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Asymptotically Necessary and Sufficient Quadratic Stability Conditions of T-S Fuzzy Systems Using Staircase Membership Function and Basic Inequality

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Abstract

Asymptotically necessary and sufficient quadratic stability conditions of Takagi-Sugeno (T-S) fuzzy systems are obtained by utilizing staircase membership functions and a basic inequality. The information of the membership functions is incorporated in the stability analysis by approximating the original continuous membership functions with staircase membership functions. The stability of the T-S fuzzy systems was investigated based on a quadratic Lyapunov function. The asymptotically necessary and sufficient stability conditions in terms of linear matrix inequalities were derived using a basic inequality. A fuzzy controller was also designed based on the stability results. The derivation process of the stability results is straightforward and easy to understand. Case studies confirmed the validity of the obtained stability results.

Keywords

Takagi-Sugeno Fuzzy System, Quadratic Stability, Fuzzy Controller, Staircase Membership Function

1. Introduction

The Takagi-Sugeno (T-S) fuzzy system provides an effective approach for the control synthesis of nonlinear systems [1] [2] [3] [4]. The T-S fuzzy system represents a nonlinear system by fuzzily combining a series of linear subsystems with membership functions, which makes it possible to apply control synthesis methods for linear systems to nonlinear systems. The membership functions, including the nonlinear factors of the original system, play an important role in constructing T-S fuzzy systems [5] [6]. The stability and control synthesis of T-S fuzzy systems have been extensively studied based on Lyapunov stability theory [7] [8] [9] [10]. In early studies, the stability and stabilizability conditions were usually obtained without considering the membership functions, which led to conservative control design. Recent studies have indicated that an efficient way to lower the conservatism of the stability conditions of T-S fuzzy systems is incorporating the information of membership functions in the stability analysis [5] [11] [12]. The upper bounds on the cross products of membership functions were utilized to relax the stability and performance conditions of T-S fuzzy systems [13]. To incorporate more information on the membership functions in the stability analysis, the local boundary information of the membership functions was employed by dividing the operation region of the membership functions into sub-regions [14]. The stability conditions were also relaxed by incorporating the shape information of the membership functions in the form of polynomial constraints [15].

Approximating the membership functions with some special functions is an effective way to incorporate the information of the membership functions in stability analysis [5] [6] [12] [16]. Staircase membership functions were proposed to approximate the continuous membership functions of a T-S fuzzy system and a fuzzy controller in [17], and sufficient stability conditions in terms of linear matrix inequalities (LMIs) were derived by introducing some slack matrices. A new relaxed stability condition of T-S fuzzy control systems was obtained by using both staircase membership functions and quadratic fuzzy Lyapunov functions in the stability analysis [18]. Membership-function-dependent controller synthesis was investigated in [12] [19], where the information of the membership functions was utilized by approximating the original membership functions with staircase membership functions, and the controller was designed based on piecewise Lyapunov functions.

Previous studies usually found sufficient conditions for the stability of T-S fuzzy systems, and it was often needed to introduce slag matrices, which led to complicated stability results. In this article, the asymptotically necessary and sufficient quadratic stability conditions of T-S fuzzy systems are derived using the staircase membership function and a basic inequality. The derivation process of the stability results is simple and easy to understand. The paper is organized as follows. In Section 2, the T-S fuzzy model and fuzzy controller are briefly described. In Section 3, the stability of T-S fuzzy systems is investigated based on quadratic Lyapunov functions. The asymptotically necessary and sufficient quadratic stability conditions are derived using the staircase membership functions and a basic inequality. Case studies are given to verify the validity of the obtained stability results. In Section 4, a fuzzy controller is designed based on the obtained stability results. Finally, Section 5 presents a summary and discussion.

2. Preliminaries

A T-S fuzzy model system is described as

Model Rule i: IF $z_1(t)$ is M_1^i and … and $z_p(t)$ is M_p^i, **THEN**

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad i = 1,2,\cdots,r, \tag{1}$$

where the dot denotes the derivative with respect to time *t*, $z_j(t)$ ($j = 1,2,\cdots,p$) are the premise variables, M_j^i ($i = 1,2,\cdots,r; j = 1,2,\cdots,p$) are the fuzzy sets, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector.

The linear systems (1) are called the subsystems of the T-S fuzzy system. The T-S fuzzy system is obtained by combining the subsystems (1),

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right), \tag{2}$$

where $z(t) = \left[z_1(t),\cdots,z_p(t)\right]^{\mathrm{T}} \in \Omega \subset R^p$ are the vector of premise variables, Ω is the operation domain of the premise variables,

$$h_i(z(t)) = \frac{\prod_{j=1}^{p} M_j^i(z_j(t))}{\sum_{i=1}^{r}\prod_{j=1}^{p} M_j^i(z_j(t))},$$

the term $M_j^i(z_j(t))$ represents the grade of membership of $z_j(t)$ in the fuzzy set M_j^i. One has $\sum_{i=1}^{r} h_i(z(t)) = 1$ and $h_i(z(t)) \geq 0$ for all *i*.

The parallel distributed compensation (PDC) is used to close the controlled fuzzy system. The PDC reads

$$u(t) = -\sum_{i=1}^{r} h_i(z(t)) F_i x(t), \tag{3}$$

where F_i ($i = 1,\cdots,r$) are the feedback strength matrices, which are determined by the control design.

Substituting Equation (3) into Equation (2) leads to the closed-loop fuzzy system

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))\left(A_i - B_i\left(\sum_{j=1}^{r} h_j(z(t)) F_j\right)\right)x(t) \\ &= \sum_{i=1}^{r} h_i(z(t)) A_i x(t) - \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) B_i F_j x(t).\end{aligned} \tag{4}$$

The stability of Equation (4) is investigated by considering the quadratic Lyapunov function

$$V(x(t)) = x^{\mathrm{T}}(t) P x(t). \tag{5}$$

where $P = P^{\mathrm{T}}$ is a positive matrix.

From Equations (4) and (5), one has

$$\begin{aligned}\dot{V}(x(t)) &= \dot{x}^{\mathrm{T}}(t) P x(t) + x^{\mathrm{T}}(t) P \dot{x}(t) \\ &= x^{\mathrm{T}}(t)\Bigg[\sum_{i=1}^{r} h_i(z(t))\left(A_i^{\mathrm{T}} P + P A_i\right) \\ &\quad - \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t))\left(\left(B_i F_j\right)^{\mathrm{T}} P + P\left(B_i F_j\right)\right)\Bigg] x(t) \\ &= x^{\mathrm{T}}(t)\Bigg[\sum_{i=1}^{r} h_i(z(t))\left(A_i^{\mathrm{T}} P + P A_i\right) \\ &\quad - \sum_{i \leq j} h_i(z(t)) h_j(z(t))\left(G_{ij}^{\mathrm{T}} P + P G_{ij}\right)\Bigg] x(t) \\ &= x^{\mathrm{T}}(t) Q_1(z(t)) x(t)\end{aligned} \tag{6}$$

where $G_{ii} = B_i F_i$, $G_{ij} = B_i F_j + B_j F_i$ ($i < j$), and

$$Q_1\left(z(t)\right) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i^{\mathrm{T}} P + PA_i\right) - \sum_{i \le j} h_i\left(z(t)\right) h_j\left(z(t)\right)\left(G_{ij}^{\mathrm{T}} P + PG_{ij}\right).$$

The controlled system (4) is stable if there exists a positive matrix P such that the inequalities $Q_1\left(z(t)\right) < 0$ hold for any $z(t) \in \Omega$. However, there are infinite inequalities $Q_1\left(z(t)\right) < 0$, since $h_i\left(z(t)\right)$ is a continuous function of the premise variables, and one cannot determine the stability of system (4) by checking the infinite inequalities. To overcome this difficulty, the continuous membership functions $h_i\left(z(t)\right)$ are approximated with staircase membership functions. Thus, the infinite inequalities $Q_1\left(z(t)\right) < 0$ are replaced by finite inequalities. Then, the asymptotically necessary and sufficient quadratic stability conditions of the controlled system (4) are obtained using a basic inequality.

3. Asymptotically Necessary and Sufficient Quadratic Stability Condition

First, a basic inequality that plays a key role in deriving the main results is introduced.

Lemma 1: Assume that A and B are two isotype matrices. Then, for any $\varepsilon > 0$, the following basic inequality holds:

$$-\left(A^{\mathrm{T}} B + B^{\mathrm{T}} A\right) \le \varepsilon A^{\mathrm{T}} A + \frac{1}{\varepsilon} B^{\mathrm{T}} B. \tag{7}$$

Proof: The following equality holds:

$$\left(\sqrt{\varepsilon} A + \frac{1}{\sqrt{\varepsilon}} B\right)^{\mathrm{T}} \left(\sqrt{\varepsilon} A + \frac{1}{\sqrt{\varepsilon}} B\right) = \varepsilon A^{\mathrm{T}} A + \frac{1}{\varepsilon} B^{\mathrm{T}} B + \left(A^{\mathrm{T}} B + B^{\mathrm{T}} A\right).$$

$\left(\sqrt{\varepsilon} A + \frac{1}{\sqrt{\varepsilon}} B\right)^{\mathrm{T}} \left(\sqrt{\varepsilon} A + \frac{1}{\sqrt{\varepsilon}} B\right) \ge 0$ implies that

$\varepsilon A^{\mathrm{T}} A + \frac{1}{\varepsilon} B^{\mathrm{T}} B \ge -\left(A^{\mathrm{T}} B + B^{\mathrm{T}} A\right)$, which concludes the proof.

For the sake of converting the infinite inequalities $Q\left(z(t)\right) < 0$ to finite inequalities, it is required to approximate the continuous membership functions $h_i\left(z(t)\right)$ with staircase membership functions. To this end, the operation domain Ω of the premise variables is divided into a series of subdomains Ω_k ($k = 1, \cdots, N$), where

$$\bigcup_{k=1}^{N} \Omega_k = \Omega, \quad \Omega_{k_1} \cap \Omega_{k_2} = \varnothing\left(k_1 \ne k_2\right),$$

$\varnothing$ represents the empty set. Let $d_{\max}$ denote the maximum diameter of the subdomains Ω_i and $\lim_{N \to +\infty} d_{\max} = 0$. The staircase membership functions are defined as

$$\bar{h}_i\left(z(t)\right) = h_i\left(\zeta_k\right), \quad z(t) \in \Omega_k, \tag{8}$$

where ζ_k is the center point of Ω_k ($k = 1, 2, \cdots, N$). The staircase membership functions $\bar{h}_i\left(z(t)\right)$ have a finite number of values rather than the infinite number of values of the original continuous membership functions $h_i\left(z(t)\right)$, and one has $\sum_{i=1}^{r}\bar{h}_i\left(z(t)\right)=1$ and $\bar{h}_i\left(z(t)\right)\geq 0$ for all *i*.

The following notation is introduced

$$\Delta_i = h_i\left(z(t)\right)-\bar{h}_i\left(z(t)\right),\quad \Delta_{ij} = h_i\left(z(t)\right)h_j\left(z(t)\right)-\bar{h}_i\left(z(t)\right)\bar{h}_j\left(z(t)\right),$$
$$\delta_i = \max_{z(t)\in\Omega}\left|\Delta_i\right|,\qquad \delta_{ij} = \max_{z(t)\in\Omega}\left|\Delta_{ij}\right|.$$

The following equalities hold:

$$\lim_{N\to+\infty}\delta_i = 0,\quad \lim_{N\to+\infty}\delta_{ij} = 0. \tag{9}$$

For brevity, $h_i\left(z(t)\right)$ are represented by h_i below.

Theorem 1: The closed-loop system (4) is stable if there exist a positive matrix $P=P^{\mathrm{T}}$ and positive real numbers ε_i ($i=1,\cdots,r$) and ε_{ij} ($i\leq j,i,j=1,\cdots,r$), such that

$$\begin{bmatrix}
Q_2\left(z(t)\right)+\sum_{i=1}^{r}\varepsilon_i A_i^{\mathrm{T}}A_i+\sum_{i\leq j}\varepsilon_{ij}G_{ij}^{\mathrm{T}}G_{ij} & P & \cdots & P & P & \cdots & P & P & \cdots & P \\
P & -\frac{\varepsilon_1}{\delta_1^2}E & & & & & & & & \\
\vdots & & \ddots & & & & & & & \\
P & & & -\frac{\varepsilon_r}{\delta_r^2}E & & & & & & \\
P & & & & -\frac{\varepsilon_{11}}{\delta_{11}^2}E & & & & & \\
\vdots & & & & & \ddots & & & & \\
P & & & & & & -\frac{\varepsilon_{1r}}{\delta_{1r}^2}E & & & \\
P & & & & & & & -\frac{\varepsilon_{22}}{\delta_{22}^2}E & & \\
\vdots & & & & & & & & \ddots & \\
P & & & & & & & & & -\frac{\varepsilon_{rr}}{\delta_{rr}^2}E
\end{bmatrix}<0 \tag{10}$$

where $Q_2\left(z(t)\right)=\sum_{i=1}^{r}\bar{h}_i\left(A_i^{\mathrm{T}}P+PA_i\right)-\sum_{i\leq j}\bar{h}_i\bar{h}_j\left(G_{ij}^{\mathrm{T}}P+PG_{ij}^{\mathrm{T}}\right)$, and E is the identity matrix.

Proof: The closed-loop system (4) is rewritten as

$$\dot{x}(t)=\sum_{i=1}^{r}\bar{h}_i A_i x(t)-\sum_{i\le j}\bar{h}_i\bar{h}_j G_{ij}x(t)+\sum_{i=1}^{r}\Delta_i A_i x(t)-\sum_{i\le j}\Delta_{ij}G_{ij}x(t). \tag{11}$$

The time-derivative of the quadratic Lyapunov function (5) is

$$\begin{aligned}\dot{V}(x(t))&=\dot{x}^{\mathrm{T}}(t)Px(t)+x^{\mathrm{T}}(t)P\dot{x}(t)\\&=x^{\mathrm{T}}(t)\Bigg[\sum_{i=1}^{r}\bar{h}_i\left(A_i^{\mathrm{T}}P+PA_i\right)-\sum_{i\le j}\bar{h}_i\bar{h}_j\left(G_{ij}^{\mathrm{T}}P+PG_{ij}\right)\\&\quad+\sum_{i=1}^{r}\Delta_i\left(A_i^{\mathrm{T}}P+PA_i\right)-\sum_{i\le j}\Delta_{ij}\left(G_{ij}^{\mathrm{T}}P+PG_{ij}\right)\Bigg]x(t)\\&=x^{\mathrm{T}}(t)\Bigg[Q_2(z(t))+\sum_{i=1}^{r}\Delta_i\left(A_i^{\mathrm{T}}P+PA_i\right)-\sum_{i\le j}\Delta_{ij}\left(G_{ij}^{\mathrm{T}}P+PG_{ij}\right)\Bigg]x(t).\end{aligned} \tag{12}$$

The following inequalities are derived from the results of Lemma 1,

$$\Delta_i\left(A_i^{\mathrm{T}}P+PA_i\right)\le\varepsilon_i A_i^{\mathrm{T}}A_i+\frac{\Delta_i\Delta_i}{\varepsilon_i}PP\le\varepsilon_i A_i^{\mathrm{T}}A_i+\frac{\delta_i^2}{\varepsilon_i}PP, \tag{13}$$

$$\begin{aligned}-\Delta_{ij}\left(G_{ij}^{\mathrm{T}}P+PG_{ij}^{\mathrm{T}}\right)&\le\varepsilon_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\frac{\Delta_{ij}\Delta_{ij}}{\varepsilon_{ij}}PP\\&\le\varepsilon_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\frac{\delta_{ij}^2}{\varepsilon_{ij}}PP.\end{aligned} \tag{14}$$

Substituting Equations (13) and (14) into Equation (12) yields

$$\begin{aligned}\dot{V}(x(t))&\le x^{\mathrm{T}}(t)\Bigg[Q_2(z(t))+\sum_{i=1}^{r}\varepsilon_i A_i^{\mathrm{T}}A_i+\sum_{i\le j}\varepsilon_{ij}G_{ij}^{\mathrm{T}}G_{ij}\\&\quad+\sum_{i=1}^{r}\frac{\delta_i^2}{\varepsilon_i}PP+\sum_{i\le j}\frac{\delta_{ij}^2}{\varepsilon_{ij}}PP\Bigg]x(t)\\&=x^{\mathrm{T}}(t)\Lambda_1(z(t))x(t),\end{aligned} \tag{15}$$

where

$$\Lambda_1(z(t))=Q_2(z(t))+\sum_{i=1}^{r}\varepsilon_i A_i^{\mathrm{T}}A_i+\sum_{i\le j}\varepsilon_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\sum_{i=1}^{r}\frac{\delta_i^2}{\varepsilon_i}PP+\sum_{i\le j}\frac{\delta_{ij}^2}{\varepsilon_{ij}}PP.$$

Applying Schur complements [1] on $\Lambda_1(z(t))<0$ leads to the inequalities (10), which concludes the proof.

It must be pointed out that the matrices on the left side of the inequalities (10) are ill-conditioned because the parameters δ_i and δ_{ij} are small when N is large. This makes it difficult to solve the LMIs (10) with the conventional convex-programming techniques. To reduce the condition number of the matrices, let $\varepsilon_i=\delta_i$ and $\varepsilon_{ij}=\delta_{ij}$ in the inequalities (13) and (14), respectively. Then, we have the following results.

Theorem 2: The closed-loop system (4) is stable if there exists a positive matrix $P=P^{\mathrm{T}}$, such that

$$\begin{bmatrix} Q_2\left(z(t)\right)+\sum_{i=1}^{r}\delta_i A_i^{\mathrm{T}} A_i+\sum_{i\le j}\delta_{ij} G_{ij}^{\mathrm{T}} G_{ij} & \sqrt{\sum_{i=1}^{r}\delta_i+\sum_{i\le j}\delta_{ij}}\,P \\ \sqrt{\sum_{i=1}^{r}\delta_i+\sum_{i\le j}\delta_{ij}}\,P & -E \end{bmatrix}<0. \tag{16}$$

Proof: Substituting $\varepsilon_i=\delta_i$ and $\varepsilon_{ij}=\delta_{ij}$ into Equations (13) and (14) leads to

$$\Delta_i\left(A_i^{\mathrm{T}}P+PA_i\right)\le\delta_i A_i^{\mathrm{T}}A_i+\delta_i PP, \tag{17}$$

$$-\Delta_{ij}\left(G_{ij}^{\mathrm{T}}P+PG_{ij}^{\mathrm{T}}\right)\le\delta_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\delta_{ij}PP. \tag{18}$$

Substituting Equations (17) and (18) into Equation (12) leads to

$$\begin{aligned}\dot{V}\left(x(t)\right)&\le x^{\mathrm{T}}(t)\left[Q_2\left(z(t)\right)+\sum_{i=1}^{r}\delta_i A_i^{\mathrm{T}}A_i+\sum_{i\le j}\delta_{ij}G_{ij}^{\mathrm{T}}G_{ij}\right.\\ &\quad\left.+\left(\sum_{i=1}^{r}\delta_i+\sum_{i\le j}\delta_{ij}\right)PP\right]x(t)\\ &=x^{\mathrm{T}}(t)\Lambda_2\left(z(t)\right)x(t),\end{aligned} \tag{19}$$

where

$$\Lambda_2\left(z(t)\right)=Q_2\left(z(t)\right)+\sum_{i=1}^{r}\delta_i A_i^{\mathrm{T}}A_i+\sum_{i\le j}\delta_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\left(\sum_{i=1}^{r}\delta_i+\sum_{i\le j}\delta_{ij}\right)PP.$$

Applying the Schur complement on $\Lambda_2\left(z(t)\right)<0$ leads to the inequalities (16), which concludes the proof.

Theorem 3: The quadratic stability conditions proposed in Theorems 1 and 2 are asymptotically necessary and sufficient for the stability of the system (4).

Proof: It is only necessary to prove that the conditions proposed in Theorems 1 and 2 are asymptotically necessary stability conditions of the system (4). Let P be the positive symmetric matrix, such that $Q_1\left(z(t)\right)<0$, which implies that $Q_2\left(z(t)\right)<0$. There exists a small positive number μ_0, such that $Q_2\left(z(t)\right)+\mu_0 E<0$. Since ε_i and ε_{ij} are adjustable positive numbers, without loss of generality, let $\varepsilon_i=\delta_i$ and $\varepsilon_{ij}=\delta_{ij}$. Then,

$$\begin{aligned}\Lambda_1\left(z(t)\right)&=\Lambda_2\left(z(t)\right)\\ &=Q_2\left(z(t)\right)+\sum_{i=1}^{r}\delta_i A_i^{\mathrm{T}}A_i+\sum_{i\le j}\delta_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\left(\sum_{i=1}^{r}\delta_i+\sum_{i\le j}\delta_{ij}\right)PP.\end{aligned}$$

Since $\lim_{N\to+\infty}\delta_i=0$ and $\lim_{N\to+\infty}\delta_{ij}=0$, there is a positive number N_0, such that

$$\sum_{i=1}^{r}\delta_i A_i^{\mathrm{T}}A_i+\sum_{i\le j}\delta_{ij}G_{ij}^{\mathrm{T}}G_{ij}+\left(\sum_{i=1}^{r}\delta_i+\sum_{i\le j}\delta_{ij}\right)PP\le\mu_0 E$$

holds for $N\ge N_0$. Thus, $\Lambda_1\left(z(t)\right)<0$ and $\Lambda_2\left(z(t)\right)<0$ hold for $N\ge N_0$, which concludes the proof.

Case study 1: Consider a T-S fuzzy model system composed of two rules:

Model Rule i: IF $x_1(t)$ is M_1^i, **THEN**

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad i = 1,2, \tag{20}$$

where

$$A_1 = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 15 & -10 \\ a & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

The membership functions are

$$h_1(x_1(t)) = M_1^1(x_1(t)) = \frac{10 - x_1(t)}{20}, \quad h_2(x_1(t)) = M_1^2(x_1(t)) = \frac{x_1(t) + 10}{20},$$

where $\Omega = [-10, 10]$ is the operation domain of the premise variable $x_1(t)$.

The operation domain Ω is divided into uniform subdomains Ω_i ($i = 1, \cdots, N$), and the staircase membership functions are $\bar{h}_1(x_1(t)) = \frac{10 - \zeta_i}{20}$ and $\bar{h}_2(x_1(t)) = \frac{\zeta_i + 10}{20}$ for $x_1(t) \in \Omega_i$, where ζ_i is the center of the subdomain Ω_i.

The local feedback strength matrices F_i ($i = 1, 2$) are determined such that -1 and -2 are the eigenvalues of the local subsystems (20). The feasible parameter regions of the adjustable parameters a and b are obtained by solving the LMIs (16) with $N = 4000$. The results are shown in **Figure 1**, where a comparison with previous results is given, which confirms the validity of the stability conditions in Theorems 1 and 2.

4. Fuzzy Controller Design

The fuzzy controller (3) is designed using the concept of PDC, which shares the same fuzzy rules as those of the T-S fuzzy system. The main results are summarized by the following theorem.

Theorem 4: The closed-loop system (4) is stabilized by the PDC controller (3) if there exist matrices $X = X^{\mathrm{T}} > 0$ and N_j ($j = 1, \cdots, r$) as well as real numbers

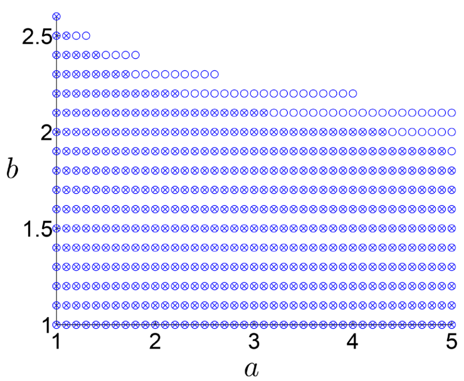

Figure 1. Stability region (Case study 1) given by the stability conditions in [12] ($\times$), and Theorem 2 ($\circ$).

$\varepsilon_i > 0$ ($i = 1, \cdots, r$) and $\varepsilon_{ij} > 0$ ($i, j = 1, \cdots, r$), such that

$$\begin{bmatrix} Q_3(z(t)) + \sum_{i=1}^{r} \varepsilon_i A_i A_i^{\mathrm{T}} + \sum_{i=1}^{r}\sum_{j=1}^{r} \varepsilon_{ij} B_i B_i^{\mathrm{T}} & X & \cdots & X & N_1^{\mathrm{T}} & \cdots & N_r^{\mathrm{T}} & N_1^{\mathrm{T}} & \cdots & N_r^{\mathrm{T}} \\ X & -\frac{\varepsilon_1}{\delta_1^2}E & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ X & & & -\frac{\varepsilon_r}{\delta_r^2}E & & & & & & \\ N_1 & & & & -\frac{\varepsilon_{11}}{\delta_{11}^2}E & & & & & \\ \vdots & & & & & \ddots & & & & \\ N_r & & & & & & -\frac{\varepsilon_{1r}}{\delta_{1r}^2}E & & & \\ N_1 & & & & & & & -\frac{\varepsilon_{21}}{\delta_{21}^2}E & & \\ \vdots & & & & & & & & \ddots & \\ N_r & & & & & & & & & -\frac{\varepsilon_{kk}}{\delta_{kk}^2}E \end{bmatrix} < 0 \tag{21}$$

where $Q_3(z(t)) = \sum_{i=1}^{r} \bar{h}_i (A_i X + X A_i^{\mathrm{T}}) - \sum_{i=1}^{r}\sum_{j=1}^{r} \bar{h}_i \bar{h}_j (B_i N_j + N_j^{\mathrm{T}} B_i^{\mathrm{T}})$. The feedback strength matrices are $F_i = N_i X^{-1}$ ($i = 1, \cdots, r$).

Proof: Letting $X = P^{-1}$, $x(t) = X y(t)$, and $N_i = F_i X$ ($i = 1, \cdots, r$) in Equation (12) yields

$$\begin{aligned} \dot{V}(x(t)) &= y^{\mathrm{T}}(t)\Bigg[\sum_{i=1}^{r} \bar{h}_i (A_i X + X A_i^{\mathrm{T}}) - \sum_{i=1}^{r}\sum_{j=1}^{r} \bar{h}_i \bar{h}_j (B_i N_j + N_j^{\mathrm{T}} B_i^{\mathrm{T}}) \\ &\quad + \sum_{i=1}^{r} \Delta_i (A_i X + X A_i^{\mathrm{T}}) - \sum_{i=1}^{r}\sum_{j=1}^{r} \Delta_{ij} (B_i N_j + N_j^{\mathrm{T}} B_i^{\mathrm{T}})\Bigg] y(t) \\ &= y^{\mathrm{T}}(t)\Bigg[Q_3(z(t)) + \sum_{i=1}^{r} \Delta_i (A_i X + X A_i^{\mathrm{T}}) - \sum_{i=1}^{r}\sum_{j=1}^{r} \Delta_{ij} (B_i N_j + N_j^{\mathrm{T}} B_i^{\mathrm{T}})\Bigg] y(t). \end{aligned} \tag{22}$$

The following inequalities are derived from the results of Lemma 1:

$$\Delta_i (A_i X + X A_i^{\mathrm{T}}) \le \varepsilon_i A_i A_i^{\mathrm{T}} + \frac{\Delta_i \Delta_i}{\varepsilon_i} XX \le \varepsilon_i A_i A_i^{\mathrm{T}} + \frac{\delta_i^2}{\varepsilon_i} XX, \tag{23}$$

$$-\Delta_{ij}\left(B_iN_j+N_j^{\mathrm{T}}B_i^{\mathrm{T}}\right)\le\varepsilon_{ij}B_iB_i^{\mathrm{T}}+\frac{\Delta_{ij}\Delta_{ij}}{\varepsilon_{ij}}N_j^{\mathrm{T}}N_j\le\varepsilon_{ij}B_iB_i^{\mathrm{T}}+\frac{\delta_{ij}^2}{\varepsilon_{ij}}N_j^{\mathrm{T}}N_j. \tag{24}$$

Substituting Equations (23) and (24) into Equation (22) yields

$$\begin{aligned}\dot{V}\left(x(t)\right)\le y^{\mathrm{T}}(t)\Bigg[&Q_3\left(z(t)\right)+\sum_{i=1}^r\varepsilon_iA_iA_i^{\mathrm{T}}+\sum_{i=1}^r\sum_{j=1}^r\varepsilon_{ij}B_iB_i^{\mathrm{T}}\\&+\sum_{i=1}^r\frac{\delta_i^2}{\varepsilon_i}XX+\sum_{i=1}^r\sum_{j=1}^r\frac{\delta_{ij}^2}{\varepsilon_{ij}}N_j^{\mathrm{T}}N_j\Bigg]y(t)\\=\;&y^{\mathrm{T}}(t)\Lambda_3\left(z(t)\right)y(t),\end{aligned} \tag{25}$$

where

$$\Lambda_3\left(z(t)\right)=Q_3\left(z(t)\right)+\sum_{i=1}^r\varepsilon_iA_iA_i^{\mathrm{T}}+\sum_{i=1}^r\sum_{j=1}^r\varepsilon_{ij}B_iB_i^{\mathrm{T}}+\sum_{i=1}^r\frac{\delta_i^2}{\varepsilon_i}XX+\sum_{i=1}^r\sum_{j=1}^r\frac{\delta_{ij}^2}{\varepsilon_{ij}}N_j^{\mathrm{T}}N_j.$$

Applying the Schur complements on $\Lambda_3\left(z(t)\right)<0$ leads to Equation (21), which concludes the proof.

To reduce the condition number of the matrices on the left side of the inequalities (21), let $\varepsilon_i=\delta_i$ and $\varepsilon_{ij}=\delta_{ij}$ in Equations (23) and (24), which leads to the following results.

Theorem 5: The closed-loop system (4) is stabilized by the PDC controller (3) if there exist matrices $X=X^{\mathrm{T}}>0$ and N_j ($j=1,\cdots,r$) such that

$$\begin{bmatrix}Q_3\left(z(t)\right)+\sum_{i=1}^r\delta_iA_iA_i^{\mathrm{T}}+\sum_{i=1}^r\sum_{j=1}^r\delta_{ij}B_iB_i^{\mathrm{T}} & \sqrt{\sum_{i=1}^r\delta_i}\,X & \sqrt{\sum_{i=1}^r\delta_{i1}}\,N_1^{\mathrm{T}} & \cdots & \sqrt{\sum_{i=1}^r\delta_{ir}}\,N_r^{\mathrm{T}}\\ \sqrt{\sum_{i=1}^r\delta_i}\,X & -E & & & \\ \sqrt{\sum_{i=1}^r\delta_{i1}}\,N_1 & & -E & & \\ \vdots & & & \ddots & \\ \sqrt{\sum_{i=1}^r\delta_{ir}}\,N_r & & & & -E\end{bmatrix}<0. \tag{26}$$

The feedback strength matrices are $F_i=N_iX^{-1}$ ($i=1,\cdots,r$).

Proof: Substituting $\varepsilon_i=\delta_i$ and $\varepsilon_{ij}=\delta_{ij}$ into Equations (23) and (24) leads to

$$\Delta_i\left(A_iX+XA_i^{\mathrm{T}}\right)\le\delta_iA_iA_i^{\mathrm{T}}+\delta_iXX, \tag{27}$$

$$-\Delta_{ij}\left(B_iN_j+N_j^{\mathrm{T}}B_i^{\mathrm{T}}\right)\le\delta_{ij}B_iB_i^{\mathrm{T}}+\delta_{ij}N_j^{\mathrm{T}}N_j. \tag{28}$$

Substituting Equations (27) and (28) into Equation (22) leads to

$$\begin{aligned}\dot{V}\left(x(t)\right)\le y^{\mathrm{T}}(t)\Bigg[&Q_3\left(z(t)\right)+\sum_{i=1}^r\delta_iA_iA_i^{\mathrm{T}}+\sum_{i=1}^r\sum_{j=1}^r\delta_{ij}B_iB_i^{\mathrm{T}}\\&+\sum_{i=1}^r\delta_iXX+\sum_{i=1}^r\sum_{j=1}^r\delta_{ij}N_j^{\mathrm{T}}N_j\Bigg]y(t)\\=\;&y^{\mathrm{T}}(t)\Lambda_4\left(z(t)\right)y(t),\end{aligned} \tag{29}$$

where

$$\Lambda_4(z(t)) = Q_3(z(t)) + \sum_{i=1}^{r}\delta_i A_i A_i^{\mathrm{T}} + \sum_{i=1}^{r}\sum_{j=1}^{r}\delta_{ij} B_i B_i^{\mathrm{T}} + \sum_{i=1}^{r}\delta_i XX + \sum_{i=1}^{r}\sum_{j=1}^{r}\delta_{ij} N_j^{\mathrm{T}} N_j.$$

Applying the Schur complements to $\Lambda_4(z(t)) < 0$ leads to Equation (26), which concludes the proof.

Case study 2: Consider a T-S fuzzy model system composed of three rules.

Model Rule i: IF $x_1(t)$ is M_1^i, **THEN**

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad i = 1,2,3. \tag{30}$$

where

$$A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}.$$

The membership functions are

$$h_1(x_1(t)) = 1 - \frac{1}{1+\mathrm{e}^{-(x+2)}},$$

$$h_3(x_1(t)) = 1 - \frac{1}{1+\mathrm{e}^{(x-2)}},$$

$$h_2(x_1(t)) = 1 - h_1(x_1(t)) - h_2(x_1(t)),$$

where $\Omega = [-10, 10]$ is the operation domain of the premise variable $x_1(t)$.

The staircase membership functions (8) are determined by dividing the operation domain into uniform subdomains Ω_i ($i = 1, \cdots, N$). The feasible parameter regions of the adjustable parameters a and b are obtained by solving the LMIs (26) with $N = 4000$. The feasible parameter region is shown in **Figure 2**, where a comparison with previous results is given, which confirms the validity of the stabilization conditions in Theorems 4 and 5.

5. Conclusions

Recent studies have focused on how to make use of the information of membership

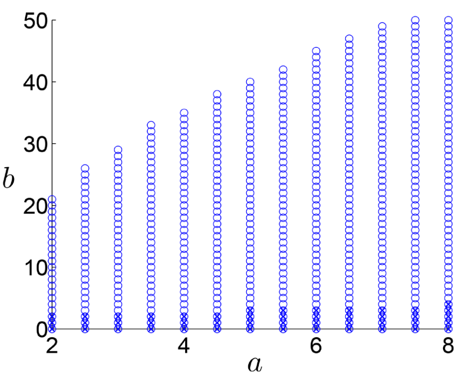

Figure 2. Stability region (Case study 2) given by the stability conditions in [7] ($\times$) and Theorem 5 ($\circ$).

functions to derive new relaxed stability and stabilization results of T-S fuzzy systems. The membership-function-approximation approaches, which can effectively incorporate the information of membership functions in stability analysis, have been frequently used to derive new relaxed stability and stabilization results. However, the stability conditions obtained in previous studies were usually sufficient for the stability of T-S fuzzy systems. In this study, asymptotically necessary and sufficient quadratic stability conditions were obtained in terms of linear matrix inequalities by utilizing staircase membership functions and a basic inequality. The derivation process of the stability results is straightforward and easy to understand.

There are some ill-conditioned matrices in the linear matrix inequalities that describe the asymptotically necessary and sufficient quadratic stability conditions of T-S fuzzy system, which makes it difficult to solve the linear matrix inequalities with conventional convex-programming techniques and severely limits the widespread use of the obtained stability results. Determining how to further reduce the condition number of the matrices in the obtained stability results is a subject worthy of future study.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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