Open Journal of Applied Sciences, 2022, 12, 1416-1438
https://www.scirp.org/journal/ojapps
ISSN Online: 2165-3925
ISSN Print: 2165-3917

Adaptive Boundary Control for the Dynamics of the Generalized Burgers-Huxley Equation

Zaki Mrzog Alaofi[1,2], Talaat Sayed Ali[3], Faisal Abd Alaal[4], Silvestru Sever Dragomir[2]

[1]Department of Mathematics, College of Science and Arts, King Khalid University, Muhayil Asir, Saudia Arabia
[2]Mathematics, College of Engineering & Science, Victoria University, Melbourne, Australia
[3]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebeen El-Kom, Egypt
[4]Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt

Email: zaki.alaofi@live.vu.edu.au, zaleawfi@kku.edu.sa, tdanaf@taibahu.edu.sa, faisal_ezz@sci.dmu.edu.eg, sever.dragomir@vu.edu.au

How to cite this paper: Alaofi, Z.M., Ali, T.S., Alaal, F.A. and Dragomir, S.S. (2022) Adaptive Boundary Control for the Dynamics of the Generalized Burgers-Huxley Equation. *Open Journal of Applied Sciences*, **12**, 1416-1438.
https://doi.org/10.4236/ojapps.2022.128098

Received: July 11, 2022
Accepted: August 22, 2022
Published: August 25, 2022

Abstract

This paper deals with the boundary control problem of the unforced generalized Burgers-Huxley equation with high order nonlinearity when the spatial domain is [0, 1]. We show that this type of equations are globally exponential stable in L^2 [0, 1] under zero Dirichlet boundary conditions. We use an adaptive nonlinear boundary controller to show the convergence of the solution to the trivial solution and to show that it achieves global asymptotic stability in time. We introduce numerical simulation for the controlled equation using the Adomian decomposition method (ADM) in order to illustrate the performance of the controller.

Keywords

Adaptive Boundary Control, Generalized Burgers-Huxley Equation, Stability, Adomian Decomposition Method

1. Introduction

Nonlinear partial differential equations (NPDE) have been widely studied by researchers over the years and have since become ubiquitous in nature [1]. Exact solutions rarely exist for nonlinear partial differential equations, and as a result of this, there has been much attention devoted recently to the search for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models [2]. Of the plethora of nonlinear partial differential equations, the Burgers-Huxley equation is finding an increasing number of useful applications in different fields. The Burgers-Huxley equation is a well-known nonlinear partial differential equation that simulates nonlinear wave phenomena in physics, biology, economics and ecology [3]. It finds application in many fields such as biology, nonlinear acoustics, metallurgy, chemistry, combustion, mathematics and engineering, as per Satsuma *et al.* [4]. It is a special type of nonlinear advection-diffusion reaction problem that is of importance in applications in mechanical engineering, material sciences, and neurophysiology. Some examples include particle transport, wall motion in liquid crystals [5], dynamics of ferroelectric materials [6], action potential propagation in nerve fibers [7]. Furthermore, some of the reaction processes have fascinating phenomena such as busting oscillation, population genetics, bifurcation, etc. [8]-[13].

The generalized Burger's-Huxley equation (GBHE) model offers applications in relation to propagating signals in the nervous system, elasticity, gas dynamics, and heat conduction [14]. The Burgers-Huxley equation was first introduced to describe turbulence in one space dimension, and has been used in several other physical contexts, including for instance sound waves in viscous media [15].

Many methods have been developed to solve the Burgers-Huxley equation such as the Adomian decomposition method (ADM) [16] [17] [18]. T. El-Danaf discussed some analytic properties of the generalized Burgers-Huxley equation such as the translation property and the steady state solution of the equation [19]. Using the first integral method, Xijun Deng studied travelling wave solutions of the generalized Burgers-Huxley equation in 2008 [20]. A year later, the homotopy analysis method (HAM) was applied to obtain the approximate analytical solutions of the generalised Burgers-Huxley and Huxley equations by A. Sami Bataineh *et al.* [21]. In 2010, N. Smaoui *et al.* designed three different adaptive control laws for the forced generalized Korteweg-de Vries-Burgers (GKdVB) equation when either the kinematic viscosity or the dynamic viscosity was unknown or when both viscosities were unknown [22]. In the same year, J. Biazar and F. Mohammadi applied the differential transform method (DTM) to the generalised Burgers-Huxley equation and some special cases of the equation like the Huxley equation and Fitzhugh-Nagoma equation [23]. A. G. Bratsos, in his 2011 research, proposed an implicit finite difference scheme based on fourth-order rational approximants to the matrix exponential term for the numerical solution of the Burgers-Huxley equation [24]. J.E. Macías-Díaz *et al.* (2011) developed a non-standard finite-difference scheme to approximate the solution of the generalized Burgers-Huxley equation from fluid dynamics [25]. In 2013, M. El-Kady *et al.* introduced treatments for the generalized Burgers-Huxley (GBH) equation that were dependent on cardinal Chebyshev and Legendre basis functions with the Galerkin method [26]. In the same year, S. S. Ray and A. K. Gupta solved the generalized Burgers-Huxley equation and Huxley equation using the Haar wavelet method [27]. J. Liu *et al.* (2013) used the double exp-function method to obtain a two-soliton solution of the generalized Burgers-Huxley equation [28]. A year later, A. Emad applied a relatively new semi-analytic technique, the reduced differential transform method (RDTM) to solve the generalized Burgers-Huxley equation and some special cases [29]. In 2015, V.J. Ervin *et al.* published a paper outlining a finite element scheme capable of preserving the non-negative and bounded solutions of the generalized Burgers-Huxley equation [30]. B. Inan (2016) applied an implicit exponential finite difference method to compute the numerical solutions of the nonlinear generalized Huxley equation [31]. N. Kumar and S. Singh proposed a numerical scheme for the solution of the generalized Burgers-Huxley equation using improved nodal integral method (MNIM) in 2016 [32]. In the same year, J. A. T. Machado *et al.* introduced an algorithm, based on adopting the approximate analytical solution of the Cauchy problem for the Burgers-Huxley equation [33]. In 2017, B. Inan presented an explicit exponential finite difference method to solve the generalized forms of the Huxley and Burgers-Huxley equations [34]. In 2018, I. Wasim *et al.* introduced a new numerical technique for solving nonlinear generalized Burgers-Fisher and Burgers-Huxley equations using the hybrid B-spline collocation method [35]. A. R. Appadu *et al.* (2019) obtained numerical solutions to the Burgers-Huxley equation with specified initial and boundary conditions using two novel non-standard finite difference schemes and two exponential finite difference schemes [36]. In the same year, Y. Fu discussed the persistence of travelling wavefronts in a generalized Burgers-Huxley equation with long-range diffusion [37]. A year later, L. Sun and C. Zhu developed a kind of cubic B-spline quasi-interpolation, which is used to solve Burgers-Huxley equations [38]. In 2020, M. A. Khan *et al.* demonstrated how to use the new auxiliary method for solitary wave solutions of the generalized Burgers Huxley equation (B-HE) [39]. A. Kumar and M. T. Mohan introduced an analytical global solvability as well as asymptotic analysis of stochastic generalized Burgers-Huxley (SGBH) equation perturbed by space-time white noise in a bounded interval of R in 2020 [40]. A. G. Kushner (2020) constructed such dynamics for the classical Burgers-Huxley equation and then used them to construct new exact solutions [41]. More recently, L. Ebiwareme (2021) proposed the Tanh-coth and Banach contraction methods to solve the Burgers-Huxley and Kuramoto-Sivashinsky equations [42]. In the same year, M. T. Mohan and A. Khan considered the forced generalized Burgers-Huxley equation and established the existence and uniqueness of a global weak solution using a Faedo-Galerkin approximation method [43].

Many researchers have worked on the control problems of the Burgers, Kuramoto-Sivashinsky (KS), KDV and KDVB equations (refer to [44] [45] [46] [47]). In ([48] [49] [50]), the authors obtained a nonlinear robust boundary control of the KS equations and a nonlinear robust stabilisation of the Korteweg-de Vries-Burgers equation (GKDVB) using the boundary control. In [51] and [52], Smaoui *et al.* obtained a nonlinear boundary control of the generalized Burgers and GKDVB equation. In [53] and [54], Smaoui *et al.* controlled the dynamics of Burgers and GKDVB equations using an adaptive boundary control. In [55], Smaoui and El-Gamil produced a paper dealing with the adaptive control of the unforced GKDVB equation using three different adaptive control laws.

The generalized Burgers-Huxley equation takes the form:

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right), 0\le x\le 1, t\ge 0, \tag{1}$$

where α,β,γ and δ are parameters that $\beta\ge 0,\delta>0,\gamma\in(0,1)$.

In population dynamics, $u(x,t)$ represent the population density, γ is the species carrying capacity, α stands for the speed of advection and β is a parameter that describes a nonlinear source. When a certain condition is imposed on the parameter, the generalized Burgers-Huxley equation is reduced to many parabolic evolution equations of physical insight.

These equations describe different phenomena in mathematical physics, biomathematics, chemistry and mechanics [56]. Equation (1) models the interaction between reaction mechanisms, convection effects and diffusion transports [57] [58]. The Burgers equation is a very interesting model due to the nonlinear advection $u^{\delta}u_x$ term, dissipation u_{xx} term, and the shock wave behavior when the Reynolds number is very large [59].

In this paper, an adaptive boundary control is developed for the generalized Burgers-Huxley Equations (1) with high order nonlinearity, the adomian decomposition method is investigated, to discuss the applicably of the adomian decomposition method an illustration numerical example isintroduced.

$$u_t+\alpha u^{\delta}u_x-\nu u_{xx}=\beta u\left(1-u^{\delta}\right)\left(u^{\delta}-\gamma\right),\ 0\le x\le 1,\ t\ge 0$$

with the initial condition $u(x,0)=f_0(x)$, and the boundary conditions

$$\begin{aligned} &au(0,t)+bu_x(0,t)=\omega_1(t),\\ &cu(1,t)+du_x(1,t)=\omega_2(t). \end{aligned} \tag{2}$$

2. Preliminaries

In this section, we present some basic propositions and lemmas that will become useful in the next sections.

Proposition (Gronwall-Bellman Inequality) [60].

Let $\gamma(t):[a,b]\to\mathbb{R}$ and $\alpha(t):[a,b]\to\mathbb{R}$ be two continuous functions and let $\beta(t)\ge 0$ be a non-negative integrable function on the same interval. If $\gamma(t)$ satisfies

$$\gamma(t)\le\alpha(t)+\int_a^t\beta(s)\gamma(s)\mathrm{d}s,\ a\le t\le b \tag{3}$$

andif the function $\alpha(t)$ is non-decreasing, then

$$\gamma(t)\le\alpha(t)\exp\left(\int_a^t\beta(\tau)\mathrm{d}\tau\right)\ \text{for}\ a\le t\le b. \tag{4}$$

Lemma 1. [61]

Let $\beta<0$. If $u(x,t)\in L^2(0,\infty)$, then

$$\int_0^t\exp\left(\beta(t-\tau)\right)u^2(1,\tau)\mathrm{d}\tau\to 0 \tag{5}$$

Lemma 2. [61]

Let $\beta < 0$, if $u(x,t) \in L^{2\alpha+2}(0,\infty)$, then

$$\int_0^t \exp\left(\beta(t-\tau)\right) u^{2\alpha+2}(1,\tau)\,\mathrm{d}\tau \to 0 \quad \text{as} \quad t \to \infty. \tag{6}$$

3. Global Exponential Stability of the Generalized Burgers-Huxley Equation with Zero Dirichlet Conditions

In this section, we state and prove a theorem to show these types of equations are globally exponential stable in L^2 [0, 1] under zero Dirichlet boundary conditions.

Theorem 1.

Let δ be a positive integer, $\nu > 0$ and $\gamma \le 1$; then the generalized Burgers-Huxley equation with zero Dirichlet boundary conditions is globally exponential stable in L^2 (0, 1).

Proof

Multiplying both sides of Equation (1) by $2u(x,t)$, we obtain

$$2uu_t + 2\alpha u^{\delta+1}u_x - 2\mathcal{V}uu_{xx} = -2\beta u^2\left(u^\delta - 1\right)\left(u^\delta - \gamma\right), \quad 0 \le x \le 1, \; t \ge 0 \tag{7}$$

By integrating Equation (7) from 0 to 1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u^2 \mathrm{d}x + 2\alpha\int_0^1 u^{\delta+1}u_x \mathrm{d}x - 2\nu\int_0^1 uu_{xx}\mathrm{d}x = -2\beta\int_0^1 u^2\left(u^\delta - 1\right)\left(u^\delta - \gamma\right)\mathrm{d}x, \tag{8}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{2\alpha}{\delta+2}\left[u^{\delta+2}(1,t) - u^{\delta+2}(0,t)\right] - 2\nu\left[u(1,t)u_x(1,t) - u(0,t)u_x(0,t)\right] + 2\nu\|u_x\|^2 \\ &= -2\beta\int_0^1\left(u^{2\delta+2} - (\gamma+1)u^{\delta+2} + \gamma u^2\right)\mathrm{d}x.\end{aligned} \tag{9}$$

Using the Dirichlet boundary condition $u(0,t) = u(1,t) = 0$ on Equation (9), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\nu\|u_x\|^2 = -2\beta\int_0^1\left(u^{2\delta+2} - (\gamma+1)u^{\delta+2} + \gamma u^2\right)\mathrm{d}x \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\nu\|u_x\|^2 = -2\beta\gamma\|u\|^2 + 2\beta(\gamma+1)\int_0^1 uu^{\delta+1}\mathrm{d}x - 2\beta\left\|u^{\delta+1}\right\|^2 \tag{11}$$

Using the Cauchy Shwartz and the Young inequalities, we have

$$2\beta(\gamma+1)\int_0^1 uu^{\delta+1}\mathrm{d}x \le \beta(\gamma+1)\left(\|u\|^2 + \left\|u^{\delta+1}\right\|^2\right) \tag{12}$$

From Equation (7) and inequality (12), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\nu\|u_x\|^2 &= -2\beta\gamma\|u\|^2 + 2\beta(\gamma+1)\int_0^1 uu^{\delta+1}\mathrm{d}x - 2\beta\left\|u^{\delta+1}\right\|^2 \\ &\le -2\beta\gamma\|u\|^2 + \beta(\gamma+1)\|u\|^2 + \left\|u^{\delta+1}\right\|^2 - 2\beta\left\|u^{\delta+1}\right\|^2 \\ &= (-\beta\gamma+\beta)\|u\|^2 + \left\|u^{\delta+1}\right\|^2 - 2\beta\left\|u^{\delta+1}\right\|^2\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\nu\|u_x\|^2 \le (-\beta\gamma+\beta)\|u\|^2 + (\beta\gamma-\beta)\left\|u^{\delta+1}\right\|^2 \tag{13}$$

Since $\left\|u^{\delta+1}\right\|^2 \le \left\|u^\delta\right\|^2\|u\|^2$, $\left\|u^\delta\right\|^2 \le \|u\|^{2\delta}$, we have

$$\left\|u^{\delta+1}\right\|^2 \le \|u\|^{2\delta+2}, \tag{14}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \le -2\nu\|u_x\|^2 + \beta(1-\gamma)\left[\|u\|^2 + \|u\|^{2\delta+2}\right] \tag{15}$$

Since $\|u\|^{2\delta+2} \ge \|u\|^2$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \le -2\nu\|u_x\|^2 \tag{16}$$

Using the Poincare inequality [62], we get

$$-2\nu\|u_x\|^2 \le \frac{-\upsilon}{2}\|u\|^2 \tag{17}$$

By the basic comparison of inequality (17) with the first order differential inequalities, we have

$$\|u\|^2 \le \|u_0\|^2 \exp(-2\nu t). \tag{18}$$

Therefore, $\|u(x,t)\|$ converges to zero exponentially when $t \to \infty$.

4. The Construction of the Adaptive Boundary Control for the Generalized Burgers-Huxley Equation

In this section, we build an adaptive boundary control for Equation (1) as follows.

Theorem 2.

Let $\delta > 0$, $\gamma \le 1$, then the solution $u(x,t)$ of Equation (1) with initial condition $f_0(x) \in H^3(0,1)$, which satisfying the boundary conditions (2) such that *a*, *b*, *c*, *d* are arbitrary constants has the property $\|u(.,t)\| \to 0$ as $t \to \infty$.

Proof.

If $u(0,t), u(1,t)$ are locally existing in $L^{2\alpha+2}(0,1)$ and the control functions $\omega_1(t), \omega_2(t)$ are given by

$$\omega_1(t) = k_1(t)u^{2\delta+1}(0,t) + k_2(t)u^{\delta+1}(0,t) + k_3(t)u(0,t), \tag{19}$$

$$\omega_2(t) = k_4(t)u^{2\delta+1}(1,t) + k_5(t)u^{\delta+1}(1,t) + k_6(t)u(1,t), \tag{20}$$

such that $k_n(t), n = 1,2,\cdots,6$ are bounded for any $t \ge 0$.

Now, we proceed with proving the theorem.

Consider the following Lyapunov function candidate [63]

$$V(t) = \int_0^1 u^2(x,t)\,\mathrm{d}x \tag{21}$$

Operate on (21) with the differential operator with respect to *t* and using Equation (1) gives

$$V'(t) = \int_0^1 u(x,t)u_t(x,t)\,\mathrm{d}x = \int_0^1 u\left(-\alpha u^\delta u_x + \upsilon u_{xx} + \beta u\left(u^\delta - 1\right)\left(u^\delta - \gamma\right)\right)\mathrm{d}x \tag{22}$$

Thus,

$$\begin{aligned} V'(t) &= -\alpha\int_0^1 u^{\delta+1}u_x\,\mathrm{d}x + \upsilon\int_0^1 uu_{xx}\,\mathrm{d}x + \beta\int_0^1 u^2\left(u^\delta - 1\right)\left(u^\delta - \gamma\right)\mathrm{d}x \\ &= \frac{-\alpha}{\delta+2}\left[u^{\delta+2}(1,t) - u^{\delta+2}(0,t)\right] + \upsilon\left[u(1,t)u_x(1,t) - u(0,t)u_x(0,t)\right] - \upsilon\|u_x\|^2 \\ &\quad + \beta\int_0^1\left(\left(u^{\delta+1}\right)^2 - (\gamma+1)uu^{\delta+1} + \gamma u^2\right)\mathrm{d}x. \end{aligned} \tag{23}$$

Now, using the Cauchy Shwartz and the Young inequalities, we have

$$\beta(\gamma+1)\int_0^1 uu^{\delta+1}\mathrm{d}x \le \frac{\beta(\gamma+1)}{2}\left(\|u\|^2+\|u^{\delta+1}\|^2\right) \tag{24}$$

From the Poincare inequality, we obtain

$$-\upsilon\|u_x\|^2 \le \frac{-\upsilon}{4}\|u\|^2+\frac{\upsilon}{2}u^2(0,t) \tag{25}$$

$$\begin{aligned} V'(t) \le{}& \frac{-\alpha}{\delta+2}\left[u^{\delta+2}(1,t)-u^{\delta+2}(0,t)\right]+\upsilon\left[u(1,t)u_x(1,t)-u(0,t)u_x(0,t)\right] \\ &+\frac{\upsilon}{2}u^2(0,t)-\frac{\upsilon}{4}\|u\|^2+\beta\int_0^1\left(\left(u^{\delta+1}\right)^2-(\gamma+1)uu^{\delta+1}+\gamma u^2\right)\mathrm{d}x. \end{aligned}$$

From (10),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+2\nu\|u_x\|^2=-2\beta\int_0^1\left(u^{2\delta+2}-(\gamma+1)u^{\delta+2}+\gamma u^2\right)\mathrm{d}x\,.$$

Then, we get

$$-\frac{\upsilon}{4}\|u\|^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2-\nu\|u_x\|^2$$

From (15),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \le -2\nu\|u_x\|^2+\beta(1-\gamma)\left[\|u\|^2-\|u\|^{2\delta+2}\right]$$

Then, we get

$$-\frac{\upsilon}{4}\|u\|^2-\frac{1}{2}\left(-2\nu\|u_x\|^2+\beta(1-\gamma)\left[\|u\|^2-\|u\|^{2\delta+2}\right]\right)-\nu\|u_x\|^2$$

$$-\frac{\upsilon}{4}\|u\|^2+\nu\|u_x\|^2-\frac{1}{2}\beta(1-\gamma)\left[\|u\|^2-\|u\|^{2\delta+2}\right]-\nu\|u_x\|^2$$

$$-\frac{\upsilon}{4}\|u\|^2-\frac{1}{2}\beta(1-\gamma)\|u\|^2+\frac{1}{2}\beta(1-\gamma)\|u\|^{2\delta+2}$$

$$\beta\left(1-\frac{\gamma+1}{2}\right)\|u^{\delta+1}\|^2+\left(-\beta\gamma-\frac{\upsilon}{4}-\frac{\beta\gamma}{2}+\frac{\beta}{2}\right)\|u\|^2 \tag{26}$$

Then, at $\gamma \le 1$, we have

$$\begin{aligned} V'(t) \le{}& \frac{-\alpha}{\delta+2}\left[u^{\delta+2}(1,t)-u^{\delta+2}(0,t)\right]+\upsilon\left[u(1,t)u_x(1,t)-u(0,t)u_x(0,t)\right] \\ &+\frac{\upsilon}{2}u^2(0,t)+\left(-\frac{\upsilon}{4}\right)\|u\|^2. \end{aligned} \tag{27}$$

from the first equation in Equation (2),

$$au(0,t)+bu_x(0,t)=\omega_1(t),$$

which implies to

$$u_x(0,t)=\frac{1}{b}\left(\omega_1(t)-au(0,t)\right) \quad \text{or} \quad u(0,t)=\frac{1}{a}\left(\omega_1(t)-bu_x(0,t)\right)$$

from the second equation in Equation (2),

$$cu(1,t)+du_x(1,t)=\omega_2(t),$$

which implies to

$$u_x(1,t)=\frac{1}{d}\left(\omega_2(t)-cu(1,t)\right) \quad \text{or} \quad u(1,t)=\frac{1}{c}\left(\omega_2(t)-du_x(1,t)\right)$$

then

$$u_x(0,t)=\frac{1}{b}\left(\omega_1(t)-bu(0,t)\right), \tag{28}$$

$$u_x(1,t)=\frac{1}{d}\left(\omega_2(t)-du(1,t)\right). \tag{29}$$

Using inequality (27) and Equation (28) and (29), we have

$$\begin{aligned} V'(t) \le {} & \left(-\frac{\upsilon}{4}\right)\|u\|^2+\frac{\upsilon}{2}u^2(0,t)-\frac{-\alpha}{\delta+2}\left[u^{\delta+2}(1,t)-u^{\delta+2}(0,t)\right] \\ & +\upsilon\left[u(1,t)\left(\frac{1}{d}\left(\omega_2(t)-du(1,t)\right)\right)-u(0,t)\left(\frac{1}{b}\left(\omega_1(t)-bu(0,t)\right)\right)\right]. \end{aligned} \tag{30}$$

Substituting by the suggested values of $\omega_1(t),\omega_2(t)$, we get

$$\begin{aligned} V'(t) \le {} & \left(-\frac{\upsilon}{4}\right)\|u\|^2+\frac{\upsilon}{2}u^2(0,t)-\frac{-\alpha}{\delta+2}\left[u^{\delta+2}(1,t)-u^{\delta+2}(0,t)\right] \\ & +\upsilon u(1,t)\left[\frac{1}{c}\left(k_4(t)u^{2\delta+2}(1,t)+k_5(t)u^{\delta+2}(1,t)+k_6(t)u(1,t)\right)-\frac{d}{c}u(1,t)\right] \\ & -\upsilon u(0,t)\left[\frac{1}{c}\left(k_1(t)u^{2\delta+2}(0,t)+k_2(t)u^{\delta+2}(0,t)+k_3(t)u(0,t)\right)-\frac{b}{a}u(0,t)\right]. \end{aligned} \tag{31}$$

We introduce the non-negative energy function $E(t)$, as follows.

$$\begin{aligned} E(t) = {} & V(t)+\frac{\upsilon}{2ar_1}\left(k_1(t)\right)^2+\frac{a}{2\upsilon r_2}\left(\frac{\upsilon}{a}k_2(t)-\frac{\alpha}{\delta+2}\right)^2 \\ & +\frac{a}{2\upsilon r_3}\left(\frac{\upsilon}{a}k_3(t)-\frac{\upsilon b}{a}-\frac{\upsilon}{2}\right)^2+\frac{\upsilon}{2cr_4}\left(k_4(t)\right)^2 \\ & +\frac{c}{2\upsilon r_5}\left(\frac{\upsilon}{c}k_5(t)-\frac{\alpha}{\delta+2}\right)^2+\frac{c}{2\upsilon r_6}\left(\frac{\upsilon}{c}k_6(t)-\frac{\upsilon d}{c}\right)^2 \end{aligned} \tag{32}$$

Evaluating the time derivative of $E(t)$ and substituting $V'(t)$ from inequality (31) and $k_n'(t)$ into Equation (32), we have

$$E'(t)\le\frac{-\upsilon}{4}\|u\|^2 \tag{33}$$

This implies that $E(t)\le E(0)$. Since $u(0,t)$ and $u(1,t)\in L^{2\alpha+2}(0,\infty)$, it follows that $k_j(t)$ can be defined as continuous functions on $(0,\infty)$. Then, Equation (32) and inequality (33) imply that $k_j(t), j=1,\cdots,6$ are bounded, which implies that:

$$u(i,t)\in L^2(0,\infty)\cap L^{2\alpha+2}(0,\infty),\ i=0,1$$

We also show the global asymptotic stability of Equation (1) and Equation (2). Using the Gronwall inequality on inequality (3), we have

$$V(t) \le V(0)\exp\left(\frac{-\upsilon}{4}t\right)$$
$$+\upsilon\int_0^t\left[\frac{-k_1(\tau)}{a}u^{2\delta+2}(0,\tau)+\left(\frac{\alpha}{(\delta+2)\upsilon}-\frac{k_2(\tau)}{a}\right)u^{\delta+2}(0,\tau)\right.$$
$$\left.+\left(\frac{1}{2}+\frac{b}{a}-\frac{k_3(\tau)}{a}\right)u^2(0,\tau)\right]\exp\left(\frac{-\upsilon}{4}(t-\tau)\right)\mathrm{d}\tau$$
$$+\upsilon\int_0^t\left[\frac{k_4(\tau)}{c}u^{2\delta+2}(1,\tau)+\left(\frac{-\alpha}{(\delta+2)\upsilon}+\frac{k_5(\tau)}{c}\right)u^{\delta+2}(1,\tau)\right.$$
$$\left.+\left(\frac{-d}{c}-\frac{k_6(\tau)}{c}\right)u^2(1,\tau)\right]\exp\left(\frac{-\upsilon}{4}(t-\tau)\right)\mathrm{d}\tau.$$

Next, using Lemma 1 and Lemma 2, we predict that $\|u(.,t)\| \to 0$ as $t \to \infty$.

5. Adomian Decomposition Method for the Initial Boundary Value Problem [64]

Consider the nonlinear initial boundary value problem of partial differential equation in the following general operator form:

$$Lu(x,t) = Ru(x,t) + Nu(x,t) + gu(x,t),\ \ 0<\alpha\le 1, \tag{34}$$

with the initial condition $u(x,0)=f_0(x)$, and the boundary conditions $u(0,t)=p(t)$ and $u(1,t)=q(t)$.

Where $L=\frac{\partial}{\partial t}$, is the highest partial derivative with respect to *t*, *R* is a linear operator, $N(u)$ is the nonlinear term and $g(x,t)$ is the source function. Operating on both sides of Equation (34) with the inverse operator L^{-1} gives:

$$u(x,t) = \phi + L^{-1}\left(g(x,t)\right) + L^{-1}\left(Ru(x,t)+Nu(x,t)\right) \tag{35}$$

where the first part from the right hand side of Equation (35) is obtained from the solution of the homogenous differential equation $L\phi = 0$.

The Adomian decomposition method defines the solution $u(x,t)$ as an infinite series in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{36}$$

where the components $u_n(x,t)$ can be obtained in recursive form. The nonlinear term $N(u)$ can be decomposed by an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n \tag{37}$$

The formula of Adomian polynomials is

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0},\ \ n=0,1,2,\cdots. \tag{38}$$

Substituting by Equation (36) and Equation (37) into Equation (35) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi + L^{-1}\left(g(x,t)\right) + L^{-1}\left(\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right). \tag{39}$$

Substituting the initial conditions, we can obtain the components $u_n(x,t)$ of the solution using the following formula

$$u_0(x,t) = f_0(x) + \phi + L^{-1}(g(x,t)),$$
$$u_{n+1}(x,t) = L^{-1}(Ru_n + A_n), \quad n \geq 0. \tag{40}$$

The initial solution can be written as

$$u_0(x,t) = f_0(x) \tag{41}$$

Construct a new successive approximate solution $u_n^*(x,t)$ as follows

$$u_n^*(x,t) = u_n(x,t) + (1-x)[p(t) - u_n(0,t)] + x[q(t) - u_n(1,t)], n = 0,1,2,\cdots \tag{42}$$

$$u_{n+1}^*(x,t) = L^{-1}(Ru_n^* + A_n^*), \tag{43}$$

such that

$$A_n^* = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i^*\right)\right]_{\lambda=0}, \quad n = 0,1,2,\cdots.$$

Using Equations (41-43), we obtain the approximate solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{44}$$

6. Numerical Example

Using the ADM algorithm that is presented in this section in Equation (1), when $\alpha = \beta = 1$, $\gamma = 0.001$ and $\delta = 2$, we solve the generalized Burgers-Huxley equation without control as outlined in the following tables from Tables 1-7, with time t = 0, 0.5, 1, 2, 3, 4 and t = 5. Table 8 gives the absolute errors for the generalized Burgers-Huxley equation using the Adomian decomposition method when $t = 0$ to $t = 1$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

Table 1. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 0$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

X	Numerical Solution	Exact Solution
0	0.0005	0.0005
0.1	0.000521699	0.000521699
0.2	0.000543317	0.000543317
0.3	0.000564773	0.000564773
0.4	0.000585989	0.000585989
0.5	0.00060689	0.00060689
0.6	0.000627407	0.000627407
0.7	0.000647476	0.000647476
0.8	0.000667037	0.000667037
0.9	0.000686039	0.000686039
1.0	0.000704437	0.000704437

Table 2. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 0.5$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

X	Numerical Solution	Exact Solution
0	0.000500415	0.000637703
0.1	0.000519779	0.00065752
0.2	0.000539102	0.000676802
0.3	0.000558343	0.000695501
0.4	0.000577459	0.000713576
0.5	0.000596404	0.000730993
0.6	0.000615127	0.000747726
0.7	0.000633579	0.000763754
0.8	0.000651706	0.000779064
0.9	0.000669457	0.000793651
1.0	0.000686782	0.000807512

Table 3. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 1$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

X	Numerical Solution	Exact Solution
0	0.000500772	0.00075599
0.1	0.000518010	0.000771653
0.2	0.000535386	0.000786595
0.3	0.000553023	0.000800811
0.4	0.000571005	0.000814304
0.5	0.000589367	0.00082708
0.6	0.000608090	0.000839151
0.7	0.000627102	0.000850532
0.8	0.000646283	0.00086124
0.9	0.000665479	0.000871298
1.0	0.000684512	0.000880727

Table 4. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 2$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

X	Numerical Solution	Exact Solution
0	0.000501326	0.000905649
0.1	0.000495311	0.000912814
0.2	0.00049128	0.000919482
0.3	0.000491024	0.000925683
0.4	0.000495977	0.000931441

Continued

0.5	0.000507089	0.000936784
0.6	0.000524761	0.000941736
0.7	0.000548837	0.000946323
0.8	0.000578658	0.000950567
0.9	0.000613156	0.000954492
1.0	0.000650983	0.000958119

Table 5. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 3$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

X	Numerical Solution	Exact Solution
0	0.000501689	0.000967468
0.1	0.000414376	0.000970093
0.2	0.000334647	0.000972513
0.3	0.000269394	0.000974741
0.4	0.000224215	0.000976794
0.5	0.000202957	0.000978683
0.6	0.000207464	0.000980422
0.7	0.00023755	0.000982021
0.8	0.000291162	0.000983492
0.9	0.000364719	0.000984844
1.0	0.00045356	0.000986088

Table 6. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 4$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

X	Numerical Solution	Exact Solution
0	0.000501889	0.000989263
0.1	0.000235977	0.000990147
0.2	−0.0000108612	0.00099096
0.3	−0.000221241	0.000991705
0.4	−0.000380963	0.00099239
0.5	−0.000480123	0.000993019
0.6	−0.000513747	0.000993596
0.7	−0.00048188	0.000994125
0.8	−0.00038919	0.000994611
0.9	−0.000244166	0.000995058
1.0	−0.0000580347	0.000995467

Table 7. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t=5$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

X	Numerical Solution	Exact Solution
0	0.000501954	0.000996509
0.1	−0.0000791146	0.000996799
0.2	−0.000621592	0.000997064
0.3	−0.00109026	0.000997308
0.4	−0.00145624	0.000997531
0.5	−0.00169924	0.000997736
0.6	−0.00180881	0.000997924
0.7	−0.00178456	0.000998096
0.8	−0.00163537	0.000998254
0.9	−0.00137782	0.000998399
1.0	−0.00103406	0.000998532

Table 8. The compression between the numerical and exact solution for the generalized Burgers-Huxley equation when $t=0$ to $t=1$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

t	The absolute errors
0	0.
0.1	0.0000401684
0.2	0.0000785384
0.3	0.000115102
0.4	0.00014986
0.5	0.000182822
0.6	0.00021401
0.7	0.00024346
0.8	0.000271219
0.9	0.000297349
1.0	0.000321925

To illustrate the behaviour of the numerical and exact solutions for the generalized Burgers-Huxley equation in various times, we introduce the following 2D figures from **Figures 1-7**, when $t=0,0.5,1,2,3,4,5$. The 3D figures are included in **Figures 8-12**. **Figure 13** shows the comparison between the numerical and exact solutions with control for the generalized Burgers-Huxley equation when $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$ from $t=0$ to $t=6$.

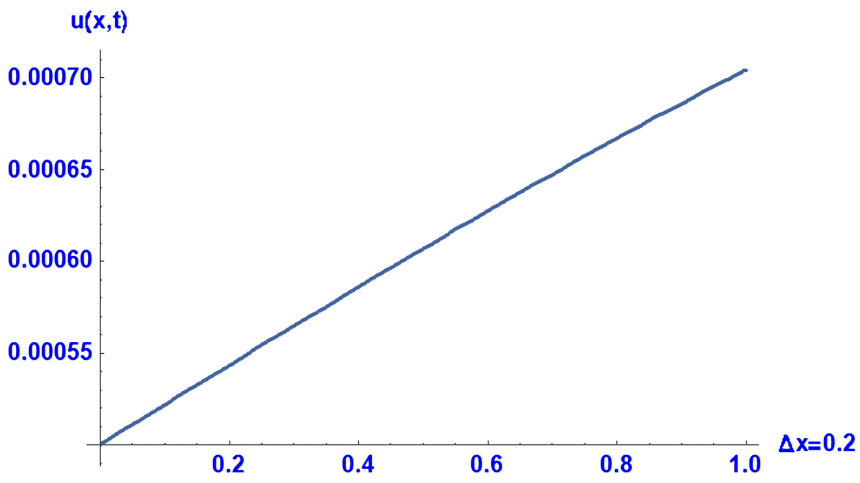

Figure 1. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t=0$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

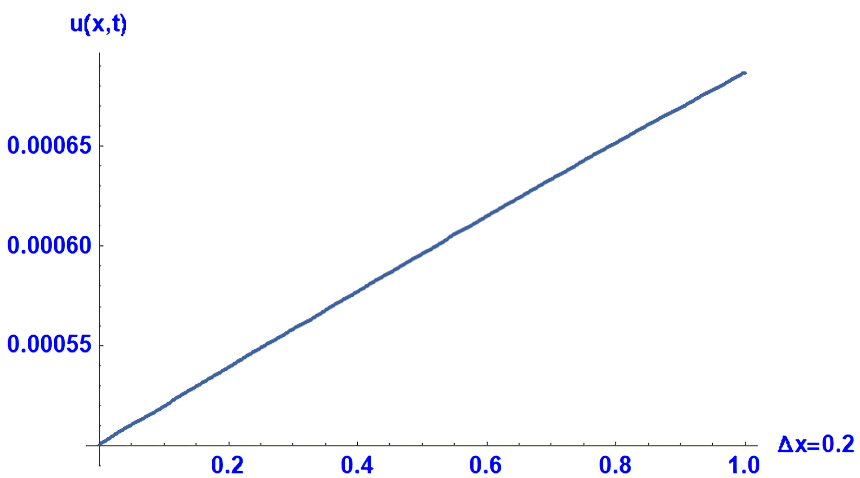

Figure 2. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t=0.5$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

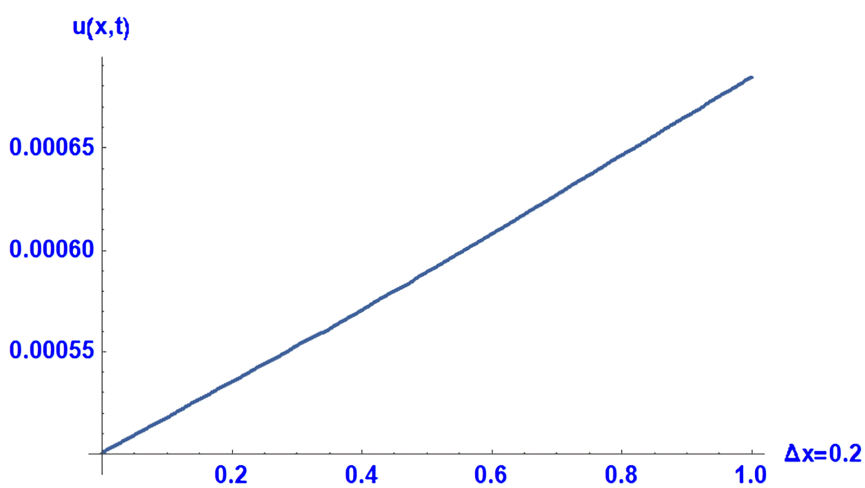

Figure 3. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t=1$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

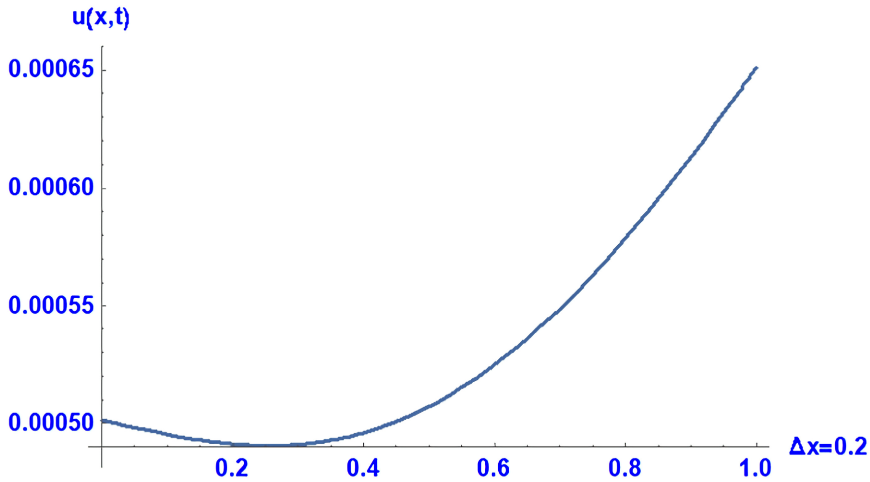

Figure 4. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 2$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

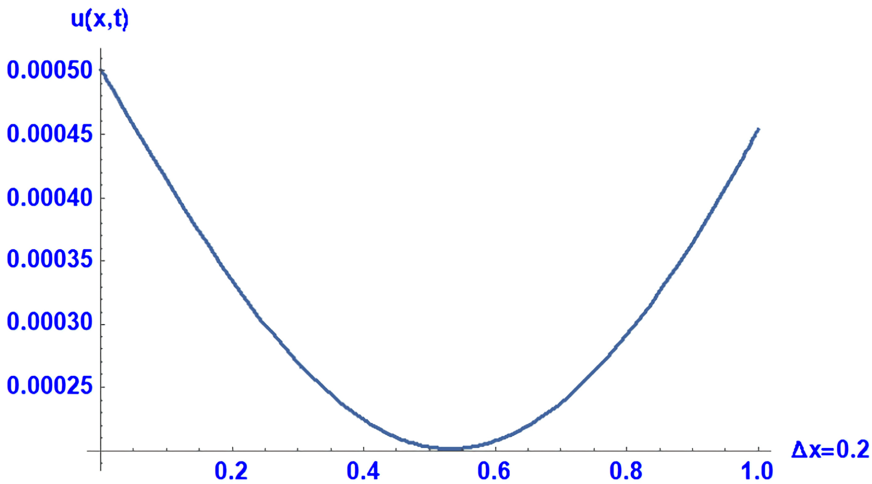

Figure 5. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 3$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

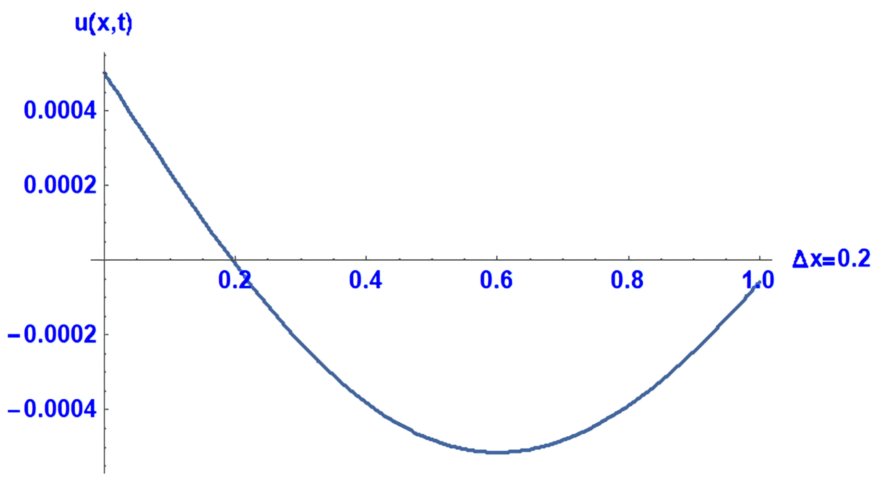

Figure 6. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t = 4$, $\gamma = 0.001$; $\delta = 2$; $\alpha = 1$; $\beta = 1$.

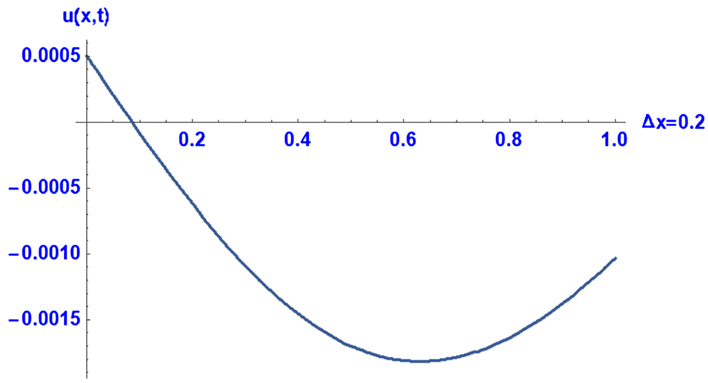

Figure 7. The comparison between the numerical and exact solution for the generalized Burgers-Huxley equation when $t=5$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

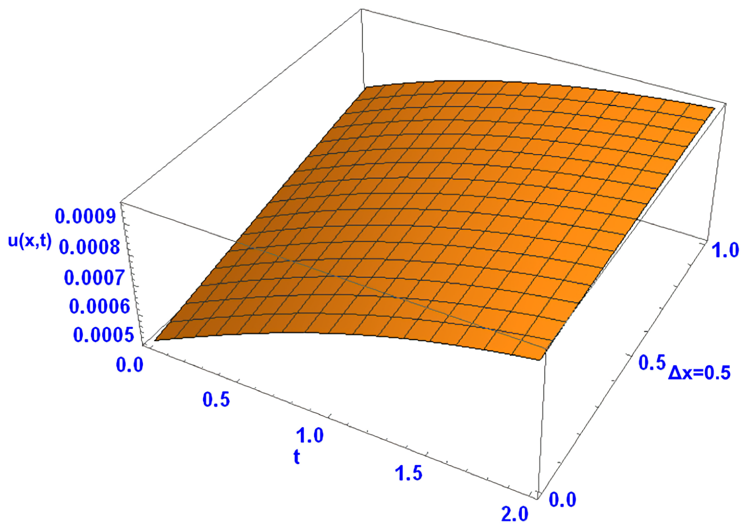

Figure 8. 3D representation of the behavior of the numerical solutions for the generalized Burgers-Huxley equation when $t=0$ to $t=2$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

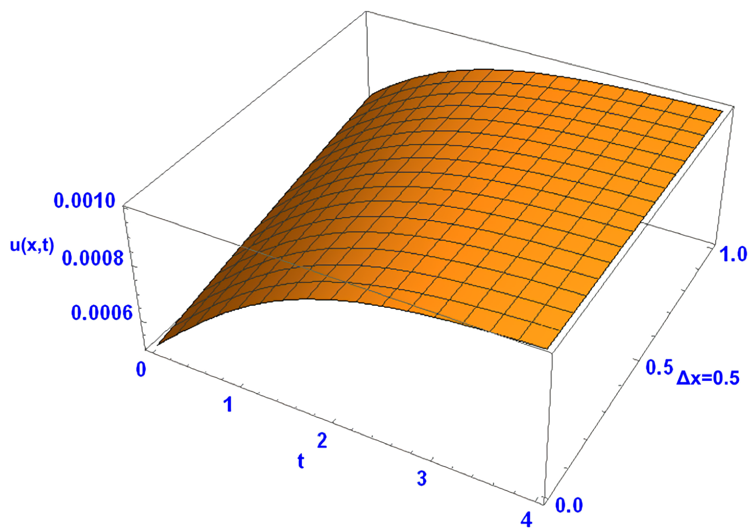

Figure 9. 3D representation of the behavior of the numerical solutions for the generalized Burgers-Huxley equation when $t=0$ to $t=4$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

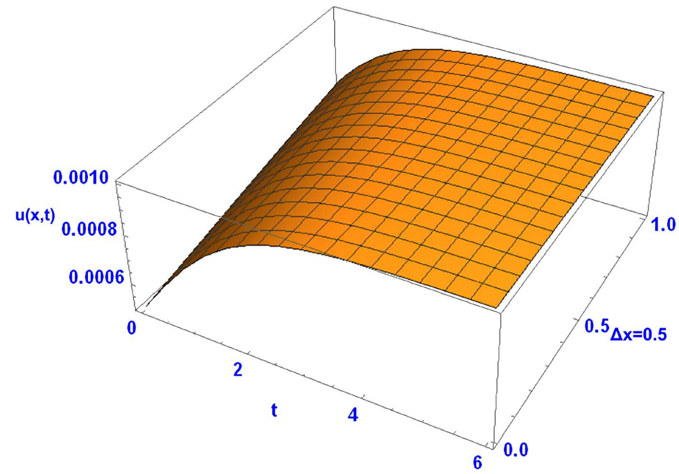

Figure 10. 3D representation of the behavior of the numerical solutions for the generalized Burgers-Huxley equation when $t=0$ to $t=6$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

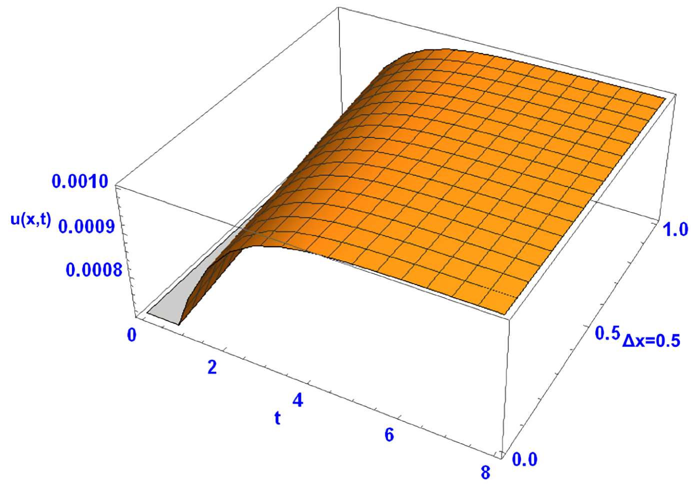

Figure 11. 3D representation of the behavior of the numerical solutions for the generalized Burgers-Huxley equation when $t=0$ to $t=8$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

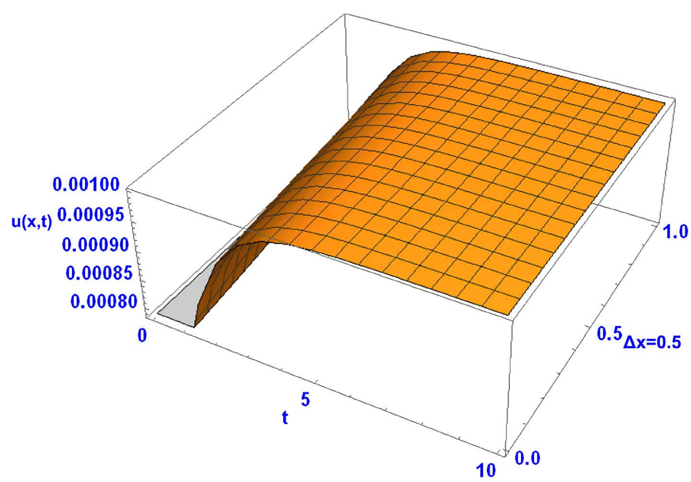

Figure 12. 3D representation of the behavior of the numerical solutions for the generalized Burgers-Huxley equation when $t=0$ to $t=10$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

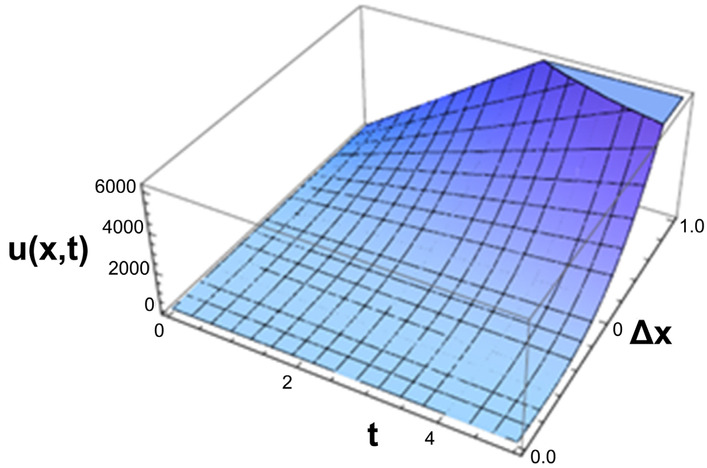

Figure 13. The ADM truncated solution $u(x,t)$ using the suggested boundary control, for the numerical and exact solution for the generalized Burgers-Huxley equation when $t=0$ to $t=6$, $\gamma=0.001$; $\delta=2$; $\alpha=1$; $\beta=1$.

7. Conclusion

In this paper, we introduce adaptive boundary control for the generalized Burgers-Huxley equation with high order nonlinearity terms. We proved that this type of generalized Burgers-Huxley equation is globally exponential stable in L^2 [0, 1], under zero Dirichlet boundary conditions. We developed an adaptive boundary control for the generalized Burgers Huxley equation, finding the solution $u(x,t)$ of the generalized Burgers-Huxley equation using initial solution $f_0(x)\in H^3(0,1)$ and some boundary conditions having the property $\|u(.,t)\|\to 0$ as $t\to\infty$. Finally, the Adomian decomposition method was used to illustrate the performance of the controller that was applied to the generalized Burgers-Huxley equations.

Author Contributions

Z.M. Alaofi planned the scheme, initiated the project, and suggested the experiments; T.A. El-Danaf conducted the experiments and analyzed the empirical results; F.E.I. Abd Alaal developed the mathematical modeling and examined the theory validation and made the mathematica programming. S.S. Dragomir developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

References

[1] Loyinmi, A.C. and Akinfe, T.K. (2020) An Algorithm for Solving the Burgers-Huxley Equation Using the Elzaki Transform. *SN Applied Sciences*, **2**, 1-17. https://doi.org/10.1007/s42452-019-1653-3

[2] Batiha, B., Noorani, M.S.M. and Hashim, I. (2008) Application of Variational Iteration Method to the Generalized Burgers-Huxley Equation. *Chaos*, *Solitons and Fractals*, **36**, 660-663. https://doi.org/10.1016/j.chaos.2006.06.080

[3] Wang, Q., Xiong, Y., Huang, W. and Romanovski, V.G. (2022) Isolated Periodic Wave Trains in a Generalized Burgers-Huxley Equation. *Electronic Journal of Qualitative Theory of Differential Equations*, No. 4, 1-16. https://doi.org/10.14232/ejqtde.2022.1.4

[4] Singh, B.K., Arora, G. and Singh, M.K. (2016) A Numerical Scheme for the Generalized Burgers-Huxley Equation. *Journal of the Egyptian Mathematical Society*, **24**, 629-637. https://doi.org/10.1016/j.joems.2015.11.003

[5] Khan, A., Mohan, M.T. and Ruiz-Baier, R. (2021) Conforming, Nonconforming and DG Methods for the Stationary Generalized Burgers-Huxley Equation. *Journal of Scientific Computing*, **88**, 1-21. https://doi.org/10.1007/s10915-021-01563-3

[6] Yefimova, O.Y. and Kudryashov, N. (2004) Exact Solutions of the Burgers-Huxley Equation. *Journal of Applied Mathematics and Mechanics*, **3**, 413-420. https://doi.org/10.1016/S0021-8928(04)00055-3

[7] Wang, X., Zhu, Z. and Lu, Y. (1990) Solitary Wave Solutions of the Generalised Burgers-Huxley Equation. *Journal of Physics A*: *Mathematical and General*, **23**, 271. https://doi.org/10.1088/0305-4470/23/3/011

[8] Bin, L., Liang, Y., Zhang, J. and Bao, X. (2020) A Robust Adaptive Grid Method for Singularly Perturbed Burger-Huxley Equations. *Electronic Research Archive*, **28**, 1439-1457. https://doi.org/10.3934/era.2020076

[9] Mohanty, R.K., Dai, W. and Liu, D. (2015) Operator Compact Method of Accuracy Two in Time and Four in Space for the Solution of Time Dependent Burgers-Huxley Equation. *Numerical Algorithms*, **70**, 591-605. http://works.bepress.com/weizhong-dai/20 https://doi.org/10.1007/s11075-015-9963-z

[10] Reza, M. (2013) B-Spline Collocation Algorithm for Numerical Solution of the Generalized Burger's-Huxley Equation. *Numerical Methods for Partial Differential Equations*, **29**, 1173-1191. https://doi.org/10.1002/num.21750

[11] Khattak, A.J. (2009) A Computational Meshless Method for the Generalized Burger's-Huxley Equation. *Applied Mathematical Modelling*, **33**, 3718-3729. https://doi.org/10.1016/j.apm.2008.12.010

[12] Mittal, R.C. and Tripathi, A. (2015) Numerical Solutions of Generalized Burgers-Fisher and Generalized Burgers-Huxley Equations Using Collocation of Cubic B-Splines. *International Journal of Computer Mathematics*, **92**, 1053-1077. https://doi.org/10.1080/00207160.2014.920834

[13] Kaushik, A. and Sharma, M.D. (2008) A Uniformly Convergent Numerical Method on Non-Uniform Mesh for Singularly Perturbed Unsteady Burger-Huxley Equation. *Applied Mathematics and Computation*, **195**, 688-706. https://doi.org/10.1016/j.amc.2007.05.067

[14] Jha, N. and Wagley, M. (2020) A Family of Quasi-Variable Meshes High-Resolution

Compact Operator Scheme for Burger's-Huxley, and Burger's-Fisher Equation. *Mathematics in Applied Sciences and Engineering*, **1**, 286-308. https://doi.org/10.5206/mase/10837

[15] Fahmy, E.S. and Bajunaid, I. (2014) Approximate Solution for the Generalized Time-Delayed Burgers-Huxley Equation.

[16] Hashim, I., Noorani, M.S.M. and Said Al-Hadidi, M.R. (2006) Solving the Generalized Burgers-Huxley Equation Using the Adomian Decomposition Method. *Mathematical and Computer Modelling*, **43**, 1404-1411. https://doi.org/10.1016/j.mcm.2005.08.017

[17] Abassy, T.A. (2010) Improved Adomian Decomposition Method. *Computers and Mathematics with Applications*, **59**, 42-54. https://doi.org/10.1016/j.camwa.2009.06.009

[18] Adomian, G. (1994) Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers, Boston. https://doi.org/10.1007/978-94-015-8289-6

[19] El-Danaf, T.S. (2007) Solitary Wave Solutions for the Generalised Burgers-Huxley Equation. *The International Journal of Nonlinear Sciences and Numerical Simulation*, **8**, 315-318. https://doi.org/10.1515/IJNSNS.2007.8.3.315

[20] Deng, X. (2008) Travelling Wave Solutions for the Generalized Burgers-Huxley Equation. *Applied Mathematics and Computation*, **204**, 733-737. https://doi.org/10.1016/j.amc.2008.07.020

[21] Sari, M. and Gürarslan, G. (2009) Numerical Solutions of the Generalized Burgers-Huxley Equation by a Differential Quadrature Method. *Mathematical Problems in Engineering*, **2009**, Article ID: 370765. https://doi.org/10.1155/2009/370765

[22] Smaoui, N., El-Kadri, A. and Zribi, M. (2010) Adaptive Boundary Control of the Forced Generalized Korteweg-de Vries-Burgers Equation. *European Journal of Control*, **16**, 72-84. https://doi.org/10.3166/ejc.16.72-84

[23] Biazar, J. and Mohammadi, F. (2010) Application of Differential Transform Method to the Generalized Burgers-Huxley Equation 2. The Model Problem. *Applications and Applied Mathematics: An International Journal*, **5**, 1726-1740.

[24] Bratsos, A.G. (2011) A Fourth Order Improved Numerical Scheme for the Generalized Burgers—Huxley Equation. *American Journal of Computational Mathematics*, **1**, 152-158. https://doi.org/10.4236/ajcm.2011.13017

[25] MacAs-Daz, J.E., Ruiz-Ramrez, J. and Villa, J. (2011) The Numerical Solution of a Generalized Burgers-Huxley Equation through a Conditionally Bounded and Symmetry-Preserving Method. *Computers & Mathematics with Applications*, **61**, 3330-3342. https://doi.org/10.1016/j.camwa.2011.04.022

[26] El-Kady, M., El-Sayed, S.M. and Fathy, H.E. (2013) Development of Galerkin Method for Solving the Generalized Burger's-Huxley Equation. *Mathematical Problems in Engineering*, **2013**, Article ID: 165492. https://doi.org/10.1155/2013/165492

[27] Saha Ray, S. and Gupta, A.K. (2013) On the Solution of Burgers-Huxley and Huxley Equation Using Wavelet Collocation Method. *CMES-Computer Modeling in Engineering & Sciences*, **91**, 409-424.

[28] Liu, J., Luo, H.Y., Mu, G., Dai, Z. and Liu, X. (2013) New Multi-Soliton Solutions for Generalized Burgers-Huxley Equation. *Thermo Scientific*, **17**, 1486-1489. https://doi.org/10.2298/TSCI1305486L

[29] Az-zo, E.A. (2014) On the Reduced Differential Transform Method and Its Application to the Generalized Burgers-Huxley Equation. *Applied Mathematical Sciences*,

8, 8823-8831. https://doi.org/10.12988/ams.2014.410835

[30] Ervin, V.J., Macías-Díaz, J.E. and Ruiz-Ramírez, J. (2015) A Positive and Bounded Finite Element Approximation of the Generalized Burgers-Huxley Equation. *Journal of Mathematical Analysis and Applications*, **424**, 1143-1160. https://doi.org/10.1016/j.jmaa.2014.11.047

[31] Inan, B. (2017) Finite Difference Methods for the Generalized Huxley and Burgers-Huxley Equations. *Kuwait Journal of Science*, **44**, 20-27.

[32] Kumar, N. and Singh, S. (2016) Numerical Solution of Burgers-Huxley Equation Using Improved Nodal Integral Method. *Ninth International Conference on Computational Fluid Dynamics* (*ICCFD9*), Istanbul, 11-15 July 2016, 1-12.

[33] Machado, J.A.T., Babaei, A. and Moghaddam, B.P. (2016) Highly Accurate Scheme for the Cauchy Problem of the Generalized Burgers-Huxley Equation. *Acta Polytechnica Hungarica*, **13**, 183-195. https://doi.org/10.12700/APH.13.6.2016.6.10

[34] Inan, B. (2016) A New Numerical Scheme for the Generalized Huxley Equation. *Bulletin of Mathematical Sciences and Applications*, **16**, 105-111. https://doi.org/10.18052/www.scipress.com/BMSA.16.105

[35] Wasim, I., Abbas, M. and Amin, M. (2018) Hybrid B-Spline Collocation Method for Solving the Generalized Burgers-Fisher and Burgers-Huxley Equations. *Mathematical Problems in Engineering*, **2018**, Article ID: 6143934. https://doi.org/10.1155/2018/6143934

[36] Appadu, A.R., Inan, B. and Tijani, Y.O. (2019) Comparative Study of Some Numerical Methods for the Burgers-Huxley Equation. *Symmetry* (*Basel*), **11**, Article No. 1333. https://doi.org/10.3390/sym11111333

[37] Fu, Y. (2019) Persistence of travelling Wavefronts in a Generalized Burgers-Huxley Equation with Long-Range Diffusion. *Journal of Applied Analysis & Computation*, **9**, 363-372. https://doi.org/10.11948/2019.363

[38] Sun, L.Y. and Zhu, C.G. (2020) Cubic B-Spline Quasi-Interpolation and an Application to Numerical Solution of Generalized Burgers-Huxley Equation. *Advances in Mechanical Engineering*, **12**, 1-8. https://doi.org/10.1177/1687814020971061

[39] Khan, M.A., Ali Akbar, M., Ali, N.H.M. and Abbas, M. (2020) The New Auxiliary Method in the Solution of the Generalized Burgers-Huxley Equation. *Journal of Prime Research in Mathematics*, **16**, 16-26.

[40] Kumar, A. and Mohan, M.T. (2021) Large Deviation Principle for Occupation Measures of Two Dimensional Stochastic Convective Brinkman-Forchheimer Equations. *Stochastic Analysis and Applications*. https://doi.org/10.1080/07362994.2021.2005626

[41] Kushner, A.G. and Matviichuk, R.I. (2020) Exact Solutions of the Burgers-Huxley Equation via Dynamics. *Journal of Geometry and Physics*, **151**, Article ID: 103615. https://doi.org/10.1016/j.geomphys.2020.103615

[42] Ebiwareme, L. (2021) Banach Contraction Method and Tanh-coth Approach for the Solitary and Exact Solutions of Burger-Huxley and Kuramoto-Sivashinsky Equations. *International Journal of Mathematics Trends and Technology*, **67**, 31-46. https://doi.org/10.14445/22315373/IJMTT-V67I4P506

[43] Mohan, M.T. and Khan, A. (2021) On the Generalized Burgers-Huxley Equation: Existence, Uniqueness, Regularity, Global Attractors and Numerical Studies. *Discrete and Continuous Dynamical Systems Ser. B*, **26**, 3943-3988. https://doi.org/10.3934/dcdsb.2020270

[44] Abergel, F. and Temam, R. (1990) On Some Control Problems in Fluid Mechanics.

Theoretical and Computational Fluid Dynamics, **1**, 303-325. https://doi.org/10.1007/BF00271794

[45] Balogh, A. and Krstić, M. (2000) Burgers' Equation with Nonlinear Boundary Feedback: H1 Stability, Well-Posedness and Simulation. *Mathematical Problems in Engineering*, **6**, 189-200. https://doi.org/10.1155/S1024123X00001320

[46] Burns, J.A. and Kang, S. (1991) A Control Problem for Burgers' Equation with Bounded Input/Output. *Nonlinear Dynamics*, **2**, 235-262. https://doi.org/10.1007/BF00045296

[47] Choi, H., Temam, R., Moin, P. and Kim, J. (1993) Feedback Control for Unsteady and Its Application to the Stochastic Burgers Equation. *Journal of Fluid Mechanics*, **253**, 509-543. https://doi.org/10.1017/S0022112093001880

[48] Ito, K. and Kang, S. (1994) A Dissipative Feedback Control Synthesis for Systems Arising in Fluid Dynamics. *SIAM Journal on Control and Optimization*, **32**, 831-854. https://doi.org/10.1137/S0363012991222619

[49] Kobayashi, T. (2001) Adaptive Regulator Design of a Viscous Burgers System by Boundary Control. *IMA Journal of Mathematical Control and Information*, **18**, 427-437. https://doi.org/10.1093/imamci/18.3.427

[50] Krstic, M. (1998) On Global Stabilization of Burgers' Equation by Boundary Control. *Proceedings of the IEEE Conference on Decision & Control*, Vol. 3, 3498-3499. https://doi.org/10.1016/S0167-6911(99)00013-4

[51] Smaoui, N. (2004) Nonlinear Boundary Control of the Generalized Burgers Equation. *Nonlinear Dynamic*, **37**, 75-86. https://doi.org/10.1023/B:NODY.0000040023.92220.09

[52] Smaoui, N., El-Kadri, A. and Zribi, M. (2010) Nonlinear Boundary Control of the Unforced Generalized Korteweg-de Vries-Burgers Equation. *Nonlinear Dynamics*, **60**, 561-574. https://doi.org/10.1007/s11071-009-9615-8

[53] Smaoui, N., El-Kadri, A. and Zribi, M. (2012) Adaptive Boundary Control of the Unforced Generalized Korteweg-de Vries-Burgers Equation. *Nonlinear Dynamic*, **69**, 1237-1253. https://doi.org/10.1007/s11071-012-0343-0

[54] Momani, S., Abuasad, S. and Odibat, Z. (2006) Variational Iteration Method for Solving Nonlinear Boundary Value Problems. *Applied Mathematics and Computation*, **183**, 1351-1358. https://doi.org/10.1016/j.amc.2006.05.138

[55] Momani, S. and Odibat, Z. (2007) Numerical Approach to Differential Equations of Fractional Order. *Journal of Computational and Applied Mathematics*, **207**, 96-110. https://doi.org/10.1016/j.cam.2006.07.015

[56] Babolian, E. and Saeidian, J. (2009) Analytic Approximate Solutions to Burgers, Fisher, Huxley Equations and Two Combined Forms of These Equations. *Communications in Nonlinear Science and Numerical Simulation*, **14**, 1984-1992. https://doi.org/10.1016/j.cnsns.2008.07.019

[57] Satsuma, J. (1986) Exact Solutions of Burgers' Equation with Reaction Terms. In: *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations*, World Science Publishing, Singapore, 255-262.

[58] Hodgkin, A.L. and Huxley, A.F. (1952) A Quantitative Description of Membrane Current and Its Applications to Conduction and Excitation in Nerve. *The Journal of Physiology*, **117**, 500-544. https://doi.org/10.1113/jphysiol.1952.sp004764

[59] Hon, Y.C. and Mao, X.Z. (1998) An Efficient Numerical Scheme for Burgers' Equation. *Applied Mathematics and Computation*, **95**, 37-50. https://doi.org/10.1016/S0096-3003(97)10060-1

[60] Coddington, E. and Levinson, N. (1955) Theory of Ordinary Differential Equations. McGraw-Hill, New York.

[61] Abramowitz, M. and Stegun, I.A. (1964) Handbook of Mathematical Functions with Formulas. Graphs, and Mathematical Tables. Dover, New York.

[62] Acosta, G. and Durán, R.G. (2003) An Optimal Poincaré Inequality in L^1 for Convex Domains. *Proceedings of the American Mathematical Society*, **132**, 195-202. https://doi.org/10.1090/S0002-9939-03-07004-7

[63] Khalil, H.K. (1996) Nonlinear Systems. Prentice-Hall, Upper Saddle River, Vol. 2, No. 5, 1-5.

[64] Ali, E.J. (2012) A New Technique of Initial Boundary Value Problems Using Adomian Decomposition Method. *International Mathematical Forum*, **7**, 799-814.