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Construction of Split-Plot Designs with General Minimum Lower Order Confounding

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Abstract

Fractional factorial split-plot design has been widely used in many fields due to its advantage of saving experimental cost. The general minimum lower order confounding criterion is usually used as one of the attractive design criterion for selecting fractional factorial split-plot design. In this paper, we are interested in the theoretical construction methods of the optimal fractional factorial split-plot designs under the general minimum lower order confounding criterion. We present the theoretical construction methods of optimal fractional factorial split-plot designs under general minimum lower order confounding criterion under several conditions.

Keywords

Fractional Factorial Design, General Minimum Lower Order Confounding Criterion, Split-Plot Design

1. Introduction

"Two-Level Regular Fractional Factorial (FF) Designs" is a class of widely used designs in practice. Such designs perform experimental runs in a completely random order. However, when there are some factors whose levels are difficult to change or control, it is infeasible to perform experimental runs in a completely random order. In these situations, the two-level regular fractional factorial split-plot (FFSP) designs are suitable choices. The FFSP design involves a two-stage randomization when performing experiments. First, randomly choose a level-setting of the hard-to-change factors, called whole plot (WP) factors, then under this level-setting, run all the level-settings of the relatively easy-to-change factors, called subplot (SP) factors, in a completely random order.

In recent years, much attention has been paid to the selection of optimal FFSP designs. Huang *et al.* [1] extended the minimum aberration (MA) criterion to

FFSP designs and proposed the MA-FFSP criterion for selecting optimal regular two-level FFSP designs. Yang *et al.* [2] applied the MA criterion to multi-level FFSP designs. Tichon *et al.* [3] proposed the theoretical construction method of MA orthogonal split-plot designs. Zhao *et al.* [4] constructed MA-FFSP designs for the design scenarios considered in [5] via complementary designs. Mukerjee *et al.* [6] proposed a criterion of minimum secondary aberration (MSA), denoted as MA-MSA-FFSP criterion, for finding the optimal FFSP designs. Yang *et al.* [7] constructed the MA-MSA-FFSP designs under weak MA. Zhao *et al.* [8] studied the mixed-level FFSP designs with a four-level factor in WP section. Zhao *et al.* [9] proposed the mixed-level FFSP designs with a four-level factor in SP section. Yang *et al.* [10] proposed a method to find the optimal FFSP designs based on clear effect criterion. Zi *et al.* [11] conducted a further study based on clear effect criterion. Han *et al.* [12] investigated the conditions for FFSP designs with two-level factors and a 2^t-level factors containing various clear effects. Han *et al.* [13] proposed the conditions for FFSP designs with *s*-level factors and an s^t-level factors containing various clear effects. Based on the principle of the effect hierarchy (see [14]), Zhang *et al.* [15] introduced aliased effect number patterns and proposed a general minimum lower order confounding (GMC) criterion for finding the optimal FF designs. Wei *et al.* [16] proposed GMC-FFSP criterion for finding the optimal FFSP designs and found some GMC-FFSP designs by computer search. However, when the number of factors is large, it is usually infeasible to search GMC-FFSP design by computer.

Although it has been noted that the GMC-FFSP designs have a wide range of applications, in addition to the research of Han *et al.* [17], there are only primitive studies on the theoretical constructions of the GMC-FFSP designs. In this paper, we propose theoretical construction methods of some $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $9N/32+1 \le n_1+n_2 \le 5N/16$, where $N=2^{(n_1+n_2)-(m_1+m_2)}$ and the notation $2^{(n_1+n_2)-(m_1+m_2)}$ will be introduced in Section 2.

The rest of the paper is organized as follows. In Section 2, we review the GMC criterion and the SOS design, which play an important role in the later theorems, and introduce some notations that we will use later in the paper. Section 3 gives the construction methods of some GMC-FFSP designs. The concluding remarks are included in Section 4.

2. Preliminaries

We usually use the notation $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP to denote a two-level regular FFSP design of n_1 WP factors and n_2 SP factors, which is determined by m_1 WP defining words and m_2 SP defining words. For a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, a defining word is called a WP defining word if it does not contain any SP factors, and a defining word is called a SP defining word if it contains at least one SP factor. Huang *et al.* [1] pointed out that a necessary condition of the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs is that the SP definition words are allowed to contain any number of WP factors, but the SP definition words are not allowed to contain only one SP factor, otherwise the split-plot structure of the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs will be destroyed. We refer to the effects that contain only WP factors as WP-type effects, and the effects that contain at least one SP factor as SP-type effects. An alias set is called an alias set of WP-type if it contains at least one WP-type effect, otherwise, it is called an alias set of SP-type.

For the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs, let ${}_{i(s)}^{\#}C_{(w)}^{(0)}$ denotes the number of *i*-factors interaction effects of SP-type which are not in any WP-type alias set and ${}_{i(s)}^{\#}C_{(w)}^{(1)}$ denotes the number of *i*-factors interaction effects of SP-type which are in WP-type alias sets. Considering the split-plot structure of the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs, there must be ${}_{1(s)}^{\#}C_{(w)}^{(0)} = n_2$. For a given $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, let ${}_{i}^{\#}C_{j}^{(k)}$ denotes the number of its *i*-factors interaction effects aliased with its k *j*-factors interaction effects, where $i, j = 1, \cdots, n$, $k = 0, 1, \cdots, K_j$ and $K_j = \binom{n}{j}$. Based on the principle of the effect hierarchy, and the assumption that the effects which involve three or more factors are negligible, a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design is called a *GMC-FFSP design* if it can sequentially maximize

$$ {}^{\#sp}C = \left({}_{1(s)}^{\#}C_{(w)}^{(0)} = n_2, {}_{1}^{\#}C_2, {}_{2}^{\#}C_2, {}_{2(s)}^{\#}C_{(w)}^{(0)} \right), \tag{1} $$

where ${}_{1}^{\#}C_2 = \left({}_{1}^{\#}C_2^{(0)}, {}_{1}^{\#}C_2^{(1)}, \cdots, {}_{1}^{\#}C_2^{(K_2)} \right)$ and ${}_{2}^{\#}C_2 = \left({}_{2}^{\#}C_2^{(0)}, {}_{2}^{\#}C_2^{(1)}, \cdots, {}_{2}^{\#}C_2^{(K_2)} \right)$.

For the convenience of presenting this work, the two-factors interactions (2fis) in the split-plot designs are divided into three categories:

1) The 2fi which involves two WP factors is called a WP-2fi;

2) The 2fi which involves two SP factors is called an SP-2fi;

3) The 2fi which involves one WP factor and one SP factor is called a WS-2fi.

Obviously, both SP-2fi and WS-2fi are SP-type 2fis.

In order to derive the construction methods in this paper, we first review some theories on GMC-FF designs which play an important role. We use the notation 2^{n-m} FF to denote an FF design with *n* factors, determined by *m* defining words. Note that a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design is a 2^{n-m} FF design which has split-plot structure, where $n = n_1 + n_2$ and $m = m_1 + m_2$. Therefore the notation ${}_{1}^{\#}C_2$ and ${}_{1}^{\#}C_2$ are also applicable to 2^{n-m} FF designs. A 2^{n-m} FF design is called a *GMC-FF design* if it can sequentially maximize

$$ {}^{\#}C = \left({}_{1}^{\#}C_2, {}_{2}^{\#}C_2 \right) \tag{2} $$

among all the 2^{n-m} designs.

Chen *et al.* [18] and Xu *et al.* [19] introduced some results on the double theory in detail. In Zhang *et al.* [20], the double theory was employed to derive theoretical construction methods of the 2^{n-m} GMC-FF designs. In the following, we briefly introduce some knowledge on double theory as it is helpful to derive the construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs in this work. Let $\boldsymbol{X}$ be an $r \times l$ matrix consisting only of elements 1 and −1. Let $\boldsymbol{\alpha}_0 = (1,1)'$, $\boldsymbol{\alpha}_1 = (1,-1)'$ and $D(\boldsymbol{X}) = (\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1) \otimes \boldsymbol{X}$, then $D(\boldsymbol{X})$ is a $2r \times 2l$ matrix obtained from $\boldsymbol{X}$ after a double, where $\otimes$ denotes the Kronecker product. Then

$$D^t(\boldsymbol{X}) = \underbrace{(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1)\otimes(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1)\otimes\cdots\otimes(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_1)}_{t\text{ times}}\otimes\boldsymbol{X}$$

is a $2^t r\times 2^t l$ matrix obtained by $\boldsymbol{X}$ after t times double. In particular, when $\boldsymbol{X}=1$

$$D^t(1)=(\boldsymbol{I},\boldsymbol{1},\boldsymbol{2},\boldsymbol{12},\cdots,\boldsymbol{12}\cdots\boldsymbol{t}),$$

is a $2^t\times 2^t$ matrix, where $\boldsymbol{I}=(1,\cdots,1)'$ with 1 being repeated 2^t times; $\boldsymbol{1}=(1,-1,\cdots,1,-1)'$ with every 1 followed by a −1, being repeated 2^{t-1} times; $\boldsymbol{2}=(1,1,-1,-1,\cdots,1,1,-1,-1)'$ with every two consecutive 1's followed by two −1's, being repeated 2^{t-2} times, ..., $\boldsymbol{t}=(1,1,\cdots,1,-1,-1,\cdots,-1)'$ with 2^{t-1} consecutive 1's followed by 2^{t-1} −1's. $\boldsymbol{12}$ is the componentwise product of vectors $\boldsymbol{1}$ and $\boldsymbol{2}$, $\boldsymbol{12}\cdots\boldsymbol{t}$ is the componentwise product of vectors $\boldsymbol{1},\boldsymbol{2},\cdots,\boldsymbol{t}$. Hereafter, let $D^t(\cdot)=D^t(1)\setminus\boldsymbol{I}$, where $\boldsymbol{I}$ belongs to $D^t(1)$. Let $\boldsymbol{X}=(\boldsymbol{b}_1,\boldsymbol{b}_2,\cdots,\boldsymbol{b}_l)$, then $D^t(\boldsymbol{X})$ can be expressed as

$$\begin{aligned} D^t(\boldsymbol{X}) &= D^t(1)\otimes\boldsymbol{X} \\ &= (\boldsymbol{I}\otimes\boldsymbol{b}_1,\cdots,\boldsymbol{I}\otimes\boldsymbol{b}_l;\boldsymbol{1}\otimes\boldsymbol{b}_1,\cdots,\boldsymbol{1}\otimes\boldsymbol{b}_l;\boldsymbol{2}\otimes\boldsymbol{b}_1,\cdots,\boldsymbol{2}\otimes\boldsymbol{b}_l; \\ &\quad \boldsymbol{12}\otimes\boldsymbol{b}_1,\cdots,\boldsymbol{12}\otimes\boldsymbol{b}_l;\cdots;\boldsymbol{12}\cdots\boldsymbol{t}\otimes\boldsymbol{b}_1,\cdots,\boldsymbol{12}\cdots\boldsymbol{t}\otimes\boldsymbol{b}_l). \end{aligned}$$

Suppose $\boldsymbol{j}_1\otimes\boldsymbol{b}_1,\boldsymbol{j}_2\otimes\boldsymbol{b}_2,\cdots,\boldsymbol{j}_k\otimes\boldsymbol{b}_k$ are k columns from $D^t(\boldsymbol{X})$ with $\boldsymbol{j}_1,\boldsymbol{j}_2,\cdots,\boldsymbol{j}_k$ belonging to $D^t(1)$ and $\boldsymbol{b}_1,\boldsymbol{b}_2,\cdots,\boldsymbol{b}_k$ belonging to $\boldsymbol{X}$, then $(\boldsymbol{j}_1\otimes\boldsymbol{b}_1)(\boldsymbol{j}_2\otimes\boldsymbol{b}_2)\cdots(\boldsymbol{j}_k\otimes\boldsymbol{b}_k)=(\boldsymbol{j}_1\boldsymbol{j}_2\cdots\boldsymbol{j}_k)\otimes(\boldsymbol{b}_1\boldsymbol{b}_2\cdots\boldsymbol{b}_k)$.

Let $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$, where $\boldsymbol{T}_W=(\boldsymbol{w}_1,\boldsymbol{w}_2,\cdots,\boldsymbol{w}_{q_1},\boldsymbol{w}_{q_1+1},\boldsymbol{w}_{q_1+2},\cdots,\boldsymbol{w}_{n_1})$ denotes the set of WP factors, $\boldsymbol{T}_S=(\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_{q_2},\boldsymbol{s}_{q_2+1},\boldsymbol{s}_{q_2+2},\cdots,\boldsymbol{s}_{n_2})$ denotes the set of SP factors, where $\boldsymbol{w}_1,\boldsymbol{w}_2,\cdots,\boldsymbol{w}_{q_1}$ are q_1 independent WP factors, $\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_{q_2}$ are q_2 independent SP factors, $q_1=n_1-m_1$ and $q_2=n_2-m_2$. Since the n factors are assigned to n columns in $D^q(\cdot)$, we do not differentiate between factors and columns hereafter. Let $q=q_1+q_2$, Yang *et al.* [10] pointed out that if and only if

$$\begin{cases}\boldsymbol{T}_W\subset\boldsymbol{H}_w,\boldsymbol{T}_S\subset\boldsymbol{H}\setminus\boldsymbol{H}_w \text{ and} \\ |\boldsymbol{T}_W|=n_1,|\boldsymbol{T}_S|=n_2,\end{cases}\tag{3}$$

then $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, where $|\cdot|$ denotes the number of columns in a design or set, $\boldsymbol{H}_w=H(\boldsymbol{w}_1,\boldsymbol{w}_2,\cdots,\boldsymbol{w}_{q_1})$ is a closed set generated by the q_1 independent columns $\boldsymbol{w}_1,\boldsymbol{w}_2,\cdots,\boldsymbol{w}_{q_1}$ from $D^q(\cdot)$, and $\boldsymbol{H}$ is a closed set generated by any q independent columns of $D^q(\cdot)$. To construct a $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP design is equivalent to choosing $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ from $D^q(\cdot)$ such that $\boldsymbol{T}$ can sequentially maximize (1).

A $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design is said to have a resolution of R if this design has no c-factors interaction that is aliased with any other interactions which involve fewer than $R-c$ factors. For a resolution III $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, there is at least one main effect aliased with one 2fi. For a resolution IV $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, there is no main effect aliased with 2fi. Unless otherwise stated, the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs mentioned in the following are of resolution IV. Note that a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design of resolution IV must sequentially maximize ${}^{\#}_{1}C_2$ as it has ${}^{\#}_{1}C_2^{(0)}=n$, where $n=n_1+n_2$. According to Zhang *et al.* [20], when $9N/32+1\le n\le 5N/16$, a 2^{n-m} FF design must belong to the unique second order saturate (SOS) design of $N=2^{n-m}$ runs and $5N/16$ factors, denoted as $\boldsymbol{S}_{(5N/16)}$. A 2^{n-m} FF design is called an SOS design if its degree of freedoms is all used to estimate the main effects and 2fis, see Block *et al.* [21] for more details on the SOS designs. In addition, the SOS design is also widely used in the field of biology, see [22]. Note that the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design can be regarded as a 2^{n-m} FF design which has split-plot structure. Therefore, the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs of resolution IV must belong to $\boldsymbol{S}_{(5N/16)}$. Let $\boldsymbol{X}_1=(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4,\boldsymbol{b}_5)$ be a 2^{5-1} design with $\boldsymbol{I}=\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3\boldsymbol{b}_4\boldsymbol{b}_5$ and $\boldsymbol{b}_i\in D^4(1)$ for $i=1,2,3,4,5$, then the unique SOS design $\boldsymbol{S}_{(5N/16)}$ can be expressed as

$$\boldsymbol{S}_{(5N/16)}=D^{q-4}(\boldsymbol{X}_1)=\left(D^{q-4}(\boldsymbol{b}_1),D^{q-4}(\boldsymbol{b}_2),D^{q-4}(\boldsymbol{b}_3),D^{q-4}(\boldsymbol{b}_4),D^{q-4}(\boldsymbol{b}_5)\right), \quad (4)$$

where $D^{q-4}(\boldsymbol{b}_i)=(\boldsymbol{I},\boldsymbol{1},\boldsymbol{2},\boldsymbol{12},\cdots,\boldsymbol{12}\cdots(\boldsymbol{q-4}))\otimes\boldsymbol{b}_i$ for $i=1,2,\cdots,5$, and $\boldsymbol{I},\boldsymbol{1},\boldsymbol{2},\boldsymbol{12},\cdots,\boldsymbol{12}\cdots(\boldsymbol{q-4})\in D^{q-4}(1)$.

With the discussions above, we obtain that choosing $\boldsymbol{T}$ from $D^q(\cdot)$ reduces to choose $\boldsymbol{T}$ from $\boldsymbol{S}_{(5N/16)}$, such that the expression (1) can be sequentially maximized. In the next section, we give the theoretical construction methods of some $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ with $9N/32+1\le n\le 5N/16$.

3. Construction Methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP Designs

Wei *et al.* [16] pointed out that a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design has

$${}^{\#}_{i(s)}C_{(w)}^{(0)}+{}^{\#}_{i(s)}C_{(w)}^{(1)}=\sum_{l=1}^{i}\binom{n_2}{l}\binom{n_1}{i-l}. \quad (5)$$

According to Equation (5), we obtain the lemma below.

Lemma 1. *For a* $2^{(n_1+n_2)-(m_1+m_2)}$ *FFSP design, there exists*

$${}^{\#}_{2(s)}C_{(w)}^{(0)}+{}^{\#}_{2(s)}C_{(w)}^{(1)}=n_1n_2+\binom{n_2}{2}. \quad (6)$$

Obviously, it is easy to draw from equation (6) that maximizing ${}^{\#}_{2(s)}C_{(w)}^{(0)}$ is equivalent to minimizing ${}^{\#}_{2(s)}C_{(w)}^{(1)}$ for a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design. Since the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs do not allow defining words which contain only one SP factor, thus no WS-2fi is aliased with any WP-2fi meaning that ${}^{\#}_{2(s)}C_{(w)}^{(0)}\ge n_1n_2$ and ${}^{\#}_{2(s)}C_{(w)}^{(1)}\le\binom{n_2}{2}$. As has been discussed, if $\boldsymbol{T}$ can sequentially maximize expression (1), then $\boldsymbol{T}\subset\boldsymbol{S}_{5N/16}$. We denote $\bar{\boldsymbol{T}}=\boldsymbol{S}_{(5N/16)}\setminus\boldsymbol{T}$.

3.1. Construction Methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP Designs with $n_1=1$ and $m_1=0$

In this section, we consider constructing $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $n_1=1$ and $m_1=0$.

Lemma 2. *Suppose* $n_1=1$ *and* $m_1=0$, *then any* $2^{(n_1+n_2)-(m_1+m_2)}$ *FFSP designs* $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ *of resolution IV must have* ${}^{\#}_{2(s)}C_{(w)}^{(0)}(\boldsymbol{T})=n_1n_2+\binom{n_2}{2}$.

Proof. If $n_1 = 1$ and $m_1 = 0$, then the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design has no WP defining words. Clearly, ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) = 0$ implying that

${}_{2(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T}) = n_1 n_2 + \binom{n_2}{2}$. This completes the proof.

Zhang *et al.* [20] gave the construction methods of GMC-FF designs for $9N/32 + 1 \le n \le 5N/16$ as stated in Lemma 3.

Lemma 3. *Up to isomorphism, the GMC* 2^{n-m} *designs with* $9N/32 + 1 \le n \le 5N/16$ *uniquely consist of the last n columns of* $\boldsymbol{S}_{(5N/16)}$.

As aforementioned, a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design can be regarded as a 2^{n-m} design that satisfies the split-plot structure. From Lemma 3, if a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design consists of the last *n* columns of $\boldsymbol{S}_{(5N/16)}$, then this design can sequentially maximize of $\left({}_1^{\#}C_2, {}_2^{\#}C_2\right)$ among all the $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs. Let $\boldsymbol{L}$ denote the set which consists of the last *n* columns in $\boldsymbol{S}_{(5N/16)}$ and $\bar{\boldsymbol{L}} = \boldsymbol{S}_{(5N/16)} \setminus \boldsymbol{L}$. With Lemma 2 and Lemma 3, we immediately obtain the construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $n_1 = 1$ and $m_1 = 0$.

Theorem 1. *Suppose* $n_1 = 1$ *and* $m_1 = 0$*, then the design* $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ *with* $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L}$ *and* $\boldsymbol{T}_S = \boldsymbol{L} \setminus \boldsymbol{T}_W$ *is a* $2^{(n_1+n_2)-(m_1+m_2)}$ *GMC-FFSP design.*

Proof. Since $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L}$ and $\boldsymbol{T}_S = \boldsymbol{L} \setminus \boldsymbol{T}_W$, then $\boldsymbol{T}_W \subset \boldsymbol{H}_w$ and $\boldsymbol{T}_S \subset \boldsymbol{H} \setminus \boldsymbol{H}_w$, where $\boldsymbol{H}_w = H(\boldsymbol{w}_1)$. Therefore, $\boldsymbol{T}$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, *i.e.*, ${}_{1(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T}) = n_2$.

Note that $\boldsymbol{T}$ consists of the last *n* columns of $\boldsymbol{S}_{(5N/16)}$, then, according to Lemma 3, we obtain that $\left({}_1^{\#}C_2, {}_2^{\#}C_2\right)$ can be sequentially maximized. According to Lemma 2, for any $\boldsymbol{T}$ with $n_1 = 1$, there exists ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) = \binom{n_2}{2} + n_1 n_2$.

Therefore, the design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP design. This completes the proof.

Example 1 below illustrates the applications of Theorem 1.

Example 1. Consider constructing a $2^{(1+9)-(0+5)}$ GMC-FFSP design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$. Since $q = (n_1 + n_2) - (m_1 + m_2) = 5$, then

$$\boldsymbol{S}_{(5N/16)} = D^1(\boldsymbol{X}_1) = (\boldsymbol{I} \otimes \boldsymbol{b}_1, \boldsymbol{1} \otimes \boldsymbol{b}_1, \boldsymbol{I} \otimes \boldsymbol{b}_2, \boldsymbol{1} \otimes \boldsymbol{b}_2, \boldsymbol{I} \otimes \boldsymbol{b}_3, \boldsymbol{1} \otimes \boldsymbol{b}_3, \boldsymbol{I} \otimes \boldsymbol{b}_4, \boldsymbol{1} \otimes \boldsymbol{b}_4, \boldsymbol{I} \otimes \boldsymbol{b}_5, \boldsymbol{1} \otimes \boldsymbol{b}_5)$$

, where both $\boldsymbol{I}$ and $\boldsymbol{1}$ are from $D^1(1)$. Note that $n = 10$, then $\boldsymbol{L} = \boldsymbol{S}_{(5N/16)}$. Let $\boldsymbol{w}_1 = \boldsymbol{I} \otimes \boldsymbol{b}_2$ be the $q_1 = 1$ WP column. Then $\boldsymbol{H}_w = H(\boldsymbol{I} \otimes \boldsymbol{b}_2) = \boldsymbol{I} \otimes \boldsymbol{b}_2$. It is obtained that $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L} = \boldsymbol{I} \otimes \boldsymbol{b}_2$ and $\boldsymbol{T}_S = \boldsymbol{L} \setminus \boldsymbol{T}_W = (\boldsymbol{I} \otimes \boldsymbol{b}_1, \boldsymbol{1} \otimes \boldsymbol{b}_1, \boldsymbol{1} \otimes \boldsymbol{b}_2, \boldsymbol{I} \otimes \boldsymbol{b}_3, \boldsymbol{1} \otimes \boldsymbol{b}_3, \boldsymbol{I} \otimes \boldsymbol{b}_4, \boldsymbol{1} \otimes \boldsymbol{b}_4, \boldsymbol{I} \otimes \boldsymbol{b}_5, \boldsymbol{1} \otimes \boldsymbol{b}_5)$. According to Theorem 1, design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ is a $2^{(1+9)-(0+5)}$ GMC-FFSP design.

3.2. Construction Methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP Designs with $m_2 = n_2 - 1$

In this section, we consider constructing $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $m_2 = n_2 - 1$.

Lemma 4. *The* $2^{(n_1+n_2)-(m_1+m_2)}$ *FFSP designs* $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ *of resolution IV* *with* $m_2 = n_2 - 1$ *must have* ${}_{2(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T}) = n_1 n_2$.

Proof. Since $m_2 = n_2 - 1$, we can obtain $q_2 = 1$ meaning that there is only one independent SP factor denoted as $\boldsymbol{s}_1$. Therefore, the non-independent SP factors $\boldsymbol{s}_2, \boldsymbol{s}_3, \cdots, \boldsymbol{s}_{n_2}$ can all be represented via $\boldsymbol{s}_i = \boldsymbol{w}_{j_1}\boldsymbol{w}_{j_2}\cdots\boldsymbol{w}_{j_k}\boldsymbol{s}_1$, where $i = 2, \cdots, n_2$, $j_1, j_2, \cdots, j_k = 1, 2, \cdots, q_1$ and $j_1, j_2, \cdots, j_k$ are mutually different. Therefore, any SP-2fi is aliased with WP-type effects. There are $\binom{n_2}{2}$ SP-2fis, so the ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \ge \binom{n_2}{2}$. According to Lemma 1, we know ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \le \binom{n_2}{2}$. Therefore, there exists ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) = \binom{n_2}{2}$ implying that any $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ with $m_2 = n_2 - 1$ has ${}_{2(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T}) = n_1 n_2$. This completes proof.

With Lemma 3 and Lemma 4, we immediately obtain the construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $m_2 = n_2 - 1$.

Theorem 2. *Suppose* $m_2 = n_2 - 1$, *then the design* $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ *with* $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L}$ *and* $\boldsymbol{T}_S = \boldsymbol{L} \setminus \boldsymbol{T}_W$ *is a* $2^{(n_1+n_2)-(m_1+m_2)}$ *GMC-FFSP design.*

Proof. Since $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L}$ and $\boldsymbol{T}_S = \boldsymbol{L} \setminus \boldsymbol{T}_W$, then $\boldsymbol{T}_W \subset \boldsymbol{H}_w$ and $\boldsymbol{T}_S \subset \boldsymbol{H} \setminus \boldsymbol{H}_w$, where $\boldsymbol{H}_w = H(\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{q_1})$. Therefore, $\boldsymbol{T}$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, *i.e.*, ${}_{1(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T}) = n_2$.

According to formula (6) and Lemma 4, it is obtained that $\boldsymbol{T}$ maximizes ${}_{2(s)}^{\#}C_{(w)}^{(0)}$. By noting that $\boldsymbol{T}$ consists of the last n columns of $\boldsymbol{S}_{(5N/16)}$, we have that $\boldsymbol{T}$ sequentially maximizes (1). This completes the proof.

Example 2 below illustrates the applications of Theorem 2.

Example 2. Consider constructing a $2^{(6+4)-(2+3)}$ GMC-FFSP design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$. Since $q = (n_1 + n_2) - (m_1 + m_2) = 5$, then

$$\boldsymbol{S}_{(5N/16)} = D^1(\boldsymbol{X}_1) = (\boldsymbol{I}\otimes\boldsymbol{b}_1, \boldsymbol{1}\otimes\boldsymbol{b}_1, \boldsymbol{I}\otimes\boldsymbol{b}_2, \boldsymbol{1}\otimes\boldsymbol{b}_2, \boldsymbol{I}\otimes\boldsymbol{b}_3, \boldsymbol{1}\otimes\boldsymbol{b}_3, \boldsymbol{I}\otimes\boldsymbol{b}_4, \boldsymbol{1}\otimes\boldsymbol{b}_4, \boldsymbol{I}\otimes\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_5),$$

where both $\boldsymbol{I}$ and $\boldsymbol{1}$ are from $D^1(1)$. Note that $n = 10$, then $\boldsymbol{L} = \boldsymbol{S}_{(5N/16)}$. Let $\boldsymbol{w}_1 = \boldsymbol{I}\otimes\boldsymbol{b}_3$, $\boldsymbol{w}_2 = \boldsymbol{I}\otimes\boldsymbol{b}_4$, $\boldsymbol{w}_3 = \boldsymbol{I}\otimes\boldsymbol{b}_5$ and $\boldsymbol{w}_4 = \boldsymbol{1}\otimes\boldsymbol{b}_5$ be the $q_1 = 4$ WP columns. Then

$$\begin{aligned}\boldsymbol{H}_w &= H(\boldsymbol{I}\otimes\boldsymbol{b}_3, \boldsymbol{I}\otimes\boldsymbol{b}_4, \boldsymbol{I}\otimes\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_5)\\ &= (\boldsymbol{I}\otimes\boldsymbol{b}_3, \boldsymbol{I}\otimes\boldsymbol{b}_4, \boldsymbol{I}\otimes\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_5, \boldsymbol{I}\otimes\boldsymbol{b}_3\boldsymbol{b}_4, \boldsymbol{I}\otimes\boldsymbol{b}_3\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_3\boldsymbol{b}_5, \boldsymbol{I}\otimes\boldsymbol{b}_4\boldsymbol{b}_5,\\ &\quad \boldsymbol{1}\otimes\boldsymbol{b}_4\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{I}, \boldsymbol{I}\otimes\boldsymbol{b}_3\boldsymbol{b}_4\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_3\boldsymbol{b}_4\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_3, \boldsymbol{1}\otimes\boldsymbol{b}_4, \boldsymbol{1}\otimes\boldsymbol{b}_3\boldsymbol{b}_4)\end{aligned}$$

. It is obtained that $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L} = (\boldsymbol{I}\otimes\boldsymbol{b}_3, \boldsymbol{I}\otimes\boldsymbol{b}_4, \boldsymbol{I}\otimes\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_5, \boldsymbol{1}\otimes\boldsymbol{b}_3, \boldsymbol{1}\otimes\boldsymbol{b}_4)$ and $\boldsymbol{T}_S = \boldsymbol{L} \setminus \boldsymbol{T}_W = (\boldsymbol{I}\otimes\boldsymbol{b}_1, \boldsymbol{1}\otimes\boldsymbol{b}_1, \boldsymbol{I}\otimes\boldsymbol{b}_2, \boldsymbol{1}\otimes\boldsymbol{b}_2)$. According to Theorem 2, the design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ is a $2^{(6+4)-(2+3)}$ GMC-FFSP design.

3.3. Construction Methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP Designs with $n_1 = 2$ and $m_1 = 0$

In this section, we consider constructing $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $n_1 = 2$ and $m_1 = 0$.

Lemma 5. *Suppose* $n_1 = 2$ *and* $m_1 = 0$, *any* $2^{(n_1+n_2)-(m_1+m_2)}$ *FFSP design* $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ *with* $\boldsymbol{T} \subset \boldsymbol{S}_{(5N/16)}$ *has*

$$_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \ge n - 2^{q-2} - 1. \qquad (7)$$

Further more, when $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_i)$, $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_j)$ *and* $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_i) \setminus \boldsymbol{w}_1$ *or* $D^{q-4}(\boldsymbol{b}_j) \setminus \boldsymbol{w}_2$, *the equality in* (7) *holds, where* $i, j = 1,2,3,4,5$ *and* $i \ne j$.

Proof. When $n_1 = 2$ and $m_1 = 0$, there are only three WP effects $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_1\boldsymbol{w}_2$. Since $\boldsymbol{T}$ has resolution IV, thus there is no SP-type 2fi which is aliased with $\boldsymbol{w}_1$ or $\boldsymbol{w}_2$. Next, we explore the number of SP-type 2fis which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$.

There are two different ways of choosing $\boldsymbol{T}_W = (\boldsymbol{w}_1, \boldsymbol{w}_2)$ from $\boldsymbol{S}_{(5N/16)}$:

1) both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are from $D^{q-4}(\boldsymbol{b}_i)$, where $i = 1,2,3,4,5$,

2) $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_i)$ and $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_j)$, where $i, j = 1,2,3,4,5$ and $i \ne j$.

For (1). Without loss of generality, we suppose both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are from $D^{q-4}(\boldsymbol{b}_1)$. Denote $\boldsymbol{w}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{b}_1$ and $\boldsymbol{w}_2 = \boldsymbol{a}_2 \otimes \boldsymbol{b}_1$, where $\boldsymbol{a}_1, \boldsymbol{a}_2 \in D^{q-4}(1)$ and $\boldsymbol{a}_1 \ne \boldsymbol{a}_2$. Then, we have $\boldsymbol{w}_1\boldsymbol{w}_2 = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{I}$, where $\boldsymbol{a}_1\boldsymbol{a}_2 \in D^{q-4}(1)$ and $\boldsymbol{I} \in D^4(1)$. By carefully checking, we can obtain that there are $2^{q-4}/2$ column-pairs, say $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_k, \boldsymbol{c}_2 \otimes \boldsymbol{b}_k)$'s, in $D^{q-4}(\boldsymbol{b}_k)$, such that $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_k)(\boldsymbol{c}_2 \otimes \boldsymbol{b}_k) = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{I}$, where $k = 1,2,3,4,5$. Therefore, $\boldsymbol{S}_{(5N/16)}$ in total $5 \cdot 2^{q-4}/2$ column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_k, \boldsymbol{c}_2 \otimes \boldsymbol{b}_k)$'s that satisfy $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_k)(\boldsymbol{c}_2 \otimes \boldsymbol{b}_k) = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{I}$. Let $\bar{n}$ denote the number of columns in $\bar{\boldsymbol{T}}$, where $\bar{n} = 5 \cdot 2^{q-4} - n$ and $0 \le \bar{n} \le 2^{q-5} - 1$. Consider deleting $\bar{n}$ columns from $\boldsymbol{S}_{(5N/16)}$ to obtain $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ with $\boldsymbol{T}_W = (\boldsymbol{w}_1, \boldsymbol{w}_2)$. By doing so, we obtain that the number of SP-type 2fis, in $\boldsymbol{T}$, which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$ is equal or larger than $5 \cdot 2^{q-4}/2 - \bar{n} - 1 = n - 5 \cdot 2^{q-5} - 1$, where the equality holds if $\bar{\boldsymbol{T}}$ shares only one column with each of any $\bar{n}$ column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_k, \boldsymbol{c}_2 \otimes \boldsymbol{b}_k)$'s, except for $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$.

For (2). Without loss of generality, we suppose $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_1)$ and $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_2)$. Denote $\boldsymbol{w}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{b}_1$ and $\boldsymbol{w}_2 = \boldsymbol{a}_2 \otimes \boldsymbol{b}_2$, where $\boldsymbol{a}_1, \boldsymbol{a}_2 \in D^{q-4}(1)$. Then, we have $\boldsymbol{w}_1\boldsymbol{w}_2 = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{b}_1\boldsymbol{b}_2$, where $\boldsymbol{a}_1\boldsymbol{a}_2 \in D^{q-4}(1)$. In $\boldsymbol{S}_{(5N/16)}$, for each column in $D^{q-4}(\boldsymbol{b}_1)$, say $\boldsymbol{c}_3 \otimes \boldsymbol{b}_1$, we can always find a column from $D^{q-4}(\boldsymbol{b}_2)$, say $\boldsymbol{c}_4 \otimes \boldsymbol{b}_2$, such that $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_1)(\boldsymbol{c}_4 \otimes \boldsymbol{b}_2) = (\boldsymbol{a}_1 \otimes \boldsymbol{b}_1)(\boldsymbol{a}_2 \otimes \boldsymbol{b}_2) = \boldsymbol{w}_1\boldsymbol{w}_2$. Therefore, there are a total of $2^{q-4} - 1$ SP-type 2fis aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$. Consider deleting $\bar{n}$ columns from $\boldsymbol{S}_{(5N/16)}$ to obtain $\boldsymbol{T}$ with $\boldsymbol{T}_W = (\boldsymbol{w}_1, \boldsymbol{w}_2)$. By doing so, we obtain that the number of SP-type 2fis, in $\boldsymbol{T}$, which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$ is equal or larger than $2^{q-4} - \bar{n} - 1 = n - 2^{q-2} - 1$, where the equality holds if $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_1) \setminus \boldsymbol{w}_1$ or $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_2) \setminus \boldsymbol{w}_2$.

Obviously, $n - 5 \cdot 2^{q-5} - 1 > n - 2^{q-2} - 1$. Therefore, we obtain that $_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \ge n - 2^{q-2} - 1$. When $n_1 = 2$ and $m_1 = 0$, any $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ with $\boldsymbol{T} \subset \boldsymbol{S}_{(5N/16)}$ has $_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \ge n - 2^{q-2} - 1$. Further more, when $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_i)$, $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_j)$ and $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_i) \setminus \boldsymbol{w}_1$ or $D^{q-4}(\boldsymbol{b}_j) \setminus \boldsymbol{w}_2$, the equality holds, where $i, j = 1,2,3,4,5$ and $i \ne j$.

This completes the proof.

With Lemma 3 and Lemma 5, we immediately obtain the construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $n_1=2$ and $m_1=0$.

Theorem 3 *Suppose* $n_1=2$ *and* $m_1=0$, *then the design* $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ *with* $\boldsymbol{T}_W=\boldsymbol{H}_w\cap\boldsymbol{L}$ *and* $\boldsymbol{T}_S=\boldsymbol{L}\setminus\boldsymbol{T}_W$ *is a* $2^{(n_1+n_2)-(m_1+m_2)}$ *GMC-FFSP design, where* $\boldsymbol{w}_1\in D^{q-4}(\boldsymbol{b}_1)\cap\boldsymbol{L}$, $\boldsymbol{w}_2\in D^{q-4}(\boldsymbol{b}_i)$ *and* $i=2,3,4$ *or* 5.

Proof. Since $\boldsymbol{T}_W=\boldsymbol{H}_w\cap\boldsymbol{L}$ and $\boldsymbol{T}_S=\boldsymbol{L}\setminus\boldsymbol{T}_W$, then $\boldsymbol{T}_W\subset\boldsymbol{H}_w$ and $\boldsymbol{T}_S\subset\boldsymbol{H}\setminus\boldsymbol{H}_w$, where $\boldsymbol{H}_w=H(\boldsymbol{w}_1,\boldsymbol{w}_2)$. Therefore, $\boldsymbol{T}$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, *i.e.*, ${}_{1(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T})=n_2$.

Because $\boldsymbol{w}_1\in D^{q-4}(\boldsymbol{b}_1)$, $\boldsymbol{w}_2\in D^{q-4}(\boldsymbol{b}_i)$ and $\bar{\boldsymbol{T}}\subset D^{q-4}(\boldsymbol{b}_1)\setminus\boldsymbol{w}_1$, according to Lemma 5, we obtain that ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T})=n-2^{q-2}-1$, *i.e.*, ${}_{2(s)}^{\#}C_{(w)}^{(0)}(\boldsymbol{T})$ is maximized, where $i=2,3,4,5$. By noting that $\boldsymbol{T}$ consists of the last *n* columns of $\boldsymbol{S}_{(5N/16)}$, we have that $\boldsymbol{T}$ sequentially maximizes (1). This completes the proof.

Example 3 below illustrates the applications of Theorem 3.

Example 3. Consider constructing a $2^{(2+17)-(0+13)}$ GMC-FFSP design $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$. Since $q=(n_1+n_2)-(m_1+m_2)=6$, then

$$\begin{aligned}\boldsymbol{S}_{(5N/16)}&=D^2(\boldsymbol{X}_1)\\&=(\boldsymbol{I}\otimes\boldsymbol{b}_1,\mathbf{1}\otimes\boldsymbol{b}_1,\mathbf{2}\otimes\boldsymbol{b}_1,\mathbf{12}\otimes\boldsymbol{b}_1,\boldsymbol{I}\otimes\boldsymbol{b}_2,\mathbf{1}\otimes\boldsymbol{b}_2,\mathbf{2}\otimes\boldsymbol{b}_2,\mathbf{12}\otimes\boldsymbol{b}_2,\boldsymbol{I}\otimes\boldsymbol{b}_3,\mathbf{1}\otimes\boldsymbol{b}_3,\ ,\\&\quad\mathbf{2}\otimes\boldsymbol{b}_3,\mathbf{12}\otimes\boldsymbol{b}_3,\boldsymbol{I}\otimes\boldsymbol{b}_4,\mathbf{1}\otimes\boldsymbol{b}_4,\mathbf{2}\otimes\boldsymbol{b}_4,\mathbf{12}\otimes\boldsymbol{b}_4,\boldsymbol{I}\otimes\boldsymbol{b}_5,\mathbf{1}\otimes\boldsymbol{b}_5,\mathbf{2}\otimes\boldsymbol{b}_5,\mathbf{12}\otimes\boldsymbol{b}_5)\end{aligned}$$

where $\boldsymbol{I},\mathbf{1},\mathbf{2}$ and $\mathbf{12}$ are from $D^2(1)$. Note that $n=19$, then $\boldsymbol{L}=\boldsymbol{S}_{(5N/16)}\setminus\boldsymbol{I}\otimes\boldsymbol{b}_1$. Let $\boldsymbol{w}_1=\mathbf{1}\otimes\boldsymbol{b}_1$ and $\boldsymbol{w}_2=\boldsymbol{I}\otimes\boldsymbol{b}_2$ be the $q_1=2$ independent WP columns. Then $\boldsymbol{H}_w=H(\mathbf{1}\otimes\boldsymbol{b}_1,\boldsymbol{I}\otimes\boldsymbol{b}_2)=(\mathbf{1}\otimes\boldsymbol{b}_1,\boldsymbol{I}\otimes\boldsymbol{b}_2,\mathbf{1}\otimes\boldsymbol{b}_1\boldsymbol{b}_2)$. It is obtained that $\boldsymbol{T}_W=\boldsymbol{H}_w\cap\boldsymbol{L}=(\mathbf{1}\otimes\boldsymbol{b}_1,\boldsymbol{I}\otimes\boldsymbol{b}_2)$ and

$$\begin{aligned}\boldsymbol{T}_S&=\boldsymbol{L}\setminus\boldsymbol{T}_W\\&=(\mathbf{2}\otimes\boldsymbol{b}_1,\mathbf{12}\otimes\boldsymbol{b}_1,\mathbf{1}\otimes\boldsymbol{b}_2,\mathbf{2}\otimes\boldsymbol{b}_2,\mathbf{12}\otimes\boldsymbol{b}_2,\boldsymbol{I}\otimes\boldsymbol{b}_3,\mathbf{1}\otimes\boldsymbol{b}_3,\mathbf{2}\otimes\boldsymbol{b}_3,\mathbf{12}\otimes\boldsymbol{b}_3,.\\&\quad\boldsymbol{I}\otimes\boldsymbol{b}_4,\mathbf{1}\otimes\boldsymbol{b}_4,\mathbf{2}\otimes\boldsymbol{b}_4,\mathbf{12}\otimes\boldsymbol{b}_4,\boldsymbol{I}\otimes\boldsymbol{b}_5,\mathbf{1}\otimes\boldsymbol{b}_5,\mathbf{2}\otimes\boldsymbol{b}_5,\mathbf{12}\otimes\boldsymbol{b}_5)\end{aligned}$$

According to Theorem 3, the design $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ is a $2^{(2+17)-(0+13)}$ GMC-FFSP design.

3.4. Construction Methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP Designs with $n_1=3$ and $m_1=0$

In this section, we consider constructing $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $n_1=3$ and $m_1=0$.

Lemma 6. *Suppose* $n_1=3$ *and* $m_1=0$, *any* $2^{(n_1+n_2)-(m_1+m_2)}$ *FFSP design* $\boldsymbol{T}=(\boldsymbol{T}_W,\boldsymbol{T}_S)$ *with* $\boldsymbol{T}\subset\boldsymbol{S}_{(5N/16)}$ *has*

$${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T})\geq 2n-3\cdot 2^{q-3}-3. \tag{8}$$

Further more, when $\boldsymbol{w}_1\in D^{q-4}(\boldsymbol{b}_i)$, $\boldsymbol{w}_2\in D^{q-4}(\boldsymbol{b}_j)$, $\boldsymbol{w}_3\in D^{q-4}(\boldsymbol{b}_k)$ and $\bar{\boldsymbol{T}}\subset D^{q-4}(\boldsymbol{b}_i)\setminus\mathrm{w}_1$, $D^{q-4}(\boldsymbol{b}_j)\setminus\boldsymbol{w}_2$ or $D^{q-4}(\boldsymbol{b}_k)\setminus\boldsymbol{w}_3$, the equality in (8) holds, where $i,j,k=1,2,3,4,5$ and are not equal to each other.

Proof. When $n_1=3$ and $m_1=0$, we have $q_1=3$, *i.e.*, there are only three WP factors and they are independent of each other. There are seven WP-type effects $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$, $\boldsymbol{w}_1\boldsymbol{w}_2$, $\boldsymbol{w}_1\boldsymbol{w}_3$, $\boldsymbol{w}_2\boldsymbol{w}_3$ and $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$ in $\boldsymbol{T}$. Note that

$\boldsymbol{T} \subset \boldsymbol{S}_{(5N/16)}$ has resolution IV which implies that no SP-type 2fi is aliased with $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ or $\boldsymbol{w}_3$. Therefore, calculating ${}_{2(s)}^{\#}C_{(w)}^{(1)}$ is equivalent to calculating the number of SP-type 2fis aliased with effects $\boldsymbol{w}_1\boldsymbol{w}_2$, $\boldsymbol{w}_1\boldsymbol{w}_3$, $\boldsymbol{w}_2\boldsymbol{w}_3$ and $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$. There are three different ways of choosing $\boldsymbol{T}_W = (\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3)$ from $\boldsymbol{S}_{5N/16}$:

1) $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$ are from $D^{q-4}(\boldsymbol{b}_i)$, where $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3 \notin \boldsymbol{T}$, otherwise an SP factor will be aliased with $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$ which is not allowed, and $i = 1,2,3,4$ or 5.

2) both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are from $D^{q-4}(\boldsymbol{b}_i)$, $\boldsymbol{w}_3 \in D^{q-4}(\boldsymbol{b}_j)$, where $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3 \notin \boldsymbol{T}$, otherwise a SP factor will be aliased with $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$ which is not allowed, and $i, j = 1,2,3,4$ or 5 and $i \neq j$.

3) $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_i)$, $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_j)$ and $\boldsymbol{w}_3 \in D^{q-4}(\boldsymbol{b}_k)$, where $i, j, k = 1,2,3,4$ or 5 and are not equal to each other.

Next, we explore the minimum values of ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T})$ in cases (1), (2) and (3) respectively.

For (1). Without loss of generality, we suppose $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$ are from $D^{q-4}(\boldsymbol{b}_1)$. Denote $\boldsymbol{w}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{b}_1$, $\boldsymbol{w}_2 = \boldsymbol{a}_2 \otimes \boldsymbol{b}_1$, $\boldsymbol{w}_3 = \boldsymbol{a}_3 \otimes \boldsymbol{b}_1$ and $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3 \notin \boldsymbol{T}$, where $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in D^{q-4}(1)$ and are not equal to each other. There are $2^{q-4}/2$ column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i, \boldsymbol{c}_2 \otimes \boldsymbol{b}_i)$'s in $D^{q-4}(\boldsymbol{b}_i)$ such that $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i)(\boldsymbol{c}_2 \otimes \boldsymbol{b}_i) = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{I}$ for $i = 1,2,3,4$ and 5, respectively, where $\boldsymbol{I}$ is from $D^4(1)$. This indicates that there are a total of $5 \cdot 2^{q-4}/2$ column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i, \boldsymbol{c}_2 \otimes \boldsymbol{b}_i)$'s in $\boldsymbol{S}_{5N/16}$ such that $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i)(\boldsymbol{c}_2 \otimes \boldsymbol{b}_i) = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{I}$, where $i = 1,2,3,4,5$. Similarly, there are a total of $5 \cdot 2^{q-4}/2$ column-pairs $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_i, \boldsymbol{c}_4 \otimes \boldsymbol{b}_i)$'s in $\boldsymbol{S}_{5N/16}$ such that $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_i)(\boldsymbol{c}_4 \otimes \boldsymbol{b}_i) = \boldsymbol{a}_1\boldsymbol{a}_3 \otimes \boldsymbol{I}$, and there are a total of $5 \cdot 2^{q-4}/2$ column-pairs $(\boldsymbol{c}_5 \otimes \boldsymbol{b}_i, \boldsymbol{c}_6 \otimes \boldsymbol{b}_i)$'s in $\boldsymbol{S}_{5N/16}$ such that $(\boldsymbol{c}_5 \otimes \boldsymbol{b}_i)(\boldsymbol{c}_6 \otimes \boldsymbol{b}_i) = \boldsymbol{a}_2\boldsymbol{a}_3 \otimes \boldsymbol{I}$, where $i = 1,2,3,4,5$ and $\boldsymbol{I}$ is from $D^4(1)$. Consider deleting $\bar{n}$ columns from $\boldsymbol{S}_{5N/16}$ to obtain $\boldsymbol{T}$ such that ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T})$ is the smaller the possible. With a similar discussion to the proofs of (1) in Lemma 5, we know that if the deleted $\bar{n}$ columns, *i.e.*, $\bar{\boldsymbol{T}}$, consist of only one column of each of any $\bar{n}$ column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i, \boldsymbol{c}_2 \otimes \boldsymbol{b}_i)$'s (which are related to $\boldsymbol{w}_1\boldsymbol{w}_2$), then there are $5 \cdot 2^{q-4}/2 - \bar{n} - 1$ SP-type 2fis in $\boldsymbol{T}$ which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$. This is always the case for $\boldsymbol{w}_1\boldsymbol{w}_3$ and $\boldsymbol{w}_2\boldsymbol{w}_3$. Note that no 2fi in $\boldsymbol{T}$ is aliased with $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$ due to $\boldsymbol{T} \subset \boldsymbol{S}_{5N/16}$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in D^{q-4}(1)$. Therefore, ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \geq 3 \cdot (5 \cdot 2^{q-4}/2 - \bar{n} - 1) = 3n - 15 \cdot 2^{q-5} - 3$. The equality holds if any two columns of $\bar{\boldsymbol{T}}$ are not in the same column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i, \boldsymbol{c}_2 \otimes \boldsymbol{b}_i)$'s, $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_i, \boldsymbol{c}_4 \otimes \boldsymbol{b}_i)$'s or $(\boldsymbol{c}_5 \otimes \boldsymbol{b}_i, \boldsymbol{c}_6 \otimes \boldsymbol{b}_i)$'s, where $i = 1,2,3,4,5$.

For (2). Without loss of generality, we suppose $\boldsymbol{w}_1, \boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_1)$ and $\boldsymbol{w}_3 \in D^{q-4}(\boldsymbol{b}_2)$. Denote $\boldsymbol{w}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{b}_1$, $\boldsymbol{w}_2 = \boldsymbol{a}_2 \otimes \boldsymbol{b}_1$, $\boldsymbol{w}_3 = \boldsymbol{a}_3 \otimes \boldsymbol{b}_2$ and $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3 \notin \boldsymbol{T}$, where $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in D^{q-4}(1)$ and $\boldsymbol{a}_1 \neq \boldsymbol{a}_2$. With a similar discussion to the proofs for (1) and the proofs of (2) in Lemma 5, we conclude that ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \geq 3n - 21 \cdot 2^{q-5} - 3$ by noting that no 2fi in $\boldsymbol{T}$ is aliased with $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$. The equality holds if $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_1) \setminus (\boldsymbol{w}_1, \boldsymbol{w}_2)$ or $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_2) \setminus \boldsymbol{w}_3$ and any two columns of $\bar{\boldsymbol{T}}$ are not in the same column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i, \boldsymbol{c}_2 \otimes \boldsymbol{b}_i)$'s that satisfy $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_i)(\boldsymbol{c}_2 \otimes \boldsymbol{b}_i) = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{I}$, where $i = 1,2,3,4,5$ and $\boldsymbol{I}$ is from $D^4(1)$.

For (3). Without loss of generality, we suppose $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_1)$, $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_2)$ and $\boldsymbol{w}_3 \in D^{q-4}(\boldsymbol{b}_3)$. Denote $\boldsymbol{w}_1 = \boldsymbol{a}_1 \otimes \boldsymbol{b}_1$, $\boldsymbol{w}_2 = \boldsymbol{a}_2 \otimes \boldsymbol{b}_2$ and $\boldsymbol{w}_3 = \boldsymbol{a}_3 \otimes \boldsymbol{b}_3$, where $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are from $D^{q-4}(1)$. There are 2^{q-4} column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_1, \boldsymbol{c}_2 \otimes \boldsymbol{b}_2)$'s in $\boldsymbol{S}_{(5N/16)}$ such that $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_1)(\boldsymbol{c}_2 \otimes \boldsymbol{b}_2) = \boldsymbol{a}_1\boldsymbol{a}_2 \otimes \boldsymbol{b}_1\boldsymbol{b}_2$, 2^{q-4} column-pairs $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_1, \boldsymbol{c}_4 \otimes \boldsymbol{b}_3)$'s in $\boldsymbol{S}_{(5N/16)}$ such that $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_1)(\boldsymbol{c}_4 \otimes \boldsymbol{b}_3) = \boldsymbol{a}_1\boldsymbol{a}_3 \otimes \boldsymbol{b}_1\boldsymbol{b}_3$, 2^{q-4} column-pairs $(\boldsymbol{c}_5 \otimes \boldsymbol{b}_2, \boldsymbol{c}_6 \otimes \boldsymbol{b}_3)$'s in $\boldsymbol{S}_{(5N/16)}$ such that $(\boldsymbol{c}_5 \otimes \boldsymbol{b}_2)(\boldsymbol{c}_6 \otimes \boldsymbol{b}_3) = \boldsymbol{a}_2\boldsymbol{a}_3 \otimes \boldsymbol{b}_2\boldsymbol{b}_3$. There are 2^{q-4} column-pairs $(\boldsymbol{c}_7 \otimes \boldsymbol{b}_4, \boldsymbol{c}_8 \otimes \boldsymbol{b}_5)$'s in $\boldsymbol{S}_{(5N/16)}$ such that $(\boldsymbol{c}_7 \otimes \boldsymbol{b}_4)(\boldsymbol{c}_8 \otimes \boldsymbol{b}_5) = \boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3 \otimes \boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$. Suppose that we delete x_1, x_2, x_3, x_4 and x_5 columns from $D^{q-4}(\boldsymbol{b}_1)$, $D^{q-4}(\boldsymbol{b}_2)$, $D^{q-4}(\boldsymbol{b}_3)$, $D^{q-4}(\boldsymbol{b}_4)$ and $D^{q-4}(\boldsymbol{b}_5)$, respectively, where $x_1 + x_2 + \cdots + x_5 = \bar{n}$. In order to minimize the total number of SP-type 2fis in $\boldsymbol{T}$ which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$, $\boldsymbol{w}_1\boldsymbol{w}_3$ and $\boldsymbol{w}_2\boldsymbol{w}_3$, any two of the to be deleted $x_1 + x_2 + x_3$ columns are not in the same column-pairs $(\boldsymbol{c}_1 \otimes \boldsymbol{b}_1, \boldsymbol{c}_2 \otimes \boldsymbol{b}_2)$'s, $(\boldsymbol{c}_3 \otimes \boldsymbol{b}_1, \boldsymbol{c}_4 \otimes \boldsymbol{b}_3)$'s or $(\boldsymbol{c}_5 \otimes \boldsymbol{b}_2, \boldsymbol{c}_6 \otimes \boldsymbol{b}_3)$'s. This can always be done noting that $\bar{n} \le 2^{q-5} - 1$. For example, we delete $x_1 + x_2 + x_3$ columns from $D^{q-4}(\boldsymbol{b}_1)$, $D^{q-4}(\boldsymbol{b}_2)$ or $D^{q-4}(\boldsymbol{b}_3)$. By doing so, there remain a total of $3 \cdot 2^{q-4} - 2(x_1 + x_2 + x_3) - 3$ SP-type 2fis in $\boldsymbol{T}$ which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2$, $\boldsymbol{w}_1\boldsymbol{w}_3$ or $\boldsymbol{w}_2\boldsymbol{w}_3$. In order to minimize the number of SP-type 2fis in $\boldsymbol{T}$ which are aliased with $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$, any two of the $x_4 + x_5$ columns (to be deleted) are not in the same column-pairs $(\boldsymbol{c}_7 \otimes \boldsymbol{b}_4, \boldsymbol{c}_8 \otimes \boldsymbol{b}_5)$'s. By doing so, there remain $2^{q-4} - (x_4 + x_5)$ SP-type 2fis in $\boldsymbol{T}$ which are aliased $\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{w}_3$. Therefore, we have

$$\begin{aligned} {}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) &\ge 4 \cdot 2^{q-4} - 2(x_1 + x_2 + x_3) - (x_4 + x_5) - 3 \\ &= 2^{q-2} - 2(\bar{n} - (x_4 + x_5)) - (x_4 + x_5) - 3 \\ &= 2^{q-2} - 2\bar{n} + (x_4 + x_5) - 3. \end{aligned}$$

Further more, when $x_4 + x_5 = 0$, we have ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \ge 2^{q-2} - 2\bar{n} - 3 = 2n - 3 \cdot 2^{q-3} - 3$ which is the minimum value for (3). When $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_1) \backslash \boldsymbol{w}_1$, $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_2) \backslash \boldsymbol{w}_2$ or $\bar{\boldsymbol{T}} \subset D^{q-4}(\boldsymbol{b}_3) \backslash \boldsymbol{w}_3$, the equation ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) = 2n - 3 \cdot 2^{q-3} - 3$ holds.

Comparing the minimum values of ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T})$ in cases (1), (2) and (3), it is clear that $3n - 15 \cdot 2^{q-5} - 3 > 3n - 21 \cdot 2^{q-5} - 3 > 2n - 3 \cdot 2^{q-3} - 3$. Therefore, when $n_1 = 3$ and $m_1 = 0$, any $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ with $\boldsymbol{T} \subset \boldsymbol{S}_{(5N/16)}$ has ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) \ge 2n - 3 \cdot 2^{q-3} - 3$. Further more, when $\boldsymbol{w}_1 \in D^{q-4}(\boldsymbol{b}_i)$, $\boldsymbol{w}_2 \in D^{q-4}(\boldsymbol{b}_j)$, $\boldsymbol{w}_3 \in D^{q-4}(\boldsymbol{b}_k)$, and $\bar{\boldsymbol{T}} \subset D^{q-4}\boldsymbol{b}_i \backslash \boldsymbol{w}_1$, $D^{q-4}(\boldsymbol{b}_j) \backslash \boldsymbol{w}_2$ or $D^{q-4}(\boldsymbol{b}_k) \backslash \boldsymbol{w}_3$, the equation ${}_{2(s)}^{\#}C_{(w)}^{(1)}(\boldsymbol{T}) = 2n - 3 \cdot 2^{q-3} - 3$ holds, where $i, j, k = 1, 2, 3, 4$ or 5, and are not equal to each other.

This completes the proof.

With Lemma 3 and Lemma 6, we immediately obtain the construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs with $n_1 = 3$ and $m_1 = 0$.

Theorem 4. *Suppose* $n_1 = 3$ *and* $m_1 = 0$, *then the design* $\boldsymbol{T} = (\boldsymbol{T}_W, \boldsymbol{T}_S)$ *with* $\boldsymbol{T}_W = \boldsymbol{H}_w \cap \boldsymbol{L}$ *and* $\boldsymbol{T}_S = \boldsymbol{L} \backslash \boldsymbol{T}_W$ *is a* $2^{(n_1+n_2)-(m_1+m_2)}$ *GMC-FFSP design,* *where* $w_1 \in D^{q-4}(b_1) \cap L$, $w_2 \in D^{q-4}(b_i)$, $w_3 \in D^{q-4}(b_j)$, $i, j = 2,3,4$ *or* 5 *and* $i \neq j$.

Proof. Since $T_W = H_w \cap L$ and $T_S = L \setminus T_W$, then $T_W \subset H_w$ and $T_S \subset H \setminus H_w$, where $H_w = H(w_1, w_2, w_3)$. Therefore, T is a $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP design, *i.e.*, ${}_{1(s)}^{\#}C_{(w)}^{(0)}(T) = n_2$.

Because $w_1 \in D^{q-4}(b_1) \cap L$, $w_2 \in D^{q-4}(b_i)$, $w_3 \in D^{q-4}(b_j)$ and $\bar{T} \subset D^{q-4}(b_1) \setminus w_1$, according to Lemma 6, we obtain that ${}_{2(s)}^{\#}C_{(w)}^{(1)}(T) = 2n - 3 \cdot 2^{q-3} - 3$, *i.e.*, ${}_{2(s)}^{\#}C_{(w)}^{(0)}(T)$ is maximized, where $i, j = 2,3,4,5$ and $i \neq j$. By noting that T consists of the last n columns of $S_{(5N/16)}$, we have that T sequentially maximizes (1). This completes the proof.

Example 4 below illustrates the applications of Theorem 4.

Example 4. Consider constructing a $2^{(3+16)-(0+13)}$ GMC-FFSP designs $T = (T_W, T_S)$. Since $q = (n_1+n_2)-(m_1+m_2) = 6$, then

$$S_{(5N/16)} = D^2(X_1)$$
$$= (I \otimes b_1, 1 \otimes b_1, 2 \otimes b_1, 12 \otimes b_1, I \otimes b_2, 1 \otimes b_2, 2 \otimes b_2, 12 \otimes b_2, I \otimes b_3, 1 \otimes b_3, 2 \otimes b_3, 12 \otimes b_3, I \otimes b_4, 1 \otimes b_4, 2 \otimes b_4, 12 \otimes b_4, I \otimes b_5, 1 \otimes b_5, 2 \otimes b_5, 12 \otimes b_5),$$

where $I, 1, 2$ and 12 are from $D^2(1)$. Note that $n = 19$, then $L = S_{(5N/16)} \setminus I \otimes b_1$. Let $w_1 = 1 \otimes b_1$, $w_2 = I \otimes b_2$ and $w_3 = I \otimes b_3$, be the $q_1 = 3$ WP columns. Then

$$H_w = H(1 \otimes b_1, I \otimes b_2, I \otimes b_3)$$
$$= (1 \otimes b_1, I \otimes b_2, I \otimes b_3, 1 \otimes b_1 b_2, 1 \otimes b_1 b_3, I \otimes b_2 b_3, 1 \otimes b_1 b_2 b_3).$$ It is obtained that $T_W = H_w \cap L = (1 \otimes b_1, I \otimes b_2, I \otimes b_3)$ and

$$T_S = L \setminus T_W$$
$$= (2 \otimes b_1, 12 \otimes b_1, 1 \otimes b_2, 2 \otimes b_2, 12 \otimes b_2, 1 \otimes b_3, 2 \otimes b_3, 12 \otimes b_3, I \otimes b_4, 1 \otimes b_4, 2 \otimes b_4, 12 \otimes b_4, I \otimes b_5, 1 \otimes b_5, 2 \otimes b_5, 12 \otimes b_5).$$

According to Theorem 4, design $T = (T_W, T_S)$ is a $2^{(3+16)-(0+13)}$ GMC-FFSP design.

4. Concluding Remarks

Two-level regular split-plot designs have wide applications in practice. To choose desirable two-level regular split-plot designs, Wei *et al.* [16] proposed the GMC-FFSP criterion. This criterion is capable of estimating as many lower order effects of interest as possible. However, the studies on theoretical construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs are still primitive.

In this paper, we explore the theoretical construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs with $9N/32 + 1 \leq n_1 + n_2 \leq 5N/16$. The theoretical construction methods of $2^{(n_1+n_2)-(m_1+m_2)}$ GMC-FFSP designs for the cases where $m_1 = 0$ with $n_1 = 1, 2$ and 3, and $m_2 = n_2 - 1$ are worked out. The construction methods are concise and easy to apply.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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