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The *g*-Good-Neighbor Connectivity of Some Cartesian Product Graphs

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Abstract

The g-good-neighbor connectivity $\kappa^g(G)$ of G is a generalization of the concept of connectivity $\kappa(G)$, which is just for $\kappa^0(G)=\kappa(G)$, and an important parameter in measuring the fault tolerance and reliability of interconnection network. Many well-known networks can be constructed by the Cartesian products of some simple graphs. In this paper, we determine the g-good-neighbor connectivity of some Cartesian product graphs. We give the exact value of g-good-neighbor connectivity of the Cartesian product of two complete graphs K_m and K_n for $0 \le g \le \left\lfloor \frac{m+n-4}{2} \right\rfloor$, mesh $P_m \times P_n$ for $0 \le g \le 2$, cylindrical grid $P_m \times C_n$ and torus $C_m \times C_n$ for $0 \le g \le 3$.

Keywords

Connectivity, The g-Good-Neighbor Connectivity, Cartesian Product

1. Introduction

We call a multiprocessor system fault-tolerant if it can keep working in case of failure. In the beginning, connectivity and edge connectivity of graph were used to measure the fault-tolerant of system. Later, people found that these two parameters had some defects since they assume that all adjacent vertices or edges of the same vertex may fail at the same time, which is unlikely in real networks. In 1996, Fàbrega and Fiol [1] made some improvements in the connectivity and proposed the concept of g-good neighbor connectivity to measure the fault-tolerant of the multiprocessor.

Let $G=(V,E)$ be a given connected graph with vertices set $V(G)$ and edges set $E(G)$. If u and v are vertices of a graph G, we say u is adjacent to v if there is an edge between u and v. We also say u and v are neighbors. For a vertex

$v \in V$, we by $N(v)$ denote the set of neighbors of v and by $N(S)$ denote the set of neighbors of every vertex in *S*. A set $F \subseteq V$ is called a *g*-good-neighbor faulty set of *G* if $|N(v) \cap (V \setminus F)| \geq g$ for every vertex v in $V - F$. A *g*-good-neighbor cut of *G* is a *g*-good-neighbor faulty set *F* such that $G - F$ is disconnected. We call the minimum cardinality of *g*-good-neighbor cuts the *g*-good-neighbor connectivity of *G*, denoted by $\kappa^g(G)$. Clearly, $\kappa^0(G) = \kappa(G)$ for any graph *G*.

In 2012, Peng *et al.* [2] determined the *g*-good-neighbor conditional diagnosability of hypercube under the PMC model. In 2016, Wang *et al.* [3] showed that 2-good-neighbor connectivity of bubble-Sort Star Graph BS_n is $8n-22$ for $n \geq 5$ and the 2-good-neighbor connectivity of BS_4 is 8. In 2017, Ren and Wang [4] [5] determined the 1-good-neighbor connectivity of locally twisted cubes and the *g*-good-neighbor diagnosability of locally twisted cubes, respectively. In 2018, Wei and Xu [6] determined the 1, 2-good-neighbor conditional diagnosabilities of regular graphs. In 2020, Wang and Wang [7] showed that the 3-good-neighbor connectivity of Modified Bubble-Sort Graphs *MBn* is $8n-24$ for $n \geq 6$. Motivated by these researches, notice that the Cartesian product is an important method to obtain large graphs from smaller ones for designing large-scale interconnection networks [8] [9] [10]. In this paper, we plan to determine the *g*-good-neighbor connectivity of the Cartesian product of graphs.

The Cartesian product of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ whose vertex set is the Cartesian product of the sets $V(G_1)$ and $V(G_2)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ precisely when either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$. In fact, many well-known networks can be constructed by the Cartesian products of some simple graphs and the Cartesian product preserves many nice properties such as regularity, existence of Hamilton cycles and Euler circuits, and transitivity of the initial graphs. See [11] [12] [13].

In this paper, we determine the *g*-good-neighbor connectivity of the Cartesian product of two complete graphs K_m and K_n for $0 \leq g \leq \left\lfloor \frac{m+n-4}{2} \right\rfloor$, mesh $P_m \times P_n$ for $0 \leq g \leq 2$, cylindrical grid $P_m \times C_n$ and torus $C_m \times C_n$ for $0 \leq g \leq 3$. As usual, we by $\Delta(G)$ and $\kappa(G)$ denote the maximum degree and the connectivity of a graph *G*, respectively. Use P_n, C_n and K_n denote path, cycle and complete graph with order *n*.

2. Main Results

In this section, we determine the *g*-good-neighbor connectivity of Cartesian product of two complete graphs K_m and K_n, mesh, cylindrical grid and torus.

Lemma 2.1. [14] *Let G be a connected graph and g be an integer. Then* $\kappa^g(G) \leq \kappa^{g+1}(G)$.

Theorem 2.2. *Let* $K_m \times K_n$ *be Cartesian product of complete graph* K_m *and* K_n *with* $1 \leq m \leq n$ *and g be non-negative integer with* $0 \leq g \leq \left\lfloor \frac{m+n-4}{2} \right\rfloor$.

Then the g-good-neighbor connectivity of $K_m \times K_n$ *is*

1) *For* $g=0$, $\kappa^g(K_m \times K_n)=\kappa(K_m \times K_n)=m+n-2$.

2) *For* $1 \le g \le \left\lfloor \frac{m+n-4}{2} \right\rfloor$,

$$\kappa^g(K_m \times K_n)=\begin{cases} \left\lceil m(g+2)-\frac{(m+2g+4-n)^2}{8} \right\rceil, & n \ge 2g+8-3m; \\ (m-1)(m-2g-6+n)+m(g+2), & n < 2g+8-3m. \end{cases}$$

Proof. Let $G=K_m \times K_n$ with $V(G)=\{w_{ij} \mid w_{ij}=(u_i,v_j) \mid u_i \in V(K_m), v_j \in V(K_n)\}$ for $1 \le i \le m$ and $1 \le j \le n$. Consider G is $m+n-2$ regular, thus, we have $\kappa^0(G)=\kappa(G)=m+n-2$. Suppose F is a g-good neighbor cut set of G with minimum cardinality and let $G-F=G_1 \cup G_2 \cup \cdots \cup G_p$.

Now, we further show that $G_k=K_{m_k} \times K_{n_k}$ with $m_k \le n_k$ for $k=1,2,\cdots,p$. In fact, it is enough if we show that whenever $(u_i,v_r),(u_i,v_s),(u_j,v_r) \in V(G_k)$, then $(u_j,v_s) \in V(G_k)$. On the contrary, if $(u_j,v_s) \in V(G-G_k)$, then we by the definition of $K_m \times K_n$ get $(u_j,v_s) \in F$. Let $F'=F-(u_j,v_s)$, then (u_j,v_s) is adjacent with $(u_i,v_s),(u_j,v_r)$ in $G-F'$ and $\left[G_k \cup \{(u_j,v_s)\}\right]$ is a component of $G-F'$ such that $|N(v) \cap (V(G) \setminus F')| \ge g$ for every $v \in V(G) \setminus F$. This implies F' is also a g-good neighbor cut set of G with $|F'|=|F|-1$. This contradicts to the fact F is of minimum cardinality. So, we have $G_k=K_{m_k} \times K_{n_k}$ with $m_k \le n_k$ for $k=1,2,\cdots,p$. Further, we by the minimality of F know that $G-F$ has exactly two components. This means $G-F=(K_{m_1} \times K_{n_1}) \cup (K_{m_2} \times K_{n_2})$.

Notice that F is a g-good neighbor cut set of G, we have $m_1+n_1 \ge g+2$ and $m_2+n_2=g+2$. Combine this with $m_1=m-m_2$, $n_1=n-n_2$, we get $m+n \ge 2(g+2)$, then $g \le \left\lfloor \frac{m+n}{2} \right\rfloor - 2$. Thus, we have

$$\begin{aligned} |F| &= \min\{m_1 n_2 + m_2 n_1\} \\ &= \min\{(m-m_2)n_2 + m_2(n-n_2)\} \\ &= \min\{(m-2m_2)(g+2-m_2)+nm_2\} \\ &= \min\{2m_2^2-(m+2g+4-n)m_2+m(g+2)\}. \end{aligned}$$

Notice that $2n_1 \ge m_1+n_1 \ge g+2$, we have $n_1 \ge \frac{g+2}{2}$. By $m_2+n_2=g+2$ and $m_1=m-m_2$, $n_1=n-n_2$, we get $m_1+n_1=m+n-(g+2)$. So $m_1 \le m+n-\frac{3(g+2)}{2}$. Thus $\frac{3(g+2)}{2}-n \le m_2 \le m-1$.

Now, let $f(x)=2x^2-(m+2g+4-n)x+m(g+2)$, the following we determine the minimum value of $f(x)$ in interval $\left[\frac{3(g+2)}{2}-n, m-1\right]$.

By $2n \ge m+n \ge 2(g+2)$, we have $n \ge g+2$. Thus

$\frac{m+2g+4-n}{4}-\left(\frac{3(g+2)}{2}-n\right)=\frac{m-n}{4}+n-g-2\geq 0$. By comparing the difference between $\frac{m+2g+4-n}{4}$ and $m-1$, we discuss the minimum value of $f(x)$.

If $n>2g+8-3m$, let $m+2g+4-n=t$, then $\min f(x)=$
$f\left(\frac{m+2g+4-n}{4}\right)=f\left(\frac{t}{4}\right)=2\left(\frac{t}{4}\right)^2-\frac{t^2}{4}+m(g+2)=\left\lceil m(g+2)-\frac{t^2}{8}\right\rceil$;

If $n\leq 2g+8-3m$, then $\min f(x)=f(m-1)=2(m-1)^2-(m+2g+4-n)(m-1)+m(g+2)=(m-1)(m-2g-6+n)+m(g+2)$.

By the above analysis, we get

$$\kappa^g(K_m\times K_n)=|F|=\begin{cases}\left\lceil m(g+2)-\frac{(m+2g+4-n)^2}{8}\right\rceil, & n>2g+8-3m;\\ (m-1)(m-2g-6+n)+m(g+2), & n\leq 2g+8-3m.\end{cases}$$

This completes the proof.

Example 1. The 1-good-neighbor connectivity of $K_3\times K_4$ is 6 with $F=\{w_{12},w_{13},w_{21},w_{24},w_{31},w_{34}\}$, which is shown in Figure 1.

Theorem 2.3. *Let g, m and n be non-negative integers with* $n\geq m\geq 2$. *Then the g-good-neighbor connectivity of mesh* $P_m\times P_n$ *is*

1) *For* $g=0$, $\kappa^g(P_m\times P_n)=\kappa(P_m\times P_n)=2$.

2) *For* $g=1$, $\kappa^g(P_m\times P_n)=\begin{cases}2, & m=2;\\ 3, & m\geq 3.\end{cases}$

3) *For* $g=2$, $\kappa^g(P_m\times P_n)=\begin{cases}m, & 2\leq m\leq 4, n\geq 5;\\ 8, & m=n=4;\\ 4, & m,n\geq 5.\end{cases}$

Proof. Let $P_m\times P_n=G$ with $V(P_m)=\{u_1,u_2,\cdots,u_m\}$ and $V(P_n)=\{v_1,v_2,\cdots,v_n\}$. Then $V(G)=\{w_{ij}\mid w_{ij}=(u_i,v_j)\mid u_i\in V(P_m) \text{ and } v_j\in V(P_n)\}$. Suppose F is a vertex cut set of G, notice that the minimum degree of $G-F$ is always less than 3, so $g=0,1,2$ and by the connectivity $\kappa(G)=2$ we have $\kappa^0(G)=2$. The following we by distinguishing cases to determine $\kappa^g(G)$.

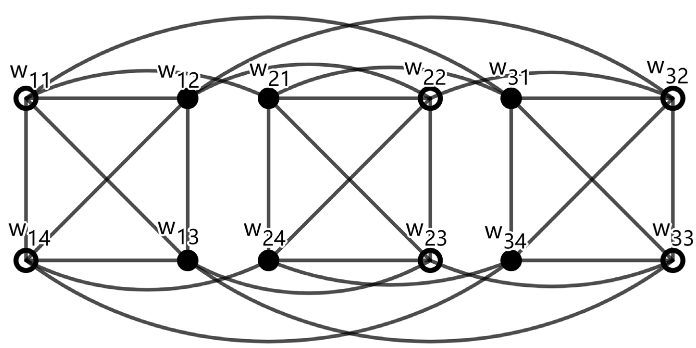

Figure 1. $\kappa^1(K_3\times K_4)=6$.

Case 1. $g=1$.

$g=1$ means $n\neq 2$, so $n\geq 3$.

Subcase 1. $m=2$.

By Lemma 2.1, we have $\kappa^1(G)\geq\kappa^0(G)=2$. On the other hand, let $F=\{w_{1j},w_{2j}\}$ for $j=2$ or $n-1$. It is clear that $G-F$ is disconnected and $\left|N(v)\cap(V(G)\setminus F)\right|\geq 1$ for every $v\in V(G)\setminus F$. By the definition of *g*-good-neighbor connectivity, we have $\kappa^1(G)\leq|F|=2$. Therefore, we get $\kappa^1(G)=2$.

Subcase 2. $m\geq 3$.

First, let $F=\{w_{12},w_{22},w_{31}\}$. Clearly, $G-F$ is disconnected and $\left|N(v)\cap(V(G)\setminus F)\right|\geq 1$ for every $v\in G-F$, so we have $\kappa^1(G)\leq|F|=3$. On the other hand, suppose $F'\subset V(G)$ is a vertex cut set of G such that $\left|N(v)\cap(V(G)\setminus F')\right|\geq 1$ for every $v\in V(G)\setminus F'$. Then $G-F'$ has a component C with $|C|\geq 2$ and the minimum degree of C is $\delta(C)=1$. Further, we have $|F'|\geq 3$. In fact, if $|F'|\leq 2$, then $\delta(G-F')=0$. This implies $\kappa^1(G)=\min|F'|\geq 3$. Therefore, $\kappa^1(G)=3$.

Case 2. $g=2$.

It is clear that $n\neq 2,3,4$ while $m=2$ and $n\neq 3,4$ while $m=3$. Now, we discuss by distinguishing three subcases.

Subcase 1. $2\leq m\leq 4$ and $n\geq 5$.

Let $F=\{w_{i3}\}$ for $1\leq i\leq m$. Notice that F is a cut set of G and $\left|N(v)\cap(V(G)\setminus F)\right|\geq 2$ for every $v\in V(G)\setminus F$, we have $\kappa^2(G)\leq|F|=m$ for $2\leq m\leq 4$. On the other hand, since $\kappa^1(G)\leq\kappa^2(G)$ and $\kappa^1(G)=m$ for $m=2,3$, so we have $\kappa^2(G)\geq m$ for $m=2,3$. Thus $\kappa^2(G)=m$ for $m=2,3$.

Similarly, consider the case for $m=4$. Suppose $F'\subset V(G)$ be a 2-good-neighbor cut of *G*, then $G-F'$ is disconnected and $\left|N(v)\cap(V(G)\setminus F')\right|\geq 2$ for each $v\in(V(G)\setminus F')$. It is not difficult find that $|C|\geq 2$ and $|N(C)|\geq 4$ for every component of $G-F'$. Thus $\kappa^2(G)\geq 4$. On the other hand, let $F_0=\{w_{13},w_{23},w_{31},w_{32}\}$. Clearly, $G-F_0$ is disconnected and $\left|N(v)\cap(V(G)\setminus F_0)\right|\geq 2$ for $v\in V(G)\setminus F_0$. So $\kappa^2(G)\leq|F_0|=4$. And thus we get $\kappa^2(G)=4$ for $m=4$.

Subcase 2. $m=n=4$.

Let $F_0=\{w_{ij}\}$ for $i=1,2$, $j=3,4$ and $i=3,4$, $j=1,2$. It is clear that $G-F_0$ is disconnected and $\left|N(v)\cap(V(G)\setminus F_0)\right|\geq 2$ for every $v\in V(G)\setminus F_0$. Thus $\kappa^2(G)\leq|F_0|=8$. On the other hand, suppose $F\subset V(G)$ is a 2-good-neighbor cut of *G*, then $G-F$ is disconnected and $\left|N(v)\cap(V(G)\setminus F)\right|\geq 2$ for $v\in V(G)\setminus F$. Then, we show $|F|\geq 8$. If not, assume $|F|\leq 7$, then by the structure of *G*, there must be $v\in V(G)\setminus F$ such that $\left|N(v)\cap(V(G)\setminus F)\right|\leq 1$, this contradicts to the choose of *F*. So $\kappa^2(G)\geq|F|\geq 8$ and thus $\kappa^2(G)=8$ for $m=n=4$.

Subcase 3. $n\geq m\geq 5$.

Let $F_0=\{w_{13},w_{23},w_{31},w_{32}\}$. It is clear that $G-F_0$ is disconnected and $\left|N(v)\cap(V(G)\setminus F_0)\right|\geq 2$ for every $v\in V(G)\setminus F_0$. Thus, we have $\kappa^2(G)\leq|F_0|=4$. On the other hand, suppose $F\subset V(G)$ is a 2-good-neighbor cut of G, then $G-F$ is disconnected and $\left|N(v)\cap(V(G)\setminus F)\right|\geq 2$ for $v\in(V(G)\setminus F)$. Notice that each component C of $G-F$ is 2-connected and $\left|N(C)\right|\geq 4$. So $\kappa^2(G)\geq 4$ and then get $\kappa^2(G)=4$.

This completes the proof.

Example 2. The 2-good-neighbor connectivity of $P_5\times P_5$ is 4 with $F=\{w_{31},w_{32},w_{13},w_{23}\}$, which is shown in Figure 2.

Theorem 2.4. *Let g, m and n be non-negative integers with* $m\geq 2, n\geq 3$. *Then the g-good-neighbor connectivity of cylindrical grid* $P_m\times C_n$ *is*

1) *For* $g=0$, $\kappa^g(P_m\times C_n)=\kappa(P_m\times C_n)=3$.

2) *For* $g=1$, $\kappa^g(P_m\times C_n)=\begin{cases}3, & n=3;\\ 4, & n\geq 4.\end{cases}$

3) *For* $g=2$, $\kappa^g(P_m\times C_n)=\begin{cases}n, & 3\leq n\leq 5;\\ 4, & m=2, n\geq 6;\\ 6, & m\geq 3, n\geq 6.\end{cases}$

4) *For* $g=3$, $\kappa^g(P_m\times C_n)=n$ for $m\geq 5$.

Proof. Similarly, let $P_m\times C_n=G$ with $V(P_m)=\{u_1,u_2,\cdots,u_m\}$, $V(C_n)=\{v_1,v_2,\cdots,v_n\}$. Then $V(G)=\{w_{ij}\mid w_{ij}=(u_i,v_j)\mid u_i\in V(P_m)\text{ and } v_j\in V(C_n)\}$. Suppose F is a vertex cut set of G, consider the minimum degree of $G-F$ is not more than 4, so $g=0,1,2$ and 3. By $\kappa(G)=3$, we directly get $\kappa^0(G)=3$. Now we distinguish three cases to determine $\kappa^g(G)$ for $g=1,2,3$.

Case 1. $g=1$.

Subcase 1. $n=3$.

Consider $n=3$ and $g=1$, here $m\geq 3$. First, let $F_0=\{w_{13},w_{21},w_{22}\}$. It is clear that $G-F_0$ is disconnected and $\left|N(v)\cap(V(G)\setminus F_0)\right|\geq 1$ for every $v\in V(G)\setminus F_0$. Thus $k^1(G)\leq 3$. On the other hand, by Lemma 2.1, we have $\kappa^1(G)\geq\kappa^0(G)=3$. So $\kappa^1(G)=3$ for $n=3$.

Subcase 2. $n\geq 4$.

Let $F_0=\{w_{13},w_{21},w_{1n},w_{22}\}$ for $n\geq 4$. It is clear that $G-F_0$ is disconnected and $\left|N(v)\cap(V(G)\setminus F_0)\right|\geq 1$ for every $v\in V(G)\setminus F_0$. Thus, we directly get

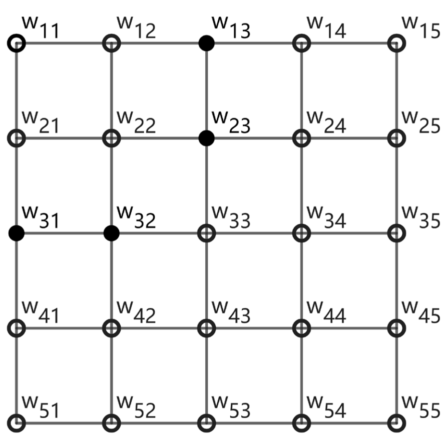

Figure 2. $\kappa^2(P_5\times P_5)=4$.

$\kappa^1(G) \le 4$. On the other hand, suppose that $F \subset V(G)$ is a 1-good-neighbor cut of *G*, then $G-F$ is disconnected and $|N(v) \cap (V(G) \setminus F)| \ge 1$ for $v \in V(G) \setminus F$. By the structure of *G*, we find that there exists a component C of $G-F$ such that $|C| \ge 2$ and $\delta(C) = 1$. Further, we find $|N(C)| \ge 4$ $\kappa^1(G) \ge 4$. Thus $\kappa^1(G) = 4$.

Case 2. $g = 2$.

Subcase 1. $3 \le n \le 5$.

Consider $g = 2$ and $3 \le n \le 4$, here $m \ge 3$. First, let $F_0 = \{w_{2j}\}$ for $1 \le j \le n$. Clearly, $G - F_0$ is disconnected and $|N(v) \cap (V(G) \setminus F_0)| \ge 2$ for every $v \in V(G) \setminus F_0$. Thus $\kappa^2(G) \le |F_0| = n$ for $3 \le n \le 5$. On the other hand, by $\kappa^1(G) \le \kappa^2(G)$ and $\kappa^1(G) = n$ for $n = 3,4$, we have $\kappa^2(G) \ge n$ for $n = 3,4$. Thus $\kappa^2(G) = n$ for $n = 3,4$.

Now, consider the case for $n = 5$ by Case 1, we directly get $\kappa^2(G) \ge \kappa^1(G) = 4$. Further, we can show $\kappa^2(G) \ne 4$. If not, assume $\kappa^2(G) = 4$, then there exists a 2-good neighbor cut set $F' \subset V(G)$ with $|F'| = 4$ such that $G - F'$ is disconnected. Combine this with the structure of *G*, there always exists a vertex $v \in (V(G) \setminus F')$ satisfies $|N(v) \cap (V(G) \setminus F')| \le 1$. This contradicts to the choose of F'. Thus $\kappa^2(G) \ne 4$ and then $\kappa^2(G) \ge 5$. So $\kappa^2(G) = n$ for $n = 5$.

Subcase 2. $m \ge 2$ and $n \ge 6$.

First, consider the case for $m = 2$ and $n \ge 6$. Let $F_0 = \{w_{i3}, w_{in}\}$ for $1 \le i \le m$. Clearly, $G - F_0$ is disconnected and $|N(v) \cap (V(G) \setminus F_0)| \ge 2$ for every $v \in V(G) \setminus F_0$. Thus, we have $\kappa^2(G) \le |F_0| = 2m$. Notice that $\kappa^1(G) = 4$, then $\kappa^2(G) \ge 4 = 2m$. So $\kappa^2(G) = 2m$ for $m = 2$.

Next, consider the case for $m \ge 3$ and $n \ge 6$. Let $F_1 = \{w_{i3}, w_{in}\}$ with $1 \le i \le m$ for $m = 3$ and $F_2 = \{w_{13}, w_{1n}, w_{23}, w_{2n}, w_{31}, w_{32}\}$ for $m > 3$. It is clear that $G - F_1$ and $G - F_2$ are disconnected and $|N(v) \cap (V(G) \setminus F_i)| \ge 2$ for every $v \in V(G) \setminus F_i$ for $i = 1,2$. Thus, we have $\kappa^2(G) \le |F_i| = 6$. On the other hand, suppose that $F \subset V(G)$ is a 2-good-neighbor cut of *G*, then $G - F$ is disconnected and $|N(v) \cap (V(G) \setminus F)| \ge 2$ for $v \in V(G) \setminus F$. This follows that each component C of $G - F$ satisfies $|C| \ge 4$ and thus $|N(C)| \ge 6$. So, we have $\kappa^2(G) \ge 6$ and thus $\kappa^2(G) = 6$ while $m \ge 3$ and $n \ge 6$.

Case 3. $g = 3$.

Suppose F' is a 3-good-neighbor cut of *G*, $g = 3$ means component of $G - F'$ is such as $P_k \times C_n$ for $k \ge 2$, so here consider $m \ge 5$. Notice that $w_{ij} \in F'$ for all $1 \le j \le n$, if $w_{ij} \in F'$ for some *j*. Thus $|F'| \ge n$ and $\kappa^3(G) \ge |F'| \ge n$. On the other hand, let $F_0 = \{w_{3j}\}$ for $1 \le j \le n$. Clearly, $G - F_0$ is disconnected and $|N(v) \cap (V(G) \setminus F_0)| \ge 3$ for every $v \in V(G) \setminus F_0$. So we have $\kappa^3(G) \le n$. Therefore, we get $\kappa^3(G) = n$.

This completes the proof.

Example: The 3-good-neighbor connectivity of $P_5 \times C_6$ is 6 with $F = \{w_{31}, w_{32}, w_{33}, w_{34}, w_{35}, w_{36}\}$, which is shown in **Figure 3**.

Theorem 2.5 *Let g, m and n be non-negative integers with* $n \ge m \ge 3$. *Then the g-good-neighbor connectivity of torus* $C_m \times C_n$ *is*

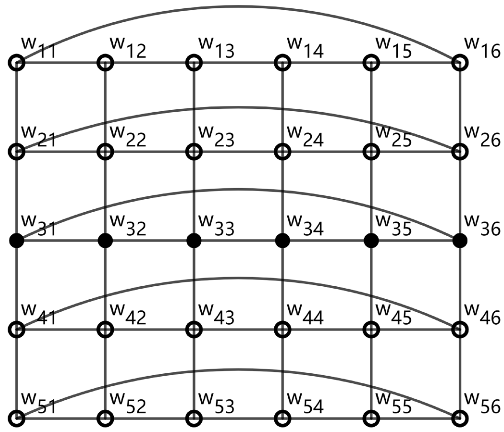

Figure 3. $\kappa^3(P_5 \times C_6) = 6$.

1) *For* $g = 0$, $\kappa^g(C_m \times C_n) = 4$.

2) *For* $g = 1$, $\kappa^g(C_m \times C_n) = \begin{cases} 5, & m = 3; \\ 6, & m \geq 4. \end{cases}$

3) *For* $g = 2$, $\kappa^g(C_m \times C_n) = \begin{cases} 2m, & 3 \leq m \leq 4; \\ 8, & m \geq 5. \end{cases}$

4) *For* $g = 3$, $\kappa^g(C_m \times C_n) = 2m$ for $n \geq 6$.

Proof. Let $C_m \times C_n = G$ and $V(C_m) = \{u_1, u_2, \cdots, u_m\}$, $V(C_n) = \{v_1, v_2, \cdots, v_n\}$. Then $V(G) = \{w_{ij} \mid w_{ij} = (u_i, v_j) \mid u_i \in V(C_m) \text{ and } v_j \in V(C_n)\}$. Suppose that F is a vertex cut set of *G*, it is not difficult find the minimum degree of $G - F$ is not more than 4. So here, we only consider $g = 0,1,2$ and 3. Notice that G is 4-regular, so we directly get $\kappa^0(G) = \kappa(G) = 4$. Now, we distinguish three cases to determine $\kappa^g(G)$ by $g = 1,2,3$.

Case 1. $g = 1$.

Subcase 1. $m = 3$.

Let $F_0 = \{w_{12}, w_{1n}, w_{21}, w_{32}, w_{3n}\}$. It is clear that $G - F_0$ is disconnected and $|N(v) \cap (V(G) \setminus F_0)| \geq 1$ for every $v \in V(G) \setminus F_0$. So we have $\kappa^1(G) \leq |F_0| = 5$. On the other hand, it is clear that $\kappa^1(G) \geq \kappa^0(G) = 4$. Further, we can show $\kappa^1(G) \neq 4$. If not, assume $\kappa^1(G) = 4$, then there exists a 1-good-neighbor cut $F \subset V(G)$ with $|F| = 4$ such that $G - F$ is disconnected. Notice that G is 4-regular, there always exists a vertex $v_0 \in V(G) \setminus F$ such that $|N(v_0) \cap (V(G) \setminus F)| = 0$. This contradicts to the choice of *F*. Thus, we get $\kappa^1(G) \geq 5$. So $\kappa^1(G) = 5$.

Subcase 2. $m \geq 4$.

Let $F_0 = \{w_{13}, w_{1n}, w_{21}, w_{22}, w_{m1}, w_{m2}\}$. Then $G - F_0$ is disconnected and $|N(v) \cap (V(G) \setminus F_0)| \geq 1$ for every $v \in (V(G) \setminus F_0)$. Thus, we get $\kappa^1(G) \leq |F_0| = 6$. Now, we show $\kappa^1(G) \geq 6$. Suppose $F \subset V(G)$ is a 1-good-neighbor cut of *G*, then $G - F$ is disconnected and $|N(v) \cap (V(G) \setminus F)| \geq 1$ for $v \in (V(G) \setminus F)$. Thus, there exist a component C with $|C| \geq 2$ such that $\delta(C) = 1$. Notice that each pair nonadjacent vertices of G has at most two common neighbor vertices and two adjacent vertices of G has no common neighbor vertices in G, then we get $N(C)\geq 6$. This means $|F'|\geq 6$. So, we get $\kappa^1(G)\geq 6$.

Case 2. $g=2$.

Subcase 1. $3\leq m\leq 4$.

Consider $g=2$, so here $n\geq 4$. Let $F_0=\{w_{ij}\}$ for $j=2,n$ and $1\leq i\leq m$. Clearly, $G-F_0$ is disconnected and $|N(v)\cap(V(G)\setminus F_0)|\geq 2$ for every $v\in V(G)\setminus F_0$. Thus $\kappa^2(G)\leq|F_0|=2m$ while $3\leq m\leq 4$. On the other hand, suppose $F\subset V(G)$ is a 2-good-neighbor cut of G, then $G-F$ has a component C with $|C|\geq 4$ and $\delta(C)=2$. Notice that G is 4-regular and each pair nonadjacent vertices has at most two common neighbor vertices and two adjacent vertices has no common neighbor vertices in G, it follows that $|N(C)|\geq 2m$ for $3\leq m\leq 4$. This means $|F|\geq 2m$ and we get $\kappa^2(G)\geq 2m$. Thus $\kappa^2(G)=2m$ for $3\leq m\leq 4$.

Subcase 2. $m\geq 5$.

Let $F_0=\{w_{ij}\}$ for $i=3,m$ while $j=1,2$ and $i=1,2$ while $j=3,n$. It is clear that $G-F_0$ is disconnected and $|N(v)\cap(V(G)\setminus F_0)|\geq 2$ for every $v\in V(G)\setminus F_0$. So we get $\kappa^2(G)\leq 8$. On the other hand, suppose $F\subset V(G)$ is a 2-good-neighbor cut of G, then $G-F$ has a component C with $|C|\geq 4$ and $\delta(C)=2$. Consider G is 4-regular, we similarly get $|N(C)|\geq 8$. Thus $\kappa^2(G)\geq 8$. So, we get $\kappa^2(G)=8$.

Case 3. $g=3$.

Consider $g=3$, so here $n\geq 6$. Let $F_0=\{w_{ij}\}$ for $j=3,n$ and $1\leq i\leq m$. Then $G-F_0$ is disconnected and $|N(v)\cap(V(G)\setminus F_0)|\geq 3$ for every $v\in V(G)\setminus F_0$. So, we have $\kappa^3(G)\leq|F_0|=2m$ for $n\geq 6$. On the other hand, if $3\leq m\leq 4$, by Lemma 2.1, we have $\kappa^3(G)\geq\kappa^2(G)=2m$. Thus $\kappa^3(G)=2m$

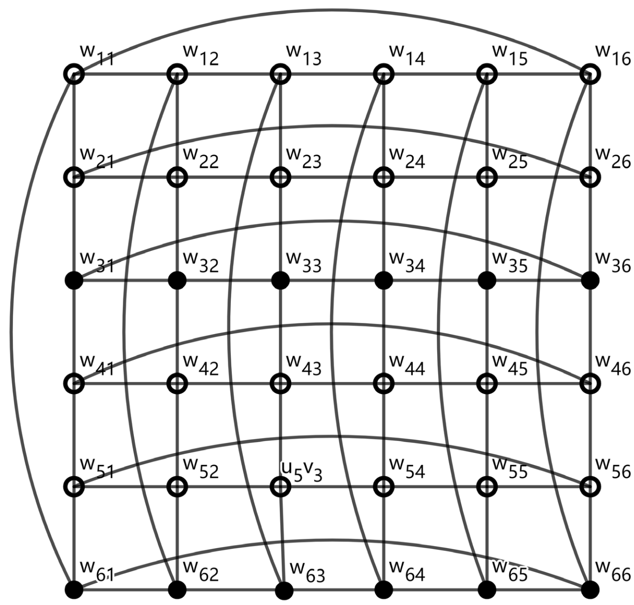

Figure 4. $\kappa^3(C_6\times C_6)=12$.

for $3 \le m \le 4$. If $m \ge 5$, suppose F is a 3-good-neighbor cut of G, it is not difficult find that $w_{ij} \in F$ for all i if $w_{ij} \in F$ for some i and j. Combine this with $\left|N(v) \cap (V(G) \setminus F)\right| \ge 3$ for every $v \in V(G) \setminus F$, we have $|F| \ge 2m$ and $\kappa^3(G) \ge 2m$. So, we get $\kappa^3(G) = 2m$.

This completes the proof.

Example: The 3-good-neighbor connectivity of $C_6 \times C_6$ is 12 with $F = \{w_{31}, w_{32}, w_{33}, w_{34}, w_{35}, w_{36}, w_{61}, w_{62}, w_{63}, w_{64}, w_{65}, w_{66}\}$, which is shown in **Figure 4**.

3. Concluding Remark

In this paper, we focus our attention on the g-good neighbor connectivity of some Cartesian product graphs. We have determined the g-good-neighbor connectivity of the Cartesian product of two complete graphs K_m and K_n for $0 \le g \le \left\lfloor \frac{m+n-4}{2} \right\rfloor$, mesh $P_m \times P_n$ for $0 \le g \le 2$, cylindrical grid $P_m \times C_n$ and torus $C_m \times C_n$ for $0 \le g \le 3$. But the g-good neighbor connectivity of the Cartesian product for the general graphs is still unknown, even for the bounds. In the future, we will devote ourselves to this research.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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