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Optimal Insurance with Background Risk and Belief Heterogeneity

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Abstract

In this paper, we study an optimal insurance model by maximizing the insurer's expected utility in the presence of background risk and belief heterogeneity. When the insurance premium is calculated by the generalized Wang's premium principle, we prove the existence and uniqueness of the optimal solution and give a necessary and sufficient condition for the optimal insurance policy. With the help of these results, we consider the optimality of no insurance and full insurance and give more concise conditions.

Keywords

Optimal Insurance, Background Risk, Belief Heterogeneity, Generalized Wang's Premium Principle, No Insurance, Full Insurance

1. Introduction

In the past half century, how to design an optimal insurance policy has attracted great attention from academics and practitioners. The study of optimal insurance design has become a cornerstone in insurance economics. As far as we know, the pioneering work is attributed to Arrow (1963), in which the optimal insurance problem is studied by maximizing the expected utility (EU) of insurer's final wealth. When the reinsurance premium is calculated by expected value principle, the stop-loss contract is shown to be the optimal solution. Later Arrow's model has been extended in two directions. One direction is to choose other reasonable premium principles for consideration. For example Young (1999) studies the optimal insurance design under Wang's premium principle. Kaluszka (2001) investigates optimal reinsurance under mean-variance premium principles. The other direction is to choose other optimization criteria. Van Heerwaarden et al. (1989) and Chi & Lin (2014) generalize Arrow's result by assuming a quite general optimization criterion that preserves the stop-loss order.

Since the 1990s, VaR, TVaR and other risk measures have been widely used by the financial and insurance. Their properties have been studied in detail, and thus more and more researchers take minimizing risk as optimization criterion in literatures such as Cai & Tan (2007), Zhuang, Weng et al. (2016) and Assa (2015).

However, the authors consider the single-risk model in all the above literatures. In fact, an insured may face multiple sources of risks, where one major risk is to be insured and other risks such as investment risk and operational risk are either uninsurable or not to be insured. These risks are often combined together and treated as background risk in insurance economics. There are many different dependence structures between insurable risk and background risk. For more detailed discussion, please refer to Dana & Scarsini (2007), Lehmann (2012), Colangelo, Hu, & Shaked (2008) and Colangelo, Scarsini, & Shaked (2005). The optimal insurance design with background risk has attracted great attention since the early work of Doherty & Schlesinger (1983). They investigate the optimal deductible level of the stop-loss insurance when the insured's initial wealth is random. Then this problem has been reconsidered by a number of authors, for example Cai & Wei (2012), Lu et al. (2012) and Chi & Wei (2020).

In the aforementioned studies, it is assumed that the insured and the insurer have the same probability belief for the underlying random loss. Actually, Savage (1972) claimed that an individual usually makes decision by his personal view of the underlying probability of the random loss. Since both the insured and the insurer possess different information about the random loss in an insurance contract, it makes sense that we should assume that insurer and insured have heterogeneous beliefs. The study of optimal insurance problem with belief heterogeneity has attracted the attention of some researchers recently. The first work devoting to the optimal insurance contract with heterogeneous beliefs is attributed to Marshall (1992). Subsequently, there are more literatures that have done some exploration in the field of optimal insurance and reinsurance design with heterogeneous beliefs. With the heterogeneous beliefs in the sense of monotone likelihood ratio (MLH) order, Jiang et al. (2018) study the pareto-optimal insurance contract by maximizing the EU of both two parties. Chi & Zhuang (2020) study the optimal reinsurance from the perspective of insurer by maximizing the EU of insurer while both the insurer and reinsurer have heterogeneous beliefs. Yu & Fang (2020) also study the optimal reinsurance from the perspective of insurer by maximizing the EU of insurer but the reinsurance premium is calculated by distortion premium principle.

Although there have been many literatures on the optimal insurance problem under the assumptions of belief heterogeneity or background risks, there are still very few literatures that put these two conditions into the insurance model at the same time. In this paper, we study an optimal insurance model by maximizing the insurer's expected utility in the presence of background risk and belief heterogeneity. This model generalizes (Chi & Wei, 2020) because of the presence of the belief heterogeneity between the insurer and the insured. We assume that there are no restrictions imposed to the form of belief heterogeneity, hence allowing for much flexibility. We give a necessary and sufficient condition for the optimal insurance policy. According to the results, we explore the optimality of no insurance and full insurance when the dependence structure between the insurable risk X and the background risk Y is assumed to be the positively quadrant dependent or negatively quadrant dependent.

The rest of the paper is organized as follows. In Section 2, we give definitions and propose an optimal insurance problem. In Section 3, when the insurance premium is calculated by the generalized Wang's premium principle, we prove the existence and uniqueness of the optimal solution and give a necessary and sufficient condition for the optimal insurance policy. With the help of these results, we consider the optimality of no insurance and full insurance and give more concise conditions.

2. Problem Formulation

Let (Ω, F) be a measurable space. We consider a one-period model with an insured and an insurer. The insured is endowed with initial wealth w_0 faces two sources of risk X and Y, where X is a non-negative bounded random variable representing an insurable risk and Y is the background risk and may be negative. The insured is endowed with beliefs given by subjective probability measure P. Both X and Y are defined on the probability space (Ω, F, P) with finite means. In order to reduce the risk exposure, the insured purchases an insurance contract for the insurable risk X, in which the insurer covers an amount of risk $f(X)$ and the insured retains the rest of the loss $I_f(X) = X - f(X)$. The losses $f(X)$ and $I_f(X)$ are called ceded loss and retained loss, while $f(x)$ and $I_f(x)$ are known as the ceded loss function and the retained loss function, respectively.

In this paper, we assume that the admissible insurance contract satisfies the principle of indemnity, which is expressed as $0 \le f(x) \le x$. This principle is widely used in insurance. However, this constraint is insufficient to exclude ex post moral hazard. In order to reduce ex post moral hazard, Huberman et al. (1983) suggest that insurance contract should satisfy the incentive compatible constraint, which means that the more the realized loss, the more paid by both the insured and the insurer. Mathematically, this implies that both the ceded loss function and the retained loss function should be increasing. Therefore, throughout the paper, we assume that the admissible set of ceded loss functions is given by

$$C = \left\{ f(x): 0 \le f(x) \le x,\ \text{both } f(x) \text{ and } I_f(x) \text{ are increasing functions} \right\}$$

It is shown in Chi & Tan (2011) that the incentive compatible constraint is equivalent to $0 \le f'(x) \le 1$, then the admissible set of ceded loss functions C is also written by

$$C = \{ f(x) : 0 \le f(x) \le x, 0 \le f'(x) \le 1 \}$$

Since the insurer covers the risk X, the insured will pay an additional cost in the form of insurance premium to the insurer. We denote the insurance premium by $\Pi(f(X))$ which corresponds to the ceded loss $f(X)$. In this paper, we assume that the insurer is risk-neutral and make use of generalized Wang's premium principles to price insurance premium. Such a premium principle is defined via distortion risk measures, more details about distortion risk measures refer to Sereda et al. (2010), and Dhaene et al. (2012). The generalized Wang's premium of a non-negative random variable X is defined as

$$\Pi(X) = (1+\rho)\int_0^{\infty} g(Q(X > x))\mathrm{d}x := (1+\rho)E_g^Q(X) \tag{1}$$

where $\rho \ge 0$ is the so-called safety loading factor. $g(\cdot)$ is a distortion function which is increasing and satisfies $g(0) = 0$ and $g(1) = 1$. Q defined in (Ω, F) is the subjective probability measure of the insurer. The probabilistic beliefs of the insurer may be different from that of the insured, so Q may be different from P.

It is worth noting, in the above definition, when $g(x) = x$, the generalized Wang's premium principle recovers the expected value premium principle. Furthermore, when the distortion function is concave and $\rho = 0$, the generalized Wang's premium principle recovers Wang's premium principle.

Let $S_X^P(x)$ and $S_X^Q(x)$ be the survival functions of X under probability measures P and Q. Define

$$x_P := \inf\{x \in R : S_X^P(x) = 0\}$$
$$x_Q := \inf\{x \in R : S_X^Q(x) = 0\}$$

then x_P and x_Q are the essential supremum of X under P and Q. Furthermore, x_P and x_Q are finite since X is a bounded random variable.

With an insurance contract $f(x)$, the wealth for the insured is given by

$$\begin{aligned} W_f(X,Y) &= w_0 - I_f(X) - Y - \Pi(f(X)) \\ &= w_0 - Y - X + f(X) - (1+\rho)E_g^Q(f(X)) \end{aligned}$$

In this paper, we assume that the insured's preference is characterized with the expected utility theory. That is to say, from the view of mathematics, the optimization problem is formulated as

$$\max_{f \in C} E^P\left[U\left(W_f(X,Y)\right)\right] \tag{2}$$

where $U(\cdot)$ is the insured's utility function. We assume that $U'(\cdot) > 0$ and $U''(\cdot) < 0$, which means the utility function is increasing and the insured is risk-averse.

3. Optimal Insurance Contract

Theorem 3.1. There exists a function $f^* \in C$ such that

$$E^P\left[U\left(W_{f^*}(X,Y)\right)\right] = \max_{f\in C} E^P\left[U\left(W_f(X,Y)\right)\right]$$

Proof. We can note that the supremum $\sup_{f\in C} E^P\left[U\left(W_f(X,Y)\right)\right]$ exists. Define $M = \sup_{f\in C} E^P\left[U\left(W_f(X,Y)\right)\right]$, then there exists a sequence $\{f_n, n\geq 1\}\subseteq C$ such that

$$\lim_{n\to\infty} E^P\left[U\left(W_{f_n}(X,Y)\right)\right] = M$$

Since it is shown in Chi & Tan (2011) that $|f(x)-f(y)|\leq |x-y|$ for every $f\in C$ and any $x,y\geq 0$, then the sequence $\{f_n, n\geq 1\}$ is equi-continuous. Furthermore, the sequence $\{f_n, n\geq 1\}$ is uniformly bounded since X is a bounded random variable. According to the Arzela-Ascoli theorem, there exists a subsequence $\{f_{n_k}, k\geq 1\}$ that converges uniformly to a continuous function f^* on the closed interval $[0, x_P]$. Define $f^*(x) = f^*(x_P)$ for any $x > x_P$. It is easy to verify that $f^*\in C$ and $W_{f_{n_k}}(X,Y)\to W_{f^*}(X,Y)$ a.s. Because $W_{f_{n_k}}(X,Y)\leq w_0 - Y$ and $U(\cdot)$ is increasing, we have

$$\begin{aligned}
&E^P\left[U(w_0-Y)\right] - E^P\left[U\left(W_{f^*}(X,Y)\right)\right]\\
&= E^P\left[\lim_{k\to\infty}\left(U(w_0-Y) - U\left(W_{f_{n_k}}(X,Y)\right)\right)\right]\\
&\leq \liminf_{k\to\infty} E^P\left[U(w_0-Y) - U\left(W_{f_{n_k}}(X,Y)\right)\right]\\
&= E^P\left[U(w_0-Y)\right] - M
\end{aligned}$$

where the first equality follows from the continuity of $U(\cdot)$ and the inequality follows from Fatou's lemma. This implies $M\leq E^P\left[U\left(W_{f^*}(X,Y)\right)\right]$. Since $M\geq E^P\left[U\left(W_{f^*}(X,Y)\right)\right]$ according to the definition of *M*, we have $M = E^P\left[U\left(W_{f^*}(X,Y)\right)\right]$. Therefore, $f^*(x)$ is a solution to the problem (2).

Theorem 3.2. If one of the following conditions is satisfied, the optimal solution to problem (2) is unique in the sense that $f_1(X) = f_2(X)$ a.s. for any two solutions f_1 and f_2.

1) $\rho > 0$ and Q is absolutely continuous with respect to *P*,

2) $P(X=0) > 0$.

Proof. Assume that f_1 and f_2 are optimal solutions to problem (2), then we have $E^P\left[U\left(W_{f_1}(X,Y)\right)\right] = E^P\left[U\left(W_{f_2}(X,Y)\right)\right] = M$. For any $\lambda\in[0,1]$, we define $f_\lambda(x) = \lambda f_1(x) + (1-\lambda) f_2(x)$. It is easy to see that $f_\lambda\in C$ and hence $E^P\left[U\left(W_{\lambda f_1+(1-\lambda)f_2}(X,Y)\right)\right]\leq M$. On the other hand, the concavity of $U(\cdot)$ leads to

$$\begin{aligned}
&E^P\left[U\left(W_{\lambda f_1+(1-\lambda)f_2}(X,Y)\right)\right]\\
&\geq \lambda E^P\left[U\left(W_{f_1}(X,Y)\right)\right] + (1-\lambda)E^P\left[U\left(W_{f_2}(X,Y)\right)\right] = M
\end{aligned}$$

Therefore, we can obtain

$$E^P\left[U\left(W_{f_\lambda}(X,Y)\right)\right]=E^P\left[U\left(W_{f_1}(X,Y)\right)\right]=E^P\left[U\left(W_{f_2}(X,Y)\right)\right]=M \quad (3)$$

This implies that $E^P\left[U\left(W_{f_\lambda}(X,Y)\right)-U\left(W_{f_1}(X,Y)\right)\right]=0$. Since $U(\cdot)$ is concave, we have $U\left(W_{f_\lambda}(X,Y)\right)-U\left(W_{f_1}(X,Y)\right)=0$ almost surely under *P*. This can imply $W_{f_1}(X,Y)=W_{f_2}(X,Y)$ almost surely under *P*, or equivalently

$$f_1(X)-(1+\rho)E_g^Q\left[f_1(X)\right]=f_2(X)-(1+\rho)E_g^Q\left[f_2(X)\right] \text{ a.s.} \quad (4)$$

If condition (1) is satisfied, we have

$$f_1(X)-(1+\rho)E_g^Q\left[f_1(X)\right]=f_2(X)-(1+\rho)E_g^Q\left[f_2(X)\right]$$

almost surely under *Q*. This leads to

$$E_g^Q\left[f_1(X)-(1+\rho)E_g^Q\left[f_1(X)\right]\right]=E_g^Q\left[f_2(X)-(1+\rho)E_g^Q\left[f_2(X)\right]\right] \quad (5)$$

Using the translation invariance of distortion risk measures and $\rho>0$, we obtain $E_g^Q\left[f_1(X)\right]=E_g^Q\left[f_2(X)\right]$. Therefore, we obtain $f_1(X)=f_2(X)$ almost surely under *P* by (4).

If condition (2) is satisfied, noting that $f_1(0)=f_2(0)=0$, we can get $E_g^Q\left[f_1(X)\right]=E_g^Q\left[f_2(X)\right]$ by letting X approximate the zero. Therefore, we obtain $f_1(X)=f_2(X)$ almost surely under *P* by (4).

Theorem 3.3. The ceded loss function $f^*(x)$ solves optimization problem (2) if and only if $f^*(x)$ satisfies the following representation

$$f^*(x)=\int_0^x h^*(t)\,\mathrm{d}t \quad (6)$$

$$h^*(t)=\begin{cases}0 & L(t)<0\\ k(t) & L(t)=0\\ 1 & L(t)>0\end{cases} \quad (7)$$

for all $x\in[0,\infty)$, where

$$L(t)=\frac{E^P\left[U'\left(W_{f^*}(X,Y)\right)I_{(X>t)}\right]}{E^P\left[U'\left(W_{f^*}(X,Y)\right)\right]}-(1+\rho)g\left(S_X^Q(x)\right) \quad (8)$$

$I_{(X>t)}$ is indicator random variable and $k(t)$ is measurable and $[0,1]$-value function on $\{t\in[0,\infty):L(t)=0\}$.

Proof. For the given ceded loss function $f^*(x)$ and any admissible ceded loss function $f(x)\in C$, we define

$$f_\beta(x)=\beta f^*(x)+(1-\beta)f(x),$$

$$\phi(\beta)=E^P\left[U\left(W_{f_\beta}(X,Y)\right)\right],\quad \beta\in[0,1].$$

It is easy to verify that $f_\beta(x)\in C$ and $\phi(\beta)$ is concave because of the concavity of $U(\cdot)$. Furthermore, since $f^*(x)$ and $f(x)$ are differentiable almost everywhere, then there exist two $[0,1]$-value functions $h^*(x)$ and $h(x)$ such that $f^*(x)=\int_0^x h^*(t)\,\mathrm{d}t$ and $f(x)=\int_0^x h(t)\,\mathrm{d}t$.

If the ceded loss function $f^*(x)$ is an optimal solution to optimization problem (2), the optimality of $f^*(x)$ implies $\phi'(\beta)\big|_{\beta=1}\ge 0$, which is equivalent to

$$\begin{aligned}\phi'(\beta)\big|_{\beta=1} &= E^P\left[U'\left(W_{f^*}(X,Y)\right)\left(f^*(X)-f(X)-(1+\rho)E_g^Q\left[f^*(X)-f(X)\right]\right)\right]\\ &= E^P\left[U'\left(W_{f^*}(X,Y)\right)\left(\int_0^X\left(h^*(t)-h(t)\right)\mathrm{d}t-(1+\rho)E_g^Q\left[\int_0^X\left(h^*(t)-h(t)\right)\mathrm{d}t\right]\right)\right]\\ &= E^P\left[U'\left(W_{f^*}(X,Y)\right)\left(\int_0^\infty\left(h^*(t)-h(t)\right)\mathrm{d}t\cdot\left(I_{(X>t)}-(1+\rho)g\left(S_X^Q(x)\right)\right)\right)\right]\\ &= \int_0^\infty E^P\left[U'\left(W_{f^*}(X,Y)\right)\cdot\left(I_{(X>t)}-(1+\rho)g\left(S_X^Q(x)\right)\right)\right]\left(h^*(t)-h(t)\right)\mathrm{d}t\\ &= E^P\left[U'\left(W_{f^*}(X,Y)\right)\right]\int_0^\infty L(t)\cdot\left(h^*(t)-h(t)\right)\mathrm{d}t\ge 0\end{aligned}$$

Note that the above inequality holds true for any $f(x)\in C$, the result (7) follows directly. If the ceded loss function $f^*(x)$ satisfies (6) and (7), then we have

$$\begin{aligned}&E^P\left[U\left(W_{f^*}(X,Y)\right)\right]-E^P\left[U\left(W_f(X,Y)\right)\right]\\ &=\phi(1)-\phi(0)\ge\phi'(\beta)\big|_{\beta=1}=E^P\left[U'\left(W_{f^*}(X,Y)\right)\right]\int_0^\infty L(t)\cdot\left(h^*(t)-h(t)\right)\mathrm{d}t\ge 0\end{aligned}$$

where the first inequality follows from the concavity of $\phi(\cdot)$. Therefore, $f^*(x)$ is an optimal solution to optimization problem (2).

It is worthwhile noting that it is challenging to derive the optimal insurance policy directly from Theorem 3.3. We can use this theorem to identify the optimality of some special ceded loss functions, for example no insurance, full insurance and stop-loss insurance. In the following, we can derive the necessity and sufficiency conditions for the optimality of no insurance and full insurance from Theorem 3.3.

Corollary 3.1. No insurance is optimal to optimization problem (2) if and only if

$$\frac{E^P\left[U'(w_0-X-Y)I_{(X>t)}\right]}{E^P\left[U'(w_0-X-Y)\right]}\le(1+\rho)g\left(S_X^Q(x)\right)\tag{9}$$

holds for all $t\in[0,\infty)$.

Corollary 3.2. Full insurance is optimal to optimization problem (2) if and only if

$$\frac{E^P\left[U'\left(w_0-Y-(1+\rho)E_g^Q(X)\right)I_{(X>t)}\right]}{E^P\left[U'\left(w_0-Y-(1+\rho)E_g^Q(X)\right)\right]}\ge(1+\rho)g\left(S_X^Q(x)\right)\tag{10}$$

holds for all $t\in[0,\infty)$.

Obviously, the solution to optimization problem (2) depends on the dependence structure between the insurable risk X and background risk Y. In order to get more concise conclusions, we introduce the definitions of positively quadrant dependent and negatively quadrant dependent.

Definition 3.1. Random variables X and Y are called positively quadrant dependent, denoted as $X\sim_{PQD}Y$, if

$$P(X > x, Y > y) \geq P(X > x)P(Y > y) \quad (11)$$

holds for all x and y.

Definition 3.2. Random variables X and Y are called negatively quadrant dependent, denoted as $X \sim_{NQD} Y$, if

$$P(X > x, Y > y) \leq P(X > x)P(Y > y) \quad (12)$$

holds for all x and y.

Under the assumption of positively quadrant dependent and negatively quadrant dependent, we can get more concise conditions for the optimality of no insurance and full insurance in the following theorems.

Theorem 3.4. If $X + Y \sim_{NQD} X$, then no insurance is optimal to optimization problem (2) if

$$S_X^P(t) \leq (1+\rho) g\left(S_X^Q(t)\right) \quad (13)$$

holds for all $t \in [0, \infty)$.

Proof. We can note that the condition $X + Y \sim_{NQD} X$ is equivalent to $E^P[v(X+Y) | X > x] \leq E^P[v(X+Y)]$ for any x and any increasing function $v(\cdot)$ such that $E^P[|v(X+Y)|] < \infty$ by the result of Shaked & Shanthikumar (2007). Since $U(x)$ is a concave utility function, then $U'(-x)$ is an increasing function. Therefore, we have

$$E^P[U'(w_0 - X - Y) | X > x] \leq E^P[U'(w_0 - X - Y)]$$

which is equivalent to

$$\frac{E^P\left[U'\left(W_{f^*}(w_0 - X - Y)\right) I_{(X>t)}\right]}{E^P\left[U'\left(W_{f^*}(w_0 - X - Y)\right)\right]} \leq S_X^P(t) \quad (14)$$

If condition (13) holds, then the result is obtained by Corollary 3.1.

Theorem 3.5. If $Y \sim_{PQD} X$, then full insurance is optimal to optimization problem (2) if

$$S_X^P(t) \geq (1+\rho) g\left(S_X^Q(t)\right) \quad (15)$$

holds for all $t \in [0, \infty)$.

Proof. Note that $Y \sim_{PQD} X$ is equivalent to $E^P[v(Y) | X > x] \geq E^P[v(Y)]$ for any x and any increasing function $v(\cdot)$ such that $E^P[|v(Y)|] < \infty$ by the result of Shaked & Shanthikumar (2007). Since $U'(-x)$ is an increasing function, then we have

$$E^P\left[U'\left(w_0 - Y - (1+\rho)E_g^Q(X)\right) | X > x\right] \geq E^P\left[U'\left(w_0 - Y - (1+\rho)E_g^Q(X)\right)\right]$$

which is equivalent to

$$\frac{E^P\left[U'\left(w_0 - Y - (1+\rho)E_g^Q(X)\right) I_{(X>t)}\right]}{E^P\left[U'\left(w_0 - Y - (1+\rho)E_g^Q(X)\right)\right]} \geq S_X^P(x) \quad (16)$$

If condition (15) holds, then the result is obtained by Corollary 3.2.

4. Conclusion

In this paper, we consider an optimal insurance problem with background risk and belief heterogeneity. We first prove the existence and uniqueness of the optimal solution. Then we give a necessary and sufficient condition for the optimal insurance policy in Theorem 3.3. With the help of Theorem 3.3, we identify the optimality of no insurance and full insurance.

Admittedly, there are unsolved problems. The optimal insurance form is still unclear when an optimal insurance problem is with background risk and belief heterogeneity. Furthermore, the influence of belief heterogeneity form on the optimal reinsurance strategy is also not discussed. We leave these for future research exploration.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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