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Stationary Measures of Three-State Quantum Walks with Defect on the One-Dimension Lattice

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Abstract

In this paper, we focus on the space-inhomogeneous three-state on the one-dimension lattice, a one-phase model and a two-phase model include. By using the transfer matrices method by Endo *et al.*, we calculate the stationary measure for initial state concrete eigenvalue. Finally we found the transfer matrices method is more effective for the three-state quantum walks than the method obtained by Kawai *et al.*

Keywords

Three-State Quantum Walks, Stationary Measure, One-Phase, Two-Phase, Transfer Matrices

1. Introduction

There is an abundance of research on discrete-time quantum walks since 1993 [1] [2] [3]. Then quantum walks as a quantum mechanical attract large number of scholars [4] [5] [6]. For instance, the two-phase quantum walks are related to the research of topological insulator [7] [8], and one-defect quantum walks are applied to quantum search algorithms [9] [10].

Recently, the asymptotic behaviors of the quantum walks have received much attention [11] [12]. Konno gave the uniform measure as a stationary measure of the one-dimensional discrete-time quantum walks [13]. Endo *et al.*, solve the eigenvalue problem and present a stationary measure by using SGF method [14]. Then Wang *et al.*, obtain the stationary measures of three-state Wojcik walk by adopting SGF method [15]. Shortly afterwards, Kawai *et al.* raised Reduced matrix method [16]. Lately, Endo *et al.* got the transfer matrices and solve the eigenvalue [17]. In this paper, we will use this method to further derive one-phase and two-phase model of space-inhomogeneous three-state quantum walks.

2. Three-State Discrete-Time Quantum Walks

In this section, we give the definition of three-state quantum walk on $\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. The discrete-time quantum walk on $\mathbb{Z}$ defined by a unitary matrix;

$$U_x = \begin{bmatrix} a_x & b_x & c_x \\ d_x & e_x & f_x \\ g_x & h_x & i_x \end{bmatrix} \tag{2.1}$$

We let $\mathbb{N}$ be the set of nonnegative integers, and $\Psi_n(x) = \left(\Psi_n^L(x), \Psi_n^O(x), \Psi_n^R(x)\right)^{\mathrm{T}}$ be the amplitude of the wave function corresponding to the chiralities "*L*", "*O*", and "*R*" at position $x \in \mathbb{Z}$ and time $n \in \mathbb{N}$. Obviously, for each position $x \in \mathbb{Z}$, the matrix U_x can be divided into three parts.

$$U_x = U_x^L + U_x^O + U_x^R \tag{2.2}$$

Through these matrix, we can define time evolution of a quantum walk in the following way:

$$\Psi_{n+1}(x) \equiv U_{x+1}^L \Psi_n(x+1) + U_x^O \Psi_n(x) + U_{x-1}^R \Psi_n(x-1) \tag{2.3}$$

Then let

$$\Psi_n = \left(\cdots, \Psi_n(-1), \Psi_n(0), \Psi_n(1)\right)^{\mathrm{T}},$$

$$U^{(s)} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & U_{-2}^O & U_{-1}^L & O & O & O & \cdots \\ \cdots & U_{-2}^R & U_{-1}^O & U_0^L & O & O & \cdots \\ \cdots & O & U_{-1}^R & U_0^O & U_1^L & O & \cdots \\ \cdots & O & O & U_0^R & U_1^R & U_2^L & \cdots \\ \cdots & O & O & O & U_1^R & U_2^O & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$

Then the sate at time n can be expressed as

$$\Psi_n = \left(U^{(s)}\right)^n \Psi_0, n \geq 0 \tag{2.5}$$

where Ψ_0 is the initial state.

Definition 2.1. The one-phase model of space-inhomogeneous three-state quantum walk is defined on the set $\mathbb{Z}$ of integers. which is characterized by a chirality-state space $\{|L\rangle, |O\rangle, |R\rangle\}$ and a position space $\{|x\rangle : x \in \mathbb{Z}\}$, and the chiralities "L", "R" and "O" express the left, right and neutral state for the motion of the walker. Its time evolution is determined by the following 3×3 unitary matrices

$$U_x = \frac{e^{i\theta_x}}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, x \in \mathbb{Z} \tag{2.6}$$

where

$$\theta_x = \begin{cases} 0, & x = \pm 1, \pm 2, \cdots, \\ 2\pi\tau, & x = 0, \end{cases}$$

with $\tau \in (0,1)$, which θ_x shows the phase $2\pi\tau$ of the walk.

Then

$$U_x^L = \frac{e^{i\theta_x}}{3}\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, U_x^O = \frac{e^{i\theta_x}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, U_x^R = \frac{e^{i\theta_x}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & -1 \end{bmatrix}.$$

Definition 2.2. The two-phase model of space-inhomogeneous three-state quantum walk is defined on the set $\mathbb{Z}$ of integers, which is characterized by a chirality-state space $\{|L\rangle, |O\rangle, |R\rangle\}$ and a position space $\{|x\rangle : x \in \mathbb{Z}\}$. Its time evolution is determined by the following unitary matrices

$$U_x = \begin{cases} U_+, & x \ge 1, \\ U_0, & x = 0, \\ U_-, & x \le -1, \end{cases} \tag{2.7}$$

where

$$U_\pm = \begin{bmatrix} -\frac{1+g_\pm}{2} & \frac{\hbar_\pm}{\sqrt{2}} & \frac{1-g_\pm}{2} \\ \frac{\hbar_\pm}{\sqrt{2}} & g_\pm & \frac{\hbar_\pm}{\sqrt{2}} \\ \frac{1-g_\pm}{2} & \frac{\hbar_\pm}{\sqrt{2}} & -\frac{1+g_\pm}{2} \end{bmatrix}, U_0 = \frac{\xi}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}. \tag{2.8}$$

where $g_\pm = \cos\gamma_\pm, \hbar_\pm = \sin\gamma_\pm, \gamma_\pm \in [0, 2\pi), \xi = e^{2\pi i\tau}, \tau \in (0,1)$. Then

$$U_x^L = \begin{cases} U_{x_+}^L, & x \ge 1, \\ U_x^L, & x = 0, \\ U_{x_-}^L, & x \le -1, \end{cases} \quad U_x^O = \begin{cases} U_{x_+}^O, & x \ge 1, \\ U_0^O, & x = 0, \\ U_{x_-}^O, & x \le -1, \end{cases} \quad U_x^R = \begin{cases} U_{x_+}^R, & x \ge 1, \\ U_0^R, & x = 0, \\ U_{x_-}^R, & x \le -1, \end{cases} \tag{2.9}$$

$$U_{x_+}^L = \begin{bmatrix} -\frac{1+g_+}{2} & \frac{\hbar_+}{\sqrt{2}} & \frac{1-g_+}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, U_{x_+}^O = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\hbar_+}{\sqrt{2}} & g_+ & \frac{\hbar_+}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}, U_{x_+}^R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1-g_+}{2} & \frac{\hbar_+}{\sqrt{2}} & -\frac{1+g_+}{2} \end{bmatrix}. \tag{2.10}$$

$$U_{x_-}^L = \begin{bmatrix} -\frac{1+g_-}{2} & \frac{\hbar_-}{\sqrt{2}} & \frac{1-g_-}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, U_{x_-}^O = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\hbar_-}{\sqrt{2}} & g_- & \frac{\hbar_-}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}, U_{x_-}^R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1-g_-}{2} & \frac{\hbar_-}{\sqrt{2}} & -\frac{1+g_-}{2} \end{bmatrix}. \tag{2.11}$$

$$U_0 = \frac{\xi}{3}\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, U_0 = \frac{\xi}{3}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, U_0 = \frac{\xi}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & -1 \end{bmatrix}. \tag{2.12}$$

3. Stationary Measure

In the present section, we first recall some fundamental notions and facts about stationary measure. Firstly, we introduced a mapping $\phi:(\mathbb{C}^3)^{\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}}$

$$\phi(x) = \left(\cdots, \|\Psi(-1)\|^2, \|\Psi(0)\|^2, \|\Psi(1)\|^2, \cdots\right)^{\mathrm{T}} \in \mathbb{R}_+^{\mathbb{Z}},$$

where $\mathbb{C}$ is the set of complex number and $\|\cdot\|$ is the norm in $\mathbb{C}^3$ and $\Psi = \left(\cdots, \Psi(-1), \Psi(0), \Psi(1), \cdots\right)^{\mathrm{T}} \in (\mathbb{C}^3)^{\mathbb{Z}}$. For every $x \in \mathbb{Z}$, we note that

$$\phi(\Psi)(x) = \|\Psi(x)\|^2, \Psi \in (\mathbb{C}^3)^{\mathbb{Z}}. \tag{3.1}$$

Then, the function $x \to \phi(\Psi)(x)$ gives a measure μ on $\mathbb{Z}$ by $\mu(\cdot) = \phi(\Psi)(\cdot)$ for Ψ.

Definition 3.1. Let

$$\mathcal{M}_s = \Big\{ \phi(\Psi_0) \in \mathbb{R}_+^{\mathbb{Z}} \setminus \{\mathbf{0}\} : \text{there exists } \Psi_0 \text{ such that } \phi\left(\left(U^{(s)}\right)^n \Psi_0\right) = \phi(\Psi_0) \text{ for any } n \ge 0 \Big\} \tag{3.2}$$

where $\mathbf{0}$ is the zero vector. We call the element of $\mathcal{M}_s$ the stationary measure of quantum walk. If $\mu \in \mathcal{M}_s$, then $\mu_n = \mu$, where $\mu_n(x) = \phi(\Psi_n(x))$ is the measure of the quantum walk at position $x \in \mathbb{Z}$ and at time $n \in \mathbb{N}$.

Next we consider the eigenvalue problem:

$$U^{(s)}\Psi = \lambda\Psi, \left(\lambda \in \mathbb{C}, |\lambda| = 1\right) \tag{3.3}$$

4. Main Results and Proofs

In this section, we obtain the stationary measure of the three-state quantum walk with one defect by following lemma.

Lemma 4.1. [17] Let $\{U_y\}_{y\in\mathbb{Z}}$ be the set of y-parameterized unitary matrices of the three-state inhomogeneous quantum walk, and $\Psi(x) = \left[\Psi^L(x), \Psi^O(x), \Psi^R(x)\right]^{\mathrm{T}}$ be the probability amplitude. Note that there is a restriction for the initial state $\Psi(0)$ [18] Then the solutions for $U^{(s)}\Psi = \lambda\Psi\left(\Psi \in Map(\mathbb{Z}, \mathbb{C}^3), \lambda \in S^1\right)$, where $S^1 = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| = 1\}$, are

$$\Psi_x = \begin{cases} \prod_{y=1}^{x} T_y^{(+)} \Psi(0), & x \ge 1, \\ \Psi(0), & x = 0, \\ \prod_{y=-1}^{x} T_y^{(-)} \Psi(0), & x \le -1, \end{cases} \tag{4.1}$$

where $T_y^{(\pm)}$ are the transfer matrices defined by

$$T_y^{(+)} = \begin{bmatrix} t_{11}^+ & t_{12}^+ & t_{13}^+ \\ t_{21}^+ & t_{22}^+ & t_{23}^+ \\ t_{31}^+ & t_{32}^+ & t_{33}^+ \end{bmatrix}, T_y^{(-)} = \begin{bmatrix} t_{11}^- & t_{12}^- & t_{13}^- \\ t_{21}^- & t_{22}^- & t_{23}^- \\ t_{31}^- & t_{32}^- & t_{33}^- \end{bmatrix}, \tag{4.2}$$

with

$$t_{11}^+ = \frac{(\lambda - e_y)(\lambda^2 - g_{y-1}c_y) - g_{y-1}b_y f_y}{\lambda\left[a_y(\lambda - e_y) + b_y d_y\right]}, t_{12}^+ = -\frac{h_{y-1}\left[b_y f_y + c_y(\lambda - e_y)\right]}{\lambda\left[a_y(\lambda - e_y) + b_y d_y\right]}$$

$$t_{13}^{+}=-\frac{i_{y-1}\left[b_y f_y+c_y\left(\lambda-e_y\right)\right]}{\lambda\left[a_y\left(\lambda-e_y\right)+b_y d_y\right]},t_{21}^{+}=\frac{\lambda^2 d_y+g_{y-1}\left(a_y f_y-c_y d_y\right)}{\lambda\left[a_y\left(\lambda-e_y\right)+b_y d_y\right]}$$

$$t_{22}^{+}=\frac{h_{y-1}\left(a_y f_y-c_y d_y\right)}{\lambda\left[a_y\left(\lambda-e_y\right)+b_y d_y\right]},t_{23}^{+}=\frac{i_{y-1}\left(a_y f_y-c_y d_y\right)}{\lambda\left[a_y\left(\lambda-e_y\right)+b_y d_y\right]}$$

$$t_{31}^{+}=\frac{g_{y-1}}{\lambda},t_{32}^{+}=\frac{h_{y-1}}{\lambda},t_{33}^{+}=\frac{i_{y-1}}{\lambda},$$

and

$$t_{11}^{-}=\frac{a_{y+1}}{\lambda},t_{12}^{-}=\frac{b_{y+1}}{\lambda},t_{13}^{-}=\frac{c_{y+1}}{\lambda},$$

$$t_{21}^{(-)}=-\frac{a_{y+1}\left(f_y-g_y-i_y d_y\right)}{\lambda\left[h_y f_y+i_y\left(\lambda-e_y\right)\right]},t_{22}^{(-)}=-\frac{b_{y+1}\left(f_y-g_y-i_y d_y\right)}{\lambda\left[h_y f_y+i_y\left(\lambda-e_y\right)\right]},$$

$$t_{23}^{(-)}=\frac{\lambda^2 f_y-c_{y+1}\left(f_y-g_y-i_y d_y\right)}{\lambda\left[h_y f_y+i_y\left(\lambda-e_y\right)\right]},t_{31}^{(-)}=-\frac{a_{y+1}\left[h_y d_y+g_y\left(\lambda-e_y\right)\right]}{\lambda\left[h_y f_y+i_y\left(\lambda-e_y\right)\right]},$$

$$t_{32}^{(-)}=-\frac{b_{y+1}\left[h_y d_y+g_y\left(\lambda-e_y\right)\right]}{\lambda\left[h_y f_y+i_y\left(\lambda-e_y\right)\right]},t_{33}^{(-)}=-\frac{\left(\lambda-e_y\right)\left(\lambda^2-g_y c_{y+1}\right)-h_y c_{y+1} d_y}{\lambda\left[h_y f_y+i_y\left(\lambda-e_y\right)\right]}.$$

We now state the stationary measure of one-phase model with one defect.

Theorem 4.1. Let $\Psi_n(x)=\left(\Psi_n(x)^L,\Psi_n(x)^O,\Psi_n(x)^R\right)$ be the wave function of probability amplitude, and $\Psi_0^L=\alpha,\Psi_0^O=\beta,\Psi_0^R=\gamma$ be the initial state. We take $\alpha=-\gamma,\beta=0$ and $\lambda=-1$. Then through the definition (2.1)

$$U_x=\frac{e^{i\theta_x}}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix},x\in\mathbb{Z}$$

where

$$\theta_x=\begin{cases}0, & x=\pm1,\pm2,\cdots,\\ 2\pi\tau, & x=0.\end{cases}$$

We obtain the stationary measure

$$\mu(x)=\begin{cases}\left(4-2\Re\xi\right)|\alpha|^2, & x=\pm1,\\ 2|\alpha|^2, & \text{others}.\end{cases}\tag{4.3}$$

Proof. Put $\alpha=\Psi^L(0),\beta=\Psi^O(0)$ and $\gamma=\Psi^R(0)$. Now we take $\alpha=-\gamma,\beta=0$ and $\lambda=-1$, then the solutions for $U^{(s)}\Psi=\lambda\Psi$ are

$$\Psi_x=\begin{cases}\prod_{y=1}^{x}T_y^{(+)}\Psi(0), & x\ge1,\\ \Psi(0), & x=0,\\ \prod_{y=-1}^{x}T_y^{(-)}\Psi(0), & x\le-1,\end{cases}$$

where $T_y^{\pm}$ are

$$T_1^{(+)}=\begin{bmatrix}1 & 0 & 0\\ \frac{2\xi}{3}-1 & \frac{2\xi}{3} & -\frac{\xi}{3}\\ -\frac{2\xi}{3} & -\frac{2\xi}{3} & \frac{\xi}{3}\end{bmatrix}, T_1^{(-)}=\begin{bmatrix}\frac{\xi}{3} & -\frac{2\xi}{3} & -\frac{2\xi}{3}\\ -\frac{\xi}{3} & \frac{2\xi}{3} & \frac{2\xi}{3}-1\\ 0 & 0 & 1\end{bmatrix}.$$

$$T_y^{(+)}=\begin{bmatrix}1 & 0 & 0\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}, T_y^{(-)}=\begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ 0 & 0 & 1\end{bmatrix}\left(|y|\ge 2\right).$$

Then through the formula (4.1), we can obtain

$$\Psi(x)=\begin{cases}\begin{bmatrix}\alpha\\ 0\\ -\alpha\end{bmatrix}, & |x|\neq 1,\\ \begin{bmatrix}\alpha\\ (\xi-1)\alpha\\ -\xi\alpha\end{bmatrix}, & x=1,\\ \begin{bmatrix}\xi\alpha\\ (1-\xi)\alpha\\ -\alpha\end{bmatrix}, & x=-1.\end{cases} \tag{4.4}$$

Therefore the corresponding stationary measure is given by

$$\mu(x)=\begin{cases}(4-2\Re\xi)|\alpha|^2, & x=\pm 1,\\ 2|\alpha|^2, & \text{others.}\end{cases}$$

□

Next we state the stationary measure of two-phase model by transfer matrices method.

Theorem 4.2. Let $\Psi_n(x)=\left(\Psi_n(x)^L,\Psi_n(x)^O,\Psi_n(x)^R\right)$ be the wave function of probability amplitude, and $\Psi_0^L=\alpha,\Psi_0^O=\beta,\Psi_0^R=\gamma$ be the initial state. We take $\alpha=-\gamma,\beta=0$ and $\lambda=-1$. Then through the definition (2.2)

$$U_x=\begin{cases}U_+, & x\ge 1,\\ U_0, & x=0,\\ U_-, & x\le -1,\end{cases}$$

where

$$U_\pm=\begin{bmatrix}-\frac{1+g_\pm}{2} & \frac{\hbar_\pm}{\sqrt{2}} & \frac{1-g_\pm}{2}\\ \frac{\hbar_\pm}{\sqrt{2}} & g_\pm & \frac{\hbar_\pm}{\sqrt{2}}\\ \frac{1-g_\pm}{2} & \frac{\hbar_\pm}{\sqrt{2}} & -\frac{1+g_\pm}{2}\end{bmatrix}, U_0=\frac{\xi}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix},$$

where $g_\pm=\cos\gamma_\pm,\hbar_\pm=\sin\gamma_\pm,\gamma_\pm\in[0,2\pi),\xi=e^{2\pi i\tau},\tau\in(0,1)$. Then the stationary measure is

$$\mu(x) = (2+\Delta)|\alpha|^2 \tag{4.5}$$

where

$$\Delta = \begin{cases} \frac{\hbar_-^2}{2}, & x<1, \\ \frac{\hbar_-^2(1-\Re\xi)}{(1+g_-)^2}, & x=-1, \\ 0, & x=0, \\ \frac{\hbar_+^2(1-\Re\xi)}{(1+g_+)^2}, & x=1, \\ \frac{\hbar_+^2}{2}, & x<1 \end{cases} \tag{4.6}$$

Proof. Put $\alpha = \Psi^L(0), \beta = \Psi^O(0)$ and $\gamma = \Psi^R(0)$. Now we take $\alpha = -\gamma, \beta = 0$ and $\lambda = -1$, then the solutions for $U^{(s)}\Psi = \lambda\Psi$ are

$$\Psi_x = \begin{cases} \prod_{y=1}^{x} T_y^{(+)}\Psi(0), & x \ge 1, \\ \Psi(0), & x=0, \\ \prod_{y=-1}^{x} T_y^{(-)}\Psi(0), & x \le -1, \end{cases}$$

where $T_y^{\pm}$ are

$$T_1^{(+)} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\hbar_+(3-2\xi)}{3\sqrt{2}(1+g_+)} & \frac{2\hbar_+\xi}{3\sqrt{2}(1+g_+)} & -\frac{\hbar_+\xi}{3\sqrt{2}(1+g_+)} \\ -\frac{2\xi}{3} & -\frac{2\xi}{3} & \frac{\xi}{3} \end{bmatrix},$$

$$T_1^{(-)} = \begin{bmatrix} \frac{\xi}{3} & -\frac{2\xi}{3} & -\frac{2\xi}{3} \\ -\frac{\hbar_-\xi}{3\sqrt{2}(1+g_-)} & \frac{2\hbar_-\xi}{3\sqrt{2}(1+g_-)} & -\frac{\hbar_-(3-2\xi)}{3\sqrt{2}(1+g_-)} \\ 0 & 0 & 1 \end{bmatrix}.$$

$$T_y^{(+)} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\hbar_+}{2\sqrt{2}} & \frac{1-g_+}{2} & -\frac{\hbar_+}{2\sqrt{2}} \\ -\frac{1-g_+}{2} & -\frac{\hbar_+}{\sqrt{2}} & \frac{1+g_+}{2} \end{bmatrix},$$

$$T_y^{(-)} = \begin{bmatrix} \frac{1+g_-}{2} & -\frac{\hbar_-}{\sqrt{2}} & -\frac{1-g_-}{2} \\ \frac{\hbar_-}{2\sqrt{2}} & \frac{1-g_-}{2} & -\frac{\hbar_-}{2\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix} (|y| \ge 2).$$

Then through the formula (4.1), we can obtain

$$\Psi(x)=\begin{cases}\begin{bmatrix}\alpha\\ \frac{\hbar_-}{\sqrt{2}}\alpha\\ -\alpha\end{bmatrix}, & x<-1,\\ \begin{bmatrix}\xi\alpha\\ \frac{\hbar_-(1-\xi)}{\sqrt{2}(1+g_-)}\alpha\\ -\alpha\end{bmatrix}, & x=-1,\\ \begin{bmatrix}\alpha\\ 0\\ -\alpha\end{bmatrix}, & x=0,\\ \begin{bmatrix}\alpha\\ \frac{\hbar_+}{\sqrt{2}(1+g_+)}\alpha\\ -\xi\alpha\end{bmatrix}, & x=1,\\ \begin{bmatrix}\alpha\\ \frac{\hbar_+}{\sqrt{2}}\alpha\\ -\alpha\end{bmatrix}, & x>1.\end{cases}$$

Therefore we obtain the stationary measure

$$\mu(x)=(2+\Delta)|\alpha|^2$$

where

$$\Delta=\begin{cases}\frac{\hbar_-^2}{2}, & x<1,\\ \frac{\hbar_-^2(1-\Re\xi)}{(1+g_-)^2}, & x=-1,\\ 0, & x=0,\\ \frac{\hbar_+^2(1-\Re\xi)}{(1+g_+)^2}, & x=1,\\ \frac{\hbar_+^2}{2}, & x<1\end{cases}$$

□

5. Summary

In this paper, we derive the stationary measure of three-state walks with one dimension via transfer matrices. As a future work, we would investigate spectral theory and localization of three-state quantum walks.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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