilbert space.

3. State of the System

We study the following 2 × 2 cooperative hyperbolic systems with Neuman conditions:

$$\begin{cases}\frac{\partial^2 y_1}{\partial t^2}+Ay_1(x)=f_1 \quad \text{in } Q,\\ \frac{\partial^2 y_2}{\partial t^2}+Ay_2(x)=f_2 \quad \text{in } Q,\\ y_1,y_2\to 0,\quad |x|\to\infty,\\ \left.\frac{\partial y_1}{\partial \upsilon_A}\right|_{\Sigma}=g_1,\ \left.\frac{\partial y_2}{\partial \upsilon_A}\right|_{\Sigma}=g_2,\\ y_1(x,0)=y_{1,0}(x),\qquad y_2(x,0)=y_{2,0}(x) \quad \text{in } \Omega,\\ \frac{\partial y_1(x,0)}{\partial t}=y_{1,1}(x),\qquad \frac{\partial y_2(x,0)}{\partial t}=y_{2,1}(x) \quad \text{in } \Omega,\end{cases} \tag{3}$$

with $y_1,y_2\in\left(L^2\left(H^{\infty}(Q)\right)\right)^2,\frac{\partial y_1}{\partial t},\frac{\partial y_2}{\partial t}\in\left(L^2\left(H^{\infty}(Q)\right)\right)^2$.

We have the following bilinear form:

$$\pi\left(t,\bar{y},\bar{\phi}\right)=\left(A\bar{y},\bar{\phi}\right),\ \forall\bar{y},\bar{\phi}\in L^2\left(H^{\infty}(Q)\right) \tag{4}$$

where A maps from $\left(L^2\left(H^{\infty}(Q)\right)\right)^2$ onto $\left(L^2\left(H^{-\infty}(Q)\right)\right)^2$, and

$$A\bar{y}(x)=(Ay_1,Ay_2)=(By_1-ay_1-by_2,By_2-cy_1-dy_2), \tag{5}$$

since $B=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}$ is an infinite order operator.

Then,

$$\begin{aligned}\pi\left(t,\bar{y},\bar{\phi}\right)=&\frac{1}{b}\int_\Omega\sum_{|\alpha|=0}^{\infty}a_\alpha D^{|\alpha|}y_1(x)D^{|\alpha|}\phi_1(x)\mathrm{d}x\\&+\frac{1}{c}\int_\Omega\sum_{|\alpha|=0}^{\infty}a_\alpha D^{|\alpha|}y_2(x)D^{|\alpha|}\phi_2(x)\mathrm{d}x\\&-\frac{a}{b}\int_\Omega y_1(x)\phi_1(x)\mathrm{d}x-\int_\Omega y_2(x)\phi_1(x)\mathrm{d}x\\&-\int_\Omega y_1(x)\phi_2(x)\mathrm{d}x-\frac{d}{c}\int_\Omega y_2(x)\phi_2(x)\mathrm{d}x.\end{aligned} \tag{6}$$

Lemma (1):

There exists a constant $\lambda_1, \lambda_2 \succ 0$, such that

$$\pi(t,\overline{y},\overline{y}) + \lambda_1 \|\overline{y}\|^2_{(L^2(\Omega))^2} \geq \lambda_2 \|\overline{y}\|^2_{(L^2(H^\infty(Q)))^2}, \tag{7}$$

that is, $\pi(t,\overline{y},\overline{\phi})$ is coercive on $(L^2(H^\infty(Q)))^2$.

Proof:

We have

$$\pi(t,\overline{y},\overline{y}) = \frac{1}{b}\int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha \left|D^{|\alpha|} y_1(x)\right|^2 \mathrm{d}x + \frac{1}{c}\int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha \left|D^{|\alpha|} y_2(x)\right|^2 \mathrm{d}x$$
$$-\frac{a}{b}\int_\Omega y_1^2 \mathrm{d}x - \frac{d}{c}\int_\Omega y_2^2 \mathrm{d}x - 2\int_\Omega y_1 y_2 \mathrm{d}x,$$

then,

$$\pi(t,\overline{y},\overline{y}) + \frac{a}{b}\int_\Omega y_1^2 \mathrm{d}x + \frac{d}{c}\int_\Omega y_2^2 \mathrm{d}x + 2\int_\Omega y_1 y_2 \mathrm{d}x$$
$$= \frac{1}{b}\int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha \left|D^{|\alpha|} y_1(x)\right|^2 \mathrm{d}x + \frac{1}{c}\int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha \left|D^{|\alpha|} y_2(x)\right|^2 \mathrm{d}x.$$

By the Cauchy Schwarz inequality, we have

$$\pi(t,\overline{y},\overline{y}) + \frac{a}{b}\int_\Omega y_1^2 \mathrm{d}x + \frac{d}{c}\int_\Omega y_2^2 \mathrm{d}x + 2\left(\int_\Omega |y_1|^2 \mathrm{d}x\right)^{1/2}\left(\int_\Omega |y_2|^2 \mathrm{d}x\right)^{1/2}$$
$$\geq \frac{1}{b}\int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha \left|D^{|\alpha|} y_1(x)\right|^2 \mathrm{d}x + \frac{1}{c}\int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha \left|D^{|\alpha|} y_2(x)\right|^2 \mathrm{d}x.$$

Hence,

$$\pi(t,\overline{y},\overline{y}) + \lambda_1 \|\overline{y}\|^2_{(L^2(\Omega))^2} \geq \lambda_2 \|\overline{y}\|^2_{(L^2(H^\infty(Q)))^2}.$$

Moreover, we assume that

$$\pi(t,\overline{y},\overline{\phi}) = \pi(t,\overline{\phi},\overline{y}),\ \forall \overline{y},\overline{\phi} \in (L^2(H^\infty(Q)))^2.$$

Lemma (2):

By satisfying (7), system (3) has a unique solution:

$$\overline{y} = (y_1, y_2) \in (L^2(H^\infty(Q)))^2.$$

Proof:

Let $\overline{\psi} = (\psi_1,\psi_2) \to L(\overline{\psi})$ is defined on $(L^2(H^\infty(Q)))^2$ by

$$L(\overline{\psi}) = \frac{1}{b}\int_Q f_1\psi_1 \mathrm{d}x\mathrm{d}t + \frac{1}{b}\int_\Sigma g_1\psi_1 \mathrm{d}\Gamma\mathrm{d}t + \frac{1}{c}\int_Q f_2\psi_2 \mathrm{d}x\mathrm{d}t + \frac{1}{c}\int_\Sigma g_2\psi_2 \mathrm{d}\Gamma\mathrm{d}t$$
$$+\frac{1}{b}\int_\Omega y_{1,1}(x)\psi_1(x,0)\mathrm{d}x + \frac{1}{c}\int_\Omega y_{2,1}(x)\psi_2(x,0)\mathrm{d}x \tag{8}$$
$$\forall \overline{\psi} = \{\psi_1,\psi_2\} \in (L^2(H^\infty(Q)))^2.$$

Then, by the Lax-Milgram lemma, $\exists! \overline{y} = \{y_1, y_2\} \in (L^2(H^\infty(Q)))^2$ such that

$$\frac{1}{b}\left(\frac{\partial^2}{\partial t^2}(y_1,\psi_1)\right) + \frac{1}{c}\left(\frac{\partial^2}{\partial t^2}(y_2,\psi_2)\right) + \pi(t,\overline{y},\overline{\psi}) = L(\overline{\psi}) \tag{9}$$

$$\forall\,\bar{\psi}=\{\psi_1,\psi_2\}\in\left(L^2\left(H^\infty\left(Q\right)\right)\right)^2.$$

Now, let us multiply system (2) by $\frac{1}{b}\psi_1$ and $\frac{1}{c}\psi_2$ as follows, and then integrate it over Q:

$$\frac{1}{b}\int_Q\left(\frac{\partial^2 y_1}{\partial t^2}+By_1\left(x\right)-ay_1\left(x\right)-by_2\left(x\right)\right)\psi_1\mathrm{d}x\mathrm{d}t=\frac{1}{b}\int_Q f_1\psi_1\mathrm{d}x\mathrm{d}t,$$

$$\frac{1}{c}\int_Q\left(\frac{\partial^2 y_2}{\partial t^2}+By_2\left(x\right)-cy_1\left(x\right)-dy_2\left(x\right)\right)\psi_2\mathrm{d}x\mathrm{d}t=\frac{1}{c}\int_Q f_2\psi_2\mathrm{d}x\mathrm{d}t.$$

Hence,

$$\frac{1}{b}\int_Q\left(\frac{\partial^2 y_1}{\partial t^2}+By_1\left(x\right)\right)\psi_1\mathrm{d}x\mathrm{d}t-\frac{1}{b}\int_Q\left(ay_1\left(x\right)+by_2\left(x\right)\right)\psi_1\mathrm{d}x\mathrm{d}t=\frac{1}{b}\int_Q f_1\psi_1\mathrm{d}x\mathrm{d}t,$$

$$\frac{1}{c}\int_Q\left(\frac{\partial^2 y_2}{\partial t^2}+By_2\left(x\right)\right)\psi_2\mathrm{d}x\mathrm{d}t-\frac{1}{c}\int_Q\left(cy_1\left(x\right)+dy_2\left(x\right)\right)\psi_2\mathrm{d}x\mathrm{d}t=\frac{1}{c}\int_Q f_2\psi_2\mathrm{d}x\mathrm{d}t.$$

By applying Green's formula, we obtain

$$\begin{aligned}&\frac{1}{b}\int_Q\frac{\partial^2\psi_1}{\partial t^2}y_1\mathrm{d}x\mathrm{d}t+\frac{1}{b}\int_Q\sum_{|\alpha|=0}^{\infty}a_\alpha D^{|\alpha|}y_1\left(x\right)D^{|\alpha|}\psi_1\left(x\right)\mathrm{d}x\\&+\frac{1}{b}\int_\Omega\frac{\partial y_1\left(x,0\right)}{\partial t}\psi_1\left(x,0\right)\mathrm{d}x+\frac{1}{b}\int_\Sigma\frac{\partial y_1}{\partial v_A}\psi_1\mathrm{d}\Gamma\mathrm{d}t+\int_Q\left(-\frac{a}{b}y_1-y_2\left(x\right)\right)\psi_1\mathrm{d}x\mathrm{d}t\\&=\frac{1}{b}\int_Q f_1\psi_1\mathrm{d}x\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}&\frac{1}{c}\int_Q\frac{\partial^2\psi_2}{\partial t^2}y_2\mathrm{d}x\mathrm{d}t+\frac{1}{c}\int_Q\sum_{|\alpha|=0}^{\infty}a_\alpha D^{|\alpha|}y_2\left(x\right)D^{|\alpha|}\psi_2\left(x\right)\mathrm{d}x\\&+\frac{1}{c}\int_\Omega\frac{\partial y_2\left(x,0\right)}{\partial t}\psi_2\left(x,0\right)\mathrm{d}x+\frac{1}{c}\int_\Sigma\frac{\partial y_2}{\partial v_A}\psi_2\mathrm{d}\Sigma+\int_Q\left(-y_1-\frac{d}{c}y_2\left(x\right)\right)\psi_2\mathrm{d}x\mathrm{d}t\\&=\frac{1}{c}\int_Q f_2\psi_2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By summing the two equations, and from (6), (8), and (9), we obtain

$$\begin{aligned}&\frac{1}{b}\int_\Omega\frac{\partial y_1\left(x,0\right)}{\partial t}\psi_1\left(x,0\right)\mathrm{d}x+\frac{1}{b}\int_\Sigma\frac{\partial y_1}{\partial v_A}\psi_1\mathrm{d}\Sigma\\&+\frac{1}{c}\int_\Omega\frac{\partial y_2\left(x,0\right)}{\partial t}\psi_2\left(x,0\right)\mathrm{d}x+\frac{1}{c}\int_\Sigma\frac{\partial y_2}{\partial v_A}\psi_2\mathrm{d}\Sigma\\&=\frac{1}{b}\int_\Omega y_{1,1}\left(x\right)\psi_1\left(x,0\right)\mathrm{d}x+\frac{1}{c}\int_\Omega y_{2,1}\left(x\right)\psi_2\left(x,0\right)\mathrm{d}x.\end{aligned}$$

Then, we deduce that

$$\left.\frac{\partial y_1}{\partial v_A}\right|_\Sigma=g_1,\ \left.\frac{\partial y_2}{\partial v_A}\right|_\Sigma=g_2,$$

$$\frac{\partial y_1\left(x,0\right)}{\partial t}=y_{1,1}\left(x\right),\quad\frac{\partial y_2\left(x,0\right)}{\partial t}=y_{2,1}\left(x\right)\quad\text{in }\Omega.$$

Thus, Equation (9) is equivalent to system (2), thereby completing the proof.

4. Control Problem When the Observation Function Is Given on *Q*

The space $U=\left(L^2(\Sigma)\right)^2$ is the space of controls $\bar{u}=(u_1,u_2)$.

The state $\bar{y}(\bar{u})=\left(y_1(u),y_2(u)\right)\in\left(L^2\left(H^\infty(Q)\right)\right)^2$ of the system is given by the solution of

$$\left.\begin{array}{l}\dfrac{\partial^2 y_1(\bar{u})}{\partial t^2}+By_1(\bar{u})=ay_1(\bar{u})+by_2(\bar{u})+f_1,\quad \text{in } Q\\ \dfrac{\partial^2 y_2(\bar{u})}{\partial t^2}+By_2(\bar{u})=cy_1(\bar{u})+dy_2(\bar{u})+f_2,\quad \text{in } Q\\ y_1,y_2\to 0,\quad |x|\to\infty\\ \left.\dfrac{\partial y_1}{\partial v}\right|_\Sigma=g_1+u_1,\ \left.\dfrac{\partial y_2}{\partial v}\right|_\Sigma=g_2+u_2\\ y_1(x,0,\bar{u})=y_{1,0}(x,\bar{u}),\ y_2(x,0,\bar{u})=y_{2,0}(x,\bar{u}),\quad x\in\Omega\\ \dfrac{\partial y_1(x,0,\bar{u})}{\partial t}=y_{1,1}(x),\quad \dfrac{\partial y_2(x,0,\bar{u})}{\partial t}=y_{2,1}(x),\quad x\in\Omega\end{array}\right\}\tag{10}$$

with $y_1,y_2\in\left(L^2\left(H^\infty(Q)\right)\right)^2,\dfrac{\partial y_1}{\partial t},\dfrac{\partial y_2}{\partial t}\in\left(L^2\left(H^\infty(Q)\right)\right)^2$.

The observation equation is given by

$$\bar{z}(\bar{u})=\{z_1(\bar{u}),z_2(\bar{u})\}=\bar{y}(\bar{u})=\{y_1(\bar{u}),y_2(\bar{u})\}\tag{11}$$

The cost function is given by

$$J(u)=\int_Q\left(y_1(\bar{u})-z_{d1}\right)^2\mathrm{d}x\mathrm{d}t+\int_Q\left(y_2(\bar{u})-z_{d2}\right)^2\mathrm{d}x\mathrm{d}t+\left(\bar{N}\bar{u},\bar{u}\right)_{\left(L^2(\Sigma)\right)^2},\tag{12}$$

where $\bar{z}_d=\{z_{d1},z_{d2}\}\in\left(L^2(Q)\right)^2$, and

$$\bar{N}=\{N_1,N_2\}\in\mathcal{L}\left(\left(L^2(\Sigma)\right)^2,\left(L^2(\Sigma)\right)^2\right)$$

is a Hermitian positive definite operator:

$$\left(\bar{N}\bar{u},\bar{u}\right)\geq c\|\bar{u}\|,\ c\succ 0\tag{13}$$

Then, the control problem is to minimize *J* over U_{ad}, which is a closed convex subset of $U=\left(L^2(\Sigma)\right)^2$.

i.e., to determine $\bar{u}$ such that

$$J(\bar{u})=\inf_{\bar{v}\in U_{ad}}J(\bar{v}),\quad \bar{v}=\{v_1,v_2\}.$$

Moreover, we have the following theorem:

Theorem 1:

Assuming that (7), (12), and (13) hold, ∃! the optimal control $\bar{u}=\{u_1,u_2\}\in U_{ad}$, such that $J(\bar{u})\leq J(\bar{v}),\forall\bar{v}=\{v_1,v_2\}\in U_{ad}$ if the following equations and inequalities are satisfied:

$$\left.\begin{aligned}
&\frac{\partial^2 p_1(\bar{u})}{\partial t^2}+Bp_1(\bar{u})-ap_1(\bar{u})-cp_2(\bar{u})=y_1(\bar{u})-z_{d1} \ \text{ in } Q\\
&\frac{\partial^2 p_2(\bar{u})}{\partial t^2}+Bp_2(\bar{u})-bp_1(\bar{u})-dp_2(\bar{u})=y_2(\bar{u})-z_{d2} \ \text{ in } Q\\
&p_1,p_2\to 0 \ \text{ as } |x|\to\infty\\
&\frac{\partial p_1(\bar{u})}{\partial \upsilon}=0,\frac{\partial p_2(\bar{u})}{\partial \upsilon}=0 \quad \text{on } \Sigma\\
&p_1(x,t,\bar{u})=0, p_2(x,t,\bar{u})=0, \quad \text{in } \Omega\\
&\frac{\partial p_1(x,t,\bar{u})}{\partial t}=\frac{\partial p_2(x,t,\bar{u})}{\partial t}=0 \quad \text{in } \Omega
\end{aligned}\right\} \tag{14}$$

with $p_1(\bar{u}),p_2(\bar{u})\in L^2(Q),\frac{\partial p_1(x,t,\bar{u})}{\partial t},\frac{\partial p_2(x,t,\bar{u})}{\partial t}\in L^2(Q)$,

$$\left(\bar{p}(\bar{u})+\bar{N}\bar{u},\bar{v}-\bar{u}\right)_{\left(L^2(\Sigma)\right)^2}\ge 0, \tag{15}$$

together with (10), where $\bar{u}=\{u_1,u_2\}\in U_{ad}$ and $\bar{p}(\bar{u})=\left(p_1(\bar{u}),p_2(\bar{u})\right)$ is the adjoint state.

Proof:

Since $\bar{u}=\{u_1,u_2\}$ is characterized by $J'(\bar{u})\cdot(\bar{v}-\bar{u})\ge 0$, $\forall\bar{v}=\{v_1,v_2\}\in U_{ad}$, which is equivalent to

$$\begin{aligned}
&\int_0^T\left[\left(y_1(\bar{u})-z_{d1},y_1(\bar{v})-y_1(\bar{u})\right)_{L^2(\Omega)}+\left(y_2(\bar{u})-z_{d2},y_2(\bar{v})-y_2(\bar{u})\right)_{L^2(\Omega)}\right]\mathrm{d}t\\
&+\left(\bar{N}\bar{u},\bar{v}-\bar{u}\right)_{\left(L^2(\Sigma)\right)^2}\ge 0.
\end{aligned} \tag{16}$$

Now, since

$$\begin{aligned}
\left(\bar{p},A\bar{y}\right)_{\left(L^2(Q)\right)^2}=&\int_0^T\left(p_1(\bar{u}),\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}\right)y_1(\bar{u})-ay_1(\bar{u})-by_2(\bar{u})\right)_{L^2(\Omega)}\mathrm{d}t\\
&+\int_0^T\left(p_2(\bar{u}),\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}\right)y_2(\bar{u})-cy_1(\bar{u})-dy_2(\bar{u})\right)_{L^2(\Omega)}\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
A\bar{y}(\bar{u})=\Bigg(&\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}\right)y_1(\bar{u})-ay_1(\bar{u})-by_2(\bar{u}),\\
&\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}\right)y_2(\bar{u})-cy_1(\bar{u})-dy_2(\bar{u})\Bigg),
\end{aligned}$$

from (10), we obtain

$$\begin{aligned}
\left(\bar{p},A\bar{y}\right)_{\left(L^2(Q)\right)^2}=&\int_0^T\left(\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}\right)p_1(\bar{u})-ap_1(\bar{u})-cp_2(\bar{u}),y_1(\bar{u})\right)_{L^2(\Omega)}\mathrm{d}t\\
&+\int_0^T\left(\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2|\alpha|}\right)p_2(\bar{u})-bp_1(\bar{u})-dp_2(\bar{u}),y_2(\bar{u})\right)_{L^2(\Omega)}\mathrm{d}t\\
=&\left(A^*\bar{p},\bar{y}\right)_{\left(L^2(Q)\right)^2}.
\end{aligned}$$

Thus,

$$A^*\overline{p}(\overline{u}) = A^*\left(p_1(\overline{u}), p_2(\overline{u})\right)$$
$$= \left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2|\alpha|} p_1(\overline{u}) - ap_1(\overline{u}) - cp_2(\overline{u}),\right.$$
$$\left.\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2|\alpha|} p_2(\overline{u}) - bp_1(\overline{u}) - dp_2(\overline{u})\right)$$

According to the form of the adjoint equation in [1] we have proved system (14).

Now, we transform (16) by using (14) as follows:

$$\int_0^T\left(\frac{\partial^2 p_1(\overline{u})}{\partial t^2} + \left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2|\alpha|}\right)p_1(\overline{u}) - ap_1(\overline{u}) - cp_2(\overline{u}), y_1(\overline{v}) - y_1(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t$$
$$+\int_0^T\left(\frac{\partial^2 p_2(\overline{u})}{\partial t^2} + \left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2|\alpha|}\right)p_2(\overline{u}) - bp_1(\overline{u}) - dp_2(\overline{u}), y_2(\overline{v}) - y_2(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t$$
$$+\left(\overline{N}\overline{u}, \overline{v} - \overline{u}\right)_{(L^2(\Sigma))^2} \geq 0.$$

Then, we obtain

$$\int_0^T\left(p_1(\overline{u}), \left(\frac{\partial^2}{\partial t^2} + \left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2|\alpha|}\right)\right)y_1(\overline{v}) - y_1(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t$$
$$+\int_0^T -a\left(p_1(\overline{u}), y_1(\overline{v}) - y_1(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t + \int_0^T -c\left(p_2(\overline{u}), y_1(\overline{v}) - y_1(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t$$
$$-\int_{(0,T)}\left(\frac{\partial p_1(\overline{u})}{\partial \nu}, y_1(\overline{v}) - y_1(\overline{u})\right)_{L^2(\Gamma)} \mathrm{d}t + \int_{(0,T)}\left(p_1(\overline{u}), \frac{\partial\left(y_1(\overline{v}) - y_1(\overline{u})\right)}{\partial \nu}\right)_{L^2(\Gamma)} \mathrm{d}t$$
$$+\int_0^T\left(p_2(\overline{u}), \left(\frac{\partial^2}{\partial t^2} + \left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2|\alpha|}\right)\right)y_2(\overline{v}) - y_2(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t$$
$$+\int_0^T -b\left(p_2(\overline{u}), y_2(\overline{v}) - y_2(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t + \int_0^T -d\left(p_2(\overline{u}), y_2(\overline{v}) - y_2(\overline{u})\right)_{L^2(\Omega)} \mathrm{d}t$$
$$-\int_{(0,T)}\left(\frac{\partial p_2(\overline{u})}{\partial \nu}, y_2(\overline{v}) - y_2(\overline{u})\right)_{L^2(\Gamma)} \mathrm{d}t + \int_{(0,T)}\left(p_2(\overline{u}), \frac{\partial\left(y_2(\overline{v}) - y_2(\overline{u})\right)}{\partial \nu}\right)_{L^2(\Gamma)} \mathrm{d}t$$
$$+\left(\overline{N}\overline{u}, \overline{v} - \overline{u}\right)_{(L^2(\Sigma))^2} \geq 0.$$

Using (10), we have

$$\int_0^T\left(p_1(\overline{u}), v_1 - u_1\right)_{L^2(\Gamma)} \mathrm{d}t + \int_0^T\left(p_2(\overline{u}), v_2 - u_2\right)_{L^2(\Gamma)} \mathrm{d}t + \left(\overline{N}\overline{u}, \overline{v} - \overline{u}\right)_{(L^2(\Sigma))^2} \geq 0,$$

which is equivalent to

$$\left(\overline{p}(\overline{u}) + \overline{N}\overline{u}, \overline{v} - \overline{u}\right)_{(L^2(\Sigma))^2} \geq 0.$$

Thus, the proof is complete.

5. Boundary Observation Function

Let us define the operator $M \in \mathcal{L}\left(\left(L^2(\Sigma)\right)^2, \left(L^2(\Sigma)\right)^2\right)$ as follows:

$$M\left(y(\overline{u})\big|_{\Sigma}\right)=M\left(\left(y_1(\overline{u})\big|_{\Sigma}\right),\left(y_2(\overline{u})\big|_{\Sigma}\right)\right)=\left(Z_1(\overline{u}),Z_2(\overline{u})\right)=Z(\overline{u});$$

therefore, $Z(\overline{u})$ is the observation equation on Σ.

The cost function $J(v)$ is defined by

$$J(\overline{v})=\left\|y_1(\overline{u})\big|_{\Sigma}-z_{d1}\right\|^2_{L^2(\Sigma)}+\left\|y_2(\overline{u})\big|_{\Sigma}-z_{d2}\right\|^2_{L^2(\Sigma)}+\left(N\overline{v},\overline{v}\right)_{\left(L^2(\Sigma)\right)^2},\tag{17}$$

where $N\in\mathcal{L}\left(\left(L^2(\Sigma)\right)^2,\left(L^2(\Sigma)\right)^2\right)$ is defined as in (13), and

$z_d=\left(z_{d1},z_{d2}\right)\in\left(L^2(\Sigma)\right)^2$.

Then, the control problem is to minimize *J* over U_{ad}, which is a closed convex subset of $U=\left(L^2(\Sigma)\right)^2$, *i.e.*, to determine $\overline{u}=(u_1,u_2)\in U_{ad}$ such that $J(\overline{u})\le J(\overline{v})$.

Since the cost function (16) can be written as [14]

$$J(\overline{v})=a(\overline{v},\overline{v})-2L(\overline{v})+\left\|y(0)-z_d\right\|^2_{\left(L^2(\Sigma)\right)^2},$$

$\exists!\,\overline{u}\in U_{ad}$ such that $J(\overline{u})\le J(\overline{v}),\ \forall\overline{v}\in U_{ad}$.

Based on the above considerations, we obtain the following theorem.

Theorem 2:

Assuming that (7), (13), and (17) hold, the optimal control $\overline{u}=(u_1,u_2)\in\left(L_2(\Sigma)\right)^2$ is determined by the following systems:

$$\begin{cases}\dfrac{\partial^2 p_1(\overline{u})}{\partial t^2}+(-\Delta+q)p_1(\overline{u})-ap_1(\overline{u})-cp_2(\overline{u})=0 & \text{in } Q,\\ \dfrac{\partial^2 p_2(\overline{u})}{\partial t^2}+(-\Delta+q)p_2(\overline{u})-bp_1(\overline{u})-dp_2(\overline{u})=0 & \text{in } Q,\\ p_1,p_2\to 0 \quad \text{as } |x|\to\infty, \\ \left.\dfrac{\partial p_1(\overline{u})}{\partial \nu}\right|_{\Sigma}=y_1(\overline{u})\big|_{\Sigma}-z_{d1},\quad \left.\dfrac{\partial p_2(\overline{u})}{\partial \nu}\right|_{\Sigma}=y_2(\overline{u})\big|_{\Sigma}-z_{d2}, \\ p_1(x,T,\overline{u})=p_2(x,T,\overline{u})=0 & \text{in } \Omega,\\ \dfrac{\partial p_1(x,T,\overline{u})}{\partial t}=\dfrac{\partial p_2(x,T,\overline{u})}{\partial t}=0 & \text{in } \Omega.\end{cases}\tag{18}$$

with $p_1(\overline{u}),p_2(\overline{u})\in L^2\left(H^\infty(Q)\right),\dfrac{\partial p_1(\overline{u})}{\partial t},\dfrac{\partial p_2(\overline{u})}{\partial t}\in L^2\left(H^\infty(Q)\right)$ together with (10) and (15).

Proof:

The optimal control $\overline{u}=(u_1,u_2)\in\left(L_2(\Sigma)\right)^2$ is described by [14]

$$J'(\overline{u})(\overline{v}-\overline{u})\ge 0,\ \forall\overline{v}\in U_{ad},$$

which is equivalent to

$$\begin{aligned}&\int_0^T\left[\left(y_1(\overline{u})-z_{d1},y_1(\overline{v})-y_1(\overline{u})\right)_{L^2(\Gamma)}+\left(y_2(\overline{u})-z_{d2},y_2(\overline{v})-y_2(\overline{u})\right)_{L^2(\Gamma)}\right]\\&+\left(\overline{N}\overline{u},\overline{v}-\overline{u}\right)_{\left(L^2(\Sigma)\right)^2}\ge 0.\end{aligned}\tag{19}$$

According to the form of the adjoint equation in [1],

$$\begin{cases} \dfrac{\partial^2 p(\bar{u})}{\partial t^2} + A^* p(\bar{u}) = 0 & \text{in } Q \\ \left.\dfrac{\partial p(\bar{u})}{\partial \nu}\right|_{\Sigma} = y(\bar{u}) - z_d & \text{on } \Sigma \end{cases}$$

Then, by using theorem 1, we have a unique solution $p(u) \in \left(L^2\left(H^\infty(Q)\right)\right)^2$, which satisfies $p_1(\bar{u}), p_2(\bar{u}) \in L^2\left(H^\infty(Q)\right), \dfrac{\partial p_1(\bar{u})}{\partial t}, \dfrac{\partial p_2(\bar{u})}{\partial t} \in L^2\left(H^\infty(Q)\right)$.

This proves system (18).

Now, from (20) and (18), we have

$$\int_0^T \left(\frac{\partial p_1(\bar{u})}{\partial \nu}, y_1(\bar{v}) - y_1(\bar{u}) \right)_{L^2(\Gamma)} \mathrm{d}t + \int_0^T \left(\frac{\partial p_2(\bar{u})}{\partial \nu}, y_2(\bar{v}) - y_2(\bar{u}) \right)_{L^2(\Gamma)} \mathrm{d}t$$
$$+ \left(\bar{N}\bar{u}, \bar{v} - \bar{u} \right)_{\left(L^2(\Sigma)\right)^2} \ge 0.$$

Using the Green formula, we obtain

$$\int_0^T \left(p_1(\bar{u}), \frac{\partial y_1(\bar{v})}{\partial \nu} - \frac{\partial y_1(\bar{u})}{\partial \nu} \right)_{L^2(\Gamma)} \mathrm{d}t + \int_0^T \left(p_2(\bar{u}), \frac{\partial y_2(\bar{v})}{\partial \nu} - \frac{\partial y_2(\bar{u})}{\partial \nu} \right)_{L^2(\Gamma)} \mathrm{d}t$$
$$+ \left(\bar{N}\bar{u}, \bar{v} - \bar{u} \right)_{\left(L^2(\Sigma)\right)^2} \ge 0.$$

Using (10), we have

$$\int_0^T \left(p_1(\bar{u}), v_1 - u_1 \right)_{L^2(\Gamma)} \mathrm{d}t + \int_0^T \left(p_2(\bar{u}), v_2 - u_2 \right)_{L^2(\Gamma)} \mathrm{d}t + \left(\bar{N}\bar{u}, \bar{v} - \bar{u} \right)_{\left(L^2(\Sigma)\right)^2} \ge 0,$$

which is equivalent to

$$\left(p(\bar{u}) + \bar{N}\bar{u}, \bar{v} - \bar{u} \right)_{\left(L^2(\Sigma)\right)^2} \ge 0.$$

Thus, the proof is complete.

6. Conclusions

In this paper, we have some important results. First of all, we proved the existence and uniqueness of the state for system (2), which is (2 × 2) cooperative hyperbolic systems involving infinite order operators (Lemma 2). Then, we found the necessary and sufficient conditions of optimality for system (10) that give the characterization of optimal control (Theorem 1). Finally, we studied the control problem when the observation function is given on the boundary (Theorem 2).

Also, it is evident that by modifying:

- the nature of the control (distributed, boundary),
- the nature of the observation (distributed, boundary),
- the initial differential system,
- the type of equation (elliptic, parabolic and hyperbolic),
- the type of system (non-cooperative, cooperative),
- the order of equation.

Many of variations on the above problem are possible to study with the help of Lions formalism.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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