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Global Existence of Solutions for Baer-Nunziato Two-Phase Flow Model in a Bounded Domain

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Abstract

In this paper, we study the global existence and uniqueness of strong solutions for the Baer-Nunziato two-phase flow model in a bounded domain with a no-slip boundary. The global existence and uniqueness of strong solutions are obtained when the initial value is near the equilibrium state in $H^2(\Omega)$. Furthermore, the exponential convergence rates of the pressure and velocity are also proved by delicate energy methods.

Keywords

Two-Phase Model, Bounded Domain, Global Existence, Energy Method

1. Introduction

1.1. Background and Motivation

In this paper, we are interested in a version of one velocity Baer-Nunziato model with dissipation for the mixture of two compressible fluids in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. The system is as follows:

$$\begin{cases} \alpha_t + (u\cdot\nabla)\alpha = 0, \quad 0\le\alpha\le 1, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ z_t + \operatorname{div}(zu) = 0, \\ ((\rho+z)u)_t + \operatorname{div}((\rho+z)u\otimes u) + \nabla P(f(\alpha)\rho, g(\alpha)z) = \operatorname{div}\mathbb{S}, \end{cases} \tag{1.1}$$

Here, $t\ge 0$ and $x=(x_1,x_2,x_3)\in\Omega$. The variables z, $u=(u^1,u^2,u^3)$ and α denote the density of the fluid, the velocity field of the fluid and the volume fraction, respectively. ρ is the density of the particles in the mixture. Where $f,g:(0,1)\mapsto[0,\infty)$ are given functions, and satisfying

$$f(\alpha):=\alpha^{1/\gamma^+}, g(\alpha):=(1-\alpha)^{1/\gamma^-}, \gamma^\pm>0; \tag{1.2}$$

$P = P(\alpha, \rho, z)$ is pressure satisfying

$$P = \alpha\rho^{\gamma^+} + (1-\alpha)z^{\gamma^-} \tag{1.3}$$

And

$$\mathbb{S} = \mu\left(\nabla u + \nabla u^{\mathrm{T}}\right) + \lambda\left(\operatorname{div}(u)\right)\mathbb{I} \tag{1.4}$$

($\mathbb{I}$ is the identity tensor) is the viscous stress tensor. The constant viscosity coefficients satisfy standard physical assumptions

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0. \tag{1.5}$$

In fact, the system (1.1) is derived by Antonin Novotry in [1] from the two velocity Baer-Nunziato system, taking the form of

$$\begin{cases} (\alpha_\pm)_t + \boldsymbol{v}_I \cdot \nabla\alpha_\pm = 0, \\ (\alpha_\pm\rho_\pm)_t + \operatorname{div}(\alpha_\pm\rho_\pm\boldsymbol{u}_\pm) = 0, \\ (\alpha_\pm\rho_\pm\boldsymbol{u}_\pm)_t + \operatorname{div}(\alpha_\pm\rho_\pm\boldsymbol{u}_\pm \otimes \boldsymbol{u}_\pm) + \nabla(\alpha_\pm P_\pm(\rho_\pm)) - P_I\nabla(\alpha_\pm) \\ = \alpha_\pm\mu_\pm(\Delta\boldsymbol{u}_\pm) + \alpha_\pm(\mu_\pm + \lambda_\pm)\nabla\operatorname{div}\boldsymbol{u}_\pm, \\ 0 \leq \alpha_\pm \leq 1, \alpha_+ + \alpha_- = 1, \end{cases} \tag{1.6}$$

in the above $\left(\alpha_\pm, \alpha_\pm\rho_\pm \geq 0, \boldsymbol{u}_\pm \in \mathbb{R}^3\right)$—concentrations, densities, velocities of the $\pm$ species—are unknown functions, $nP_\pm$ are two (different) given functions defined on $[0,\infty)$ and P_I, $\boldsymbol{v}_I$ are conveniently chosen quantities—they represent pressure and velocity at the interface. In the multifluid modeling, there are many possibilities about how the quantities $\boldsymbol{v}_I$, P_I could be chosen, and there is no consensus about this choice.

As [1] [2], under the following simplifying assumptions:

$$\mu_\pm := \mu, \lambda_\pm := \lambda, \boldsymbol{v}_I = \boldsymbol{u}_\pm := u$$

$$\alpha P_\pm(s) = F_\pm\left(f_\pm(\alpha)s\right) \text{ for all } \alpha \in (0,1), s \in [0,\infty)$$

with some functions $F_\pm$ defined on $[0,\infty)$ and functions $f_\pm$ defined on $(0,1)$. The two velocity Baer-Nunziato system reduces to the one velocity Baer-Nunziato system.

System (1.1) corresponds to the barotropic and viscous version of the five-equation model of two-phase flows derived by Allaire, Clerc and Kokh in [3] [4] by different considerations. There are many results about the numerical properties of the Baer-Nunziato two-phase model and related models. Coquel proposed a splitting method for calculating the approximate solution of the isentropic Baer-Nunziato two-phase flow model, and tested the accuracy of some approximate solutions of Baer-Nunziato model in [5]; Pan, Zhao, Tian and Wang studied the numerical calculation of Baer-Nunziato two-phase flow model and proposed a new aerodynamic scheme [6]. In [7], Li and Wang proposed an HLLC method that can avoid estimating the wave velocity, and applied it to the Baer-Nunziato model simulation of two-phase flow, which can get better simulation results. When it comes to mathematical analysis, there are few results providing insight into the existing theory and asymptotic behavior of solutions concerning the two-phase models. The first result on the existence of weak solutions to the system (1.1) was investigated by Novotny [1] for arbitrary large initial data on a large time interval in the mathematical literature. In addition, Novotny and Jin not only defined the weak solutions and dissipative weak solutions of the system (1.1) and their existence theorems in large time intervals, but also studied the strong solutions of the system and proved their existence in short time intervals in [4]. In [2], Kwon, Novotny and Cheng proved that the weak solution set is stable, and pointed out that the construction of the weak solutions of the system is still a difficult problem. Motivated by [4] [7], our aim of the paper is to establish the existence theory of strong solutions for the one velocity Baer-Nunziato model with dissipation for the mixture of two compressible fluids in a bounded domain with no-slip boundary.

The results of weak solutions to multi-fluid models are in the mathematical literature in a short supply. It is convenient to quote [8] [9] [10] [11] [12] for a few papers which are relevant to the present work. It is worth pointing out that the system (1.1) is similar to the viscous liquid-gas two-phase flow model. There is little research on Baer-Nunziato's initial value problem. Here we can refer to some relevant papers on the existence, uniqueness and large time behavior of solutions of viscous liquid-gas two-phase flow model [13]-[28]. The main difference of the viscous liquid-gas two-phase flow model from another is that the pressure term in the liquid-gas two-phase model satisfies:

$$P(\rho,z) = -b_1(\rho,z) + \sqrt{b_1(\rho,z)^2 + b_2(z)}, \tag{1.7}$$

where b_1 and b_2 are linear functions with respect to each variable. And the study of two-phase flow models in a bounded domain is becoming increasingly popular [29] [30] [31] [32]. Based on the above research background and current situation, I think it is necessary to study the global existence of the solution to the initial value problem of Baer-Nunziato two-phase flow model. So in this paper, we will study the initial value problem of the Baer-Nunziato two-phase flow model in a bounded region.

1.2. Main Results

To overcome the difficulties arising from the non-dissipation on ρ, z, we first rewrite system (1.1) in a more suitable form. The crucial idea is that instead of the variables (α,ρ,z,u), we study the variables (α,c,P,u). Let

$$c := \alpha\rho^{\gamma^+} - (1-\alpha) z^{\gamma^-}, \quad m := \rho + z. \tag{1.8}$$

By a direct calculation, from (1.3) and (1.8), we have

$$\alpha\rho^{\gamma^+} = \frac{P+c}{2}, \quad (1-\alpha) z^{\gamma^-} = \frac{P-c}{2}, \tag{1.9}$$

then the system (1.1) clearly can be written in terms of the variables (α,c,P,u), that is

$$\begin{cases} \alpha_t + (u\cdot\nabla)\alpha = 0, \quad 0 \le \alpha \le 1 \\ c_t + u\cdot\nabla c + B_1 \mathrm{div} u = 0, \\ P_t + u\cdot\nabla P + B_2 \mathrm{div} u = 0, \\ mu_t + (mu\cdot\nabla)u + \nabla P = \mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u, \end{cases} \tag{1.10}$$

with the initial and boundary conditions

$$\begin{cases} (\alpha,c,P,u)(x,0)=(\alpha_0,c_0,P_0,u_0)(x), \quad x=(x_1,x_2,x_3)\in\Omega, \\ u(x,t)\big|_{\partial\Omega}=0, \quad t\ge 0, \\ \frac{1}{|\Omega|}\int_\Omega P_0 \mathrm{d}x=\bar{P}_0, \end{cases} \tag{1.11}$$

where $B_1=\frac{\gamma^+-\gamma^-}{2}P+\frac{\gamma^++\gamma^-}{2}c$, $B_2=\frac{\gamma^++\gamma^-}{2}P+\frac{\gamma^+-\gamma^-}{2}c$ and $\bar{P}_0$ is a positive constant.

Now, we are in a position to state our main results:

Theorem 1.1. Let $\bar{P}_0>0$, $\bar{c}$ and $\bar{\alpha}$ are three constants, assume the initial boundary value $(\alpha_0,c_0,P_0-\bar{P}_0,u_0)\in H^2(\Omega)$ satisfies the compatibility conditions, *i.e.*

$$\partial_t^l u(0)\big|_{\partial\Omega}=0, \quad l=0,1,$$

where $\partial_t u(x,0)=\frac{\mu\Delta u_0+(\lambda+\mu)\nabla\mathrm{div}u_0}{m_0}-m_0(u_0\cdot\nabla)u_0-\nabla P_0$ is the lth derivative at $t=0$ of any solution of the system (1.10) - (1.11), as calculated from (1.10) to yield an expression in terms of α_0,c_0,P_0,u_0. Then there exists a constant ε_0 such that if

$$\left\|(\alpha_0-\bar{\alpha},c_0-\bar{c},P_0-\bar{P}_0,u_0)\right\|_2\le\varepsilon_0 \tag{1.12}$$

then the initial boundary value problem (1.10) - (1.11) admits a unique solution (α,c,P,u) globally in the time with $P>0$, which satisfies

$$\begin{aligned} &\alpha-\bar{\alpha},P-\bar{P},c-\bar{c}\in C^0([0,\infty);H^2(\Omega))\cap C^1([0,\infty);H^1(\Omega)), \\ &u\in C^0([0,\infty);H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,\infty);L^2(\Omega)), \end{aligned} \tag{1.13}$$

where

$$\bar{P}(t)=\frac{1}{|\Omega|}\int_\Omega P(x,t)\mathrm{d}x. \tag{1.14}$$

Moreover, there exist two positive constant C_0, η_0 such that for any $t>0$, it holds that

$$\left\|(P-\bar{P},u)(t)\right\|_2^2+\int_0^t\left\|(P-\bar{P})(\tau)\right\|_2^2+\left\|(u)(\tau)\right\|_3^2\mathrm{d}\tau\le C_0\left\|(P_0-\bar{P}_0,u_0)\right\|_2^2, \tag{1.15}$$

$$\left\|(P-\bar{P},u)(t)\right\|_2^2+\left\|\partial_t(P-\bar{P},u)(t)\right\|_{L^2}^2\le C_0\left\|(P_0-\bar{P}_0,u_0)(t)\right\|_2^2\exp\{-\eta_0 t\}, \tag{1.16}$$

$$\left\|(\alpha-\bar{\alpha})(t)\right\|_2\le C_0\left\|\alpha_0\right\|_2\exp\left\{C_0\left\|(P_0-\bar{P}_0,u_0)\right\|_2\right\}. \tag{1.17}$$

$$\left\|(c-\bar{c})(t)\right\|_2\le C_0\exp\left\{C_0\left\|(P_0-\bar{P}_0,u_0)\right\|_2\right\}\left(\left\|u_0\right\|_2+\left\|(P_0-\bar{P}_0,u_0)\right\|_2^2\right). \tag{1.18}$$

Finally, $\lim_{t\to\infty}\bar{P}(t)$ exists and let $\lim_{t\to\infty}\bar{P}(t)=\tilde{P}$, the following convergence rate holds

$$\left|\tilde{P}-\bar{P}(t)\right|\le C_0\left\|(\alpha_0,c_0,P_0-\bar{P},u_0)\right\|_2^2\exp\{-\eta_0 t\}. \tag{1.19}$$

1.3. Notations

Throughout this paper, C denotes the generic positive constant depending only on the initial data and physical coefficients but independent of time *t*. Moreover, the norms in Sobolev spaces $H^m(\Omega)$ and $W^{m,p}(\Omega)$ are denoted respectively by $\|\cdot\|_m$ and $\|\cdot\|_{m,p}$, for $m \ge 0$ and $p \ge 1$. Particularly, for $m = 0$, we will simply use $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^p}$. As usual, $\langle \cdot,\cdot \rangle$ denotes the inner-product in $L^2(\Omega)$. Finally, $\nabla = (\partial_1, \partial_2, \partial_3)$, $\partial_i = \partial_{x_i}$ $(i = 1,2,3)$ and for any integer $l \ge 0$, $\nabla^l f$ denotes all derivatives of order l of the function f.

The rest of this paper is organized in the following way. In the next section, we show some useful inequalities. In Section 3, we obtained some *a priori* estimates and hence the global existence by the energy estimate method. As a by-product, we get the time decay estimates of the solutions.

2. Preliminaries

In this section, we first introduce some Sobolev's inequalities that will be used frequently in later articles (cf. [33] [34]).

Lemma 2.1. [33] *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$ *and* $f \in H^2$. *It holds that*

$$\begin{aligned} &\text{(i)} \|f\|_{L^\infty} \le C\|f\|_2, \\ &\text{(ii)} \|f\|_{L^p} \le C\|f\|_1, 2 \le p \le 6, \end{aligned} \tag{2.1}$$

for some contants $C > 0$ depending only on Ω.

Due to the slip boundary condition, the classical energy estimates can't be applied directly to spatial derivatives. In order to get the estimates on the tangential derivatives of the solutions (P,u), we introduce the following lemmas on the stationary Stokes equations, c.f. [35].

Lemma 2.2. [36] *Let* Ω *be any bounded domain in* $\mathbb{R}^3$ *with smooth boundary. Consider the problem*

$$\begin{cases} -\mu\Delta u + \nabla P = g, \\ \mathrm{div} u = f, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$

where $f \in H^{k+1}(\Omega), g \in H^k\ (k \ge 0)$. Then the above problem has a solution $(P,u) \in H^{k+1} \times (H^{k+2} \cap H_0^1)$ which is unique modulo a constant of integration for P. Moreover, this solution satisfies:

$$\|u\|_{k+2}^2 + \|\nabla P\|_k^2 \le C\left(\|f\|_{k+1}^2 + \|g\|_k^2\right). \tag{2.3}$$

3. Global Existence

In this section, we will prove the global existence and large-time behavior of the solution with the small initial data. Firstly, we start with the local existence and uniqueness of the strong solution to the initial boundary value problem (1.10) - (1.11).

Proposition 3.1. (Local existence) Let $(\alpha_0, c_0, P_0, u_0) \in H^2(\Omega)$ such that

$$\inf_{x\in\Omega}\{P_0(x)\} > 0,\quad \partial_t^l u_0\big|_{\partial\Omega} = 0,\ l = 0,1.$$

Then there exists a positive constant T and C, such that the initial value problem (1.10) - (1.11) has a unique solution $(\alpha, c, P, u) \in C([0,T]; H^2(\Omega))$ satisfying

$$\inf_{t\in[0,T],x\in\bar{\Omega}}\{P(t,x)\} > 0,\ c_t, P_t \in C([0,T]; H^1(\Omega)),$$
$$u_t \in C([0,T]; L^2(\Omega)),\ u, \alpha \in L^2([0,T]; H^2(\Omega))$$

Furthermore, the following estimates hold,

$$\|\alpha(t)\|_2 + \|c(t)\|_2 + \|P(t)\|_2 + \|u(t)\|_2 \le C\left(\|\alpha_0\|_2 + \|c_0\|_2 + \|P_0\|_2 + \|u_0\|_2\right).$$

Next, we will establish some *a priori* estimates of the solution (α, c, P, u). We first make the *a priori* assumption that

$$\left\|(\alpha - \bar{\alpha}, c - \bar{c}, P - \bar{P}, u)(t)\right\|_2 + \left|\bar{P}(t) - \bar{P}_0\right| \le \varepsilon,\quad \text{for any } t \ge 0, \tag{3.1}$$

where $\varepsilon > 0$ is sufficiently small. By the Sobolev inequality, we have

$$\frac{1}{2}\bar{P}_0 \le P(t) \le 2\bar{P}_0,\quad \frac{1}{C} \le m(t) \le C,\quad \text{for any } t \ge 0. \tag{3.2}$$

This will be often used in the rest of paper.

In order to deduce a prior estimate, We first use the energy estimation method to estimate the lower derivative of (P, u).

Lemma 3.1. *Under the conditions of Theorem 1.1 and (3.1), there exists a positive constant C such that for any* $t \ge 0$*, it holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m|u|^2 + \frac{(P-\bar{P})^2}{\bar{B}_2}\mathrm{d}x + \mu\int_\Omega|\nabla u|^2\mathrm{d}x + (\mu+\lambda)\int_\Omega|\mathrm{div}u|^2\mathrm{d}x \\ &\le C\varepsilon\left(\|\nabla P\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).\end{aligned} \tag{3.3}$$

Proof. Multiplying $(1.10)_4$ by u, and integrating on Ω, using integration by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m|u|^2\mathrm{d}x + \mu\int_\Omega|\nabla u|^2\mathrm{d}x + (\mu+\lambda)\int_\Omega|\mathrm{div}u|^2\mathrm{d}x + \int_\Omega u\cdot\nabla P\mathrm{d}x = 0. \tag{3.4}$$

In order to get the estimate of P, we shall deduce the equation of $\bar{P}$. By integrating $(1.10)_3$ over Ω gives

$$\begin{aligned}&\int_\Omega P_t\mathrm{d}x + \int_\Omega u\cdot\nabla P\mathrm{d}x + \int_\Omega B_2\mathrm{div}u\mathrm{d}x \\ &= \int_\Omega P_t\mathrm{d}x + \int_\Omega u\cdot\nabla P\mathrm{d}x - \int_\Omega u\cdot\nabla B_2\mathrm{d}x = 0.\end{aligned} \tag{3.5}$$

Therefore,

$$\int_\Omega P_t\mathrm{d}x = \int_\Omega u\cdot\nabla(B_2 - P)\mathrm{d}x. \tag{3.6}$$

Then, we have

$$\bar{P}_t(t) = \frac{1}{|\Omega|}\int_\Omega P_t(x,t)\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega u\cdot\nabla(B_2 - P)\mathrm{d}x. \tag{3.7}$$

Combining the above equality, (3.1) and (3.2), we obtain

$$\begin{aligned}\left|\bar{P}_t\right| &\le C\|u\|_{L^2}\left(\|\nabla B_2\|_{L^2}+\|\nabla P\|_{L^2}\right)\\ &\le C\|u\|_{L^2}\left(\|\nabla c\|_{L^2}+\|\nabla P\|_{L^2}\right).\end{aligned}\tag{3.8}$$

Now, we rewrite Equation $(1.10)_3$ in the linear form as the following

$$\frac{\left(P-\bar{P}\right)_t}{\bar{B}_2}+\operatorname{div}u+\frac{\bar{P}_t+\left(B_2-\bar{B}_2\right)\operatorname{div}u+u\cdot\nabla P}{\bar{B}_2}=0.\tag{3.9}$$

Multiplying the above equality by $P-\bar{P}$ and integrating over Ω gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\left(P-\bar{P}\right)^2}{\bar{B}_2}\mathrm{d}x+\int_\Omega\operatorname{div}u\left(P-\bar{P}\right)\mathrm{d}x\\ &=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\left(P-\bar{P}\right)^2\left(\bar{B}_2\right)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}\mathrm{d}x+\int_\Omega\frac{\bar{P}_t+\left(B_2-\bar{B}_2\right)\operatorname{div}u+u\cdot\nabla P}{\bar{B}_2}\left(P-\bar{P}\right)\mathrm{d}x,\end{aligned}\tag{3.10}$$

Adding (3.4) to (3.10), we find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m|u|^2+\frac{\left(P-\bar{P}\right)^2}{\bar{B}_2}\mathrm{d}x+\mu\int_\Omega|\nabla u|^2\,\mathrm{d}x+(\mu+\lambda)\int_\Omega|\operatorname{div}u|^2\,\mathrm{d}x\\ &=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\left(P-\bar{P}\right)^2\left(\bar{B}_2\right)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}\mathrm{d}x+\int_\Omega\frac{\bar{P}_t+\left(B_2-\bar{B}_2\right)\operatorname{div}u+u\cdot\nabla P}{\bar{B}_2}\left(P-\bar{P}\right)\mathrm{d}x.\end{aligned}\tag{3.11}$$

By using (3.1), (3.2), (3.8), Lemma 2.1, Hölder's inequality and Poincaré's inequality, the right terms of the above equation can be estimated as follows:

$$\begin{aligned}\left|\int_\Omega\frac{\left(P-\bar{P}\right)^2\left(\bar{B}_2\right)_{\bar{P}}\bar{P}_t}{\bar{B}_2{}^2}\mathrm{d}x\right| &\le\left|\frac{\left(\bar{B}_2\right)_{\bar{P}}\bar{P}_t}{\bar{B}_2}\right|\left\|P-\bar{P}\right\|_{L^2}^2\\ &\le C\|u\|_{L^2}\left(\|\nabla c\|_{L^2}+\|\nabla P\|_{L^2}\right)\left\|P-\bar{P}\right\|_{L^2}^2\\ &\le C\varepsilon\|\nabla P\|_{L^2}^2.\end{aligned}\tag{3.12}$$

$$\begin{aligned}&\left|\int_\Omega\frac{\bar{P}_t+u\cdot\nabla P}{\bar{B}_2}\left(P-\bar{P}\right)\mathrm{d}x\right|\\ &\le\left|\frac{C}{\bar{B}_2}\right|\left(\left|\bar{P}_t\right|\left\|P-\bar{P}\right\|_{L^2}+\|u\|_{L^3}\|\nabla P\|_{L^2}\left\|P-\bar{P}\right\|_{L^6}\right)\\ &\le C\left(\|u\|_{L^2}\left(\|\nabla c\|_{L^2}+\|\nabla P\|_{L^2}\right)\left\|P-\bar{P}\right\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla P\|_{L^2}^2\right)\\ &\le C\varepsilon\left(\|\nabla u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right).\end{aligned}\tag{3.13}$$

$$\begin{aligned}\left|\int_\Omega\frac{\left(B_2-\bar{B}_2\right)\operatorname{div}u}{\bar{B}_2}\left(P-\bar{P}\right)\mathrm{d}x\right| &\le\int_\Omega\left|\frac{B_2-\bar{B}_2}{\bar{B}_2}\right|\operatorname{div}u\left(P-\bar{P}\right)\mathrm{d}x\\ &\le C\left\|\frac{B_2-\bar{B}_2}{\bar{B}_2}\right\|_{L^3}\|\operatorname{div}u\|_{L^2}\left\|P-\bar{P}\right\|_{L^6}\\ &\le C\varepsilon\|\operatorname{div}u\|_{L^2}\|\nabla P\|_{L^2}\\ &\le C\varepsilon\left(\|\nabla u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right).\end{aligned}\tag{3.14}$$

Plugging (3.12)-(3.14) into (3.11) yields (3.3). The proof of Lemmas 3.1 is completed.

Next, we give the energy estimate of the time derivative for (P,u).

Lemma 3.2. *Under the conditions of Theorem 1.1 and (3.1), there exists a positive constant C such that for any* $t\geq 0$ *it holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m|u_t|^2+\frac{\left(P_t-\overline{P}_t\right)^2}{\overline{B}_2}\mathrm{d}x+\mu\int_\Omega|\nabla u_t|^2\,\mathrm{d}x+(\mu+\lambda)\int_\Omega|\mathrm{div}u_t|^2\,\mathrm{d}x \\ \leq C\varepsilon\left(\|\nabla u\|_1^2+\|\nabla u_t\|_{L^2}^2\right). \tag{3.15}$$

Proof. Differentiating $(1.10)_4$ and (3.9) with respect to t, then multiplying the result by u_t and $\left(P-\overline{P}\right)_t$ respectively, then summing up and integrating on Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m|u_t|^2+\frac{\left(P_t-\overline{P}_t\right)^2}{\overline{B}_2}\mathrm{d}x+\mu\int_\Omega|\nabla u_t|^2\,\mathrm{d}x+(\mu+\lambda)\int_\Omega|\mathrm{div}u_t|^2\,\mathrm{d}x \\ =-\frac{1}{2}\int_\Omega\left(m_t|u_t|^2-\frac{\left(P_t-\overline{P}_t\right)^2\left(\overline{B}_2\right)_{\overline{P}}\overline{P}_t}{\left(\overline{B}_2\right)^2}\right)\mathrm{d}x-\int_\Omega(mu\cdot\nabla u)_t\,u_t\mathrm{d}x \\ -\int_\Omega\left(\frac{\overline{P}_t+\left(B_2-\overline{B}_2\right)\mathrm{div}u+u\cdot\nabla P}{\overline{B}_2}\right)_t\left(P_t-\overline{P}_t\right)\mathrm{d}x \tag{3.16}$$

We use the boundary conditions $u_t|_{\partial\Omega}=0$. For the first term on the right hand side of the above equality, we have from (3.1), (3.2) and Lemma 2.1, Hölder's inequality and Poincaré's inequality that

$$\left|\int_\Omega m_t|u_t|^2+\frac{\left(P_t-\overline{P}_t\right)^2\left(\overline{B}_2\right)_{\overline{P}}\overline{P}_t}{\left(\overline{B}_2\right)^2}\right|\mathrm{d}x \\ =\left|\int_\Omega \mathrm{div}(mu)|u_t|^2+\frac{\left(P_t-\overline{P}_t\right)^2\left(\overline{B}_2\right)_{\overline{P}}\overline{P}_t}{\left(\overline{B}_2\right)^2}\right|\mathrm{d}x \\ =\left|\int_\Omega 2mu\nabla u_tu_t-\frac{\left(P_t-\overline{P}_t\right)^2\left(\overline{B}_2\right)_{\overline{P}}\overline{P}_t}{\left(\overline{B}_2\right)^2}\right|\mathrm{d}x \\ \leq C\|m\|_{L^\infty}\|u\|_{L^3}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6}+\left|\frac{\left(\overline{B}_2\right)_{\overline{P}}\overline{P}_t}{\left(\overline{B}_2\right)^2}\right|\left(\|P_t\|_{L^2}^2+\|\overline{P}_t\|_{L^2}^2\right) \\ \leq C\varepsilon\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}^2+C\varepsilon\|\nabla u\|_1^2 \\ \leq C\varepsilon\left(\|\nabla u\|_1^2+\|\nabla u_t\|_{L^2}^2\right) \tag{3.17}$$

By (3.1) and $(1.10)_2$ yields

$$\|P_t\|_{L^2}\leq C\left(\|u\cdot\nabla P\|_{L^2}+\|\nabla u\|_{L^2}\right)\leq C\left(\|u\|_{L^\infty}\|\nabla P\|_{L^2}+\|\nabla u\|_{L^2}\right)\leq C\|\nabla u\|_1. \tag{3.18}$$

Similarly, for the second term, we have

$$\begin{aligned}
&\left|\int_\Omega (mu\cdot\nabla u)_t u_t \mathrm{d}x\right| = \left|\int_\Omega \left(m_t u\cdot\nabla u u_t + m u_t \nabla u u_t + mu\cdot\nabla u_t u_t\right)\mathrm{d}x\right| \\
&= \left|\int_\Omega \left(-(\nabla m u + m\mathrm{div}u)u\cdot\nabla u u_t + m u_t\nabla u u_t + mu\cdot\nabla u_t u_t\right)\mathrm{d}x\right| \\
&\le C\Big(\|\nabla m\|_{L^3}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|u_t\|_{L^6} + \|m\|_{L^\infty}\|\mathrm{div}u\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^3}\|u_t\|_{L^6} \\
&\quad + \|m\|_{L^\infty}\|\nabla u\|_{L^3}\|u_t\|_{L^3}^2 + \|m\|_{L^\infty}\|u\|_{L^3}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6}\Big) \\
&\le C\varepsilon\left(\|\nabla u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\right),
\end{aligned} \tag{3.19}$$

where is using

$$\|\nabla m\|_{L^3} \le \|\nabla m\|_1 \le C\left(\|\nabla P\|_1 + \|\nabla c\|_1\right). \tag{3.20}$$

Then, combing (3.1), $(1.10)_2$, (3.7) and (3.18), we can obtain

$$\begin{aligned}
\left|\bar{P}_{tt}\right| &\le \frac{1}{|\Omega|}\int_\Omega \left(u_t\cdot\nabla(B_2 - P) + u\cdot\nabla(B_2-P)_t\right)\mathrm{d}x \\
&\le C\left[\|u_t\|_{L^2}\left(\|\nabla P\|_{L^2} + \|\nabla c\|_{L^2}\right) + \|\nabla u\|_{L^2}\left(\|P_t\|_{L^2} + \|c_t\|_{L^2}\right)\right] \\
&\le C\varepsilon\left(\|u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|c_t\|_{L^2}\right) \\
&\le C\varepsilon\left(\|u_t\|_{L^2} + \|\nabla u\|_{L^2}\right),
\end{aligned} \tag{3.21}$$

Finally, for the last term, we have

$$\begin{aligned}
&\left|\int_\Omega \left(\frac{\bar{P}_t + (B_2 - \bar{B}_2)\mathrm{div}u + u\cdot\nabla P}{\bar{B}_2}\right)_t (P_t - \bar{P}_t)\mathrm{d}x\right| \\
&\le \left|\int_\Omega \frac{\left\{\bar{P}_{tt} + \left[(B_2)_P P_t + (B_2)_c c_t - (\bar{B}_2)_{\bar{P}}\bar{P}_t\right]\mathrm{div}u + (B_2 - \bar{B}_2)\mathrm{div}u_t\right\}(P_t - \bar{P}_t)}{\bar{B}_2}\mathrm{d}x\right| \\
&\quad + \left|\int_\Omega \frac{(u_t\cdot\nabla P + u\cdot\nabla P_t)(P_t - \bar{P}_t)}{\bar{B}_2}\mathrm{d}x\right| \\
&\quad + \left|\int_\Omega \frac{\left\{\bar{P}_t + (B_2 - \bar{B}_2)\mathrm{div}u + u\cdot\nabla P\right\}(\bar{B}_2)_{\bar{P}}\bar{P}_t(P_t - \bar{P}_t)}{(\bar{B}_2)^2}\mathrm{d}x\right| \\
&\le C\varepsilon\left(\|\nabla u\|_1^2 + \|\nabla u_t\|_{L^2}^2\right).
\end{aligned} \tag{3.22}$$

Substituting (3.17), (3.19), (3.22) into (3.16) gives (3.15).

Next, we will localize $\partial\Omega$ when estimate the boundary solutions. Refer to [35], we will revise the standard technique about separating the estimates of solution into that over the region away from the boundary and near the boundary. Let χ_0 be an arbitrary but fixed-function in $C_0^\infty(\Omega)$. Then, we have the following energy estimates on the region away from the boundary.

Lemma 3.3. *Under the conditions of Theorem 1.1 and* (*3.1*), *there exists a positive constant C such that for any* $t \ge 0$, *it holds*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m|\nabla u\chi_0|^2 + \frac{|\nabla P\chi_0|^2}{\bar{B}_2}\mathrm{d}x + \mu\int_\Omega |\nabla^2 u\chi_0|^2\mathrm{d}x + (\mu+\lambda)\int_\Omega |\nabla\mathrm{div}u\chi_0|^2\mathrm{d}x \\
&\le C\varepsilon\left(\|\nabla u\|_1^2 + \|\nabla P\|_1^2 + \|\nabla u_t\|_{L^2}^2\right) + C\|\nabla u\|_{L^2}\left(\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2}\right).
\end{aligned} \tag{3.23}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m\left|\nabla^2 u\chi_0\right|^2+\frac{\left|\nabla^2 P\chi_0\right|^2}{\bar{B}_2}\mathrm{d}x+\mu\int_\Omega\left|\nabla^3 u\chi_0\right|^2\mathrm{d}x+(\mu+\lambda)\int_\Omega\left|\nabla^2\mathrm{div}u\chi_0\right|^2\mathrm{d}x$$
$$\le C\varepsilon\left(\|\nabla u\|_2^2+\|\nabla P\|_1^2+\|\nabla u_t\|_{L^2}^2\right)+C\left\|\nabla^2 u\right\|_{L^2}\left(\left\|\nabla^3 u\right\|_{L^2}+\left\|\nabla^2 P\right\|_{L^2}\right). \tag{3.24}$$

Proof. Differentiating $(1.10)_4$ and (3.9) with respect to x_i, multiplying the resulting equations by $u_{x_i}\chi_0^2, P_{x_i}\chi_0^2$ respectively, then summing up and integrating on Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m\left|u_{x_i}\chi_0\right|^2+\frac{\left|P_{x_i}\chi_0\right|^2}{\bar{B}_2}\mathrm{d}x+\mu\int_\Omega\left|\nabla u_{x_i}\chi_0\right|^2\mathrm{d}x+(\mu+\lambda)\int_\Omega\left|\mathrm{div}u_{x_i}\chi_0\right|^2\mathrm{d}x\\
&=\frac{1}{2}\int_\Omega m_t\left|u_{x_i}\chi_0\right|^2-\frac{\left|P_{x_i}\chi_0\right|^2\left(\bar{B}_2\right)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}\mathrm{d}x-\int_\Omega\left[m_{x_i}u_t+(mu\cdot\nabla u)_{x_i}\right]u_{x_i}\chi_0^2\mathrm{d}x\\
&-\int_\Omega\left(\frac{\bar{P}_t+\left(B_2-\bar{B}_2\right)\mathrm{div}u+u\cdot\nabla P}{\bar{B}_2}\right)_{x_i}P_{x_i}\chi_0^2\mathrm{d}x-\mu\int_\Omega u_{x_i}\nabla u_{x_i}\nabla\chi_0^2\mathrm{d}x\\
&-(\mu+\lambda)\int_\Omega\mathrm{div}u_{x_i}u_{x_i}\nabla\chi_0^2\mathrm{d}x+\int_\Omega P_{x_i}u_{x_i}\nabla\chi_0^2\mathrm{d}x\\
&\le C\varepsilon\left(\|\nabla u\|_1^2+\|\nabla P\|_1^2+\|\nabla u_t\|_{L^2}^2\right)+C\|\nabla u\|_{L^2}\left(\left\|\nabla^2 u\right\|_{L^2}+\|\nabla P\|_{L^2}\right).
\end{aligned} \tag{3.25}$$

which gives (3.23). Repeating the above procedure again for 2nd order spatial derivatives we get the following

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m\left|u_{x_ix_j}\chi_0\right|^2+\frac{\left|P_{x_ix_j}\chi_0\right|^2}{\bar{B}_2}\mathrm{d}x+\mu\int_\Omega\left|\nabla u_{x_ix_j}\chi_0\right|^2\mathrm{d}x+(\mu+\lambda)\int_\Omega\left|\mathrm{div}u_{x_ix_j}\chi_0\right|^2\mathrm{d}x\\
&=\frac{1}{2}\int_\Omega m_t\left|u_{x_ix_j}\chi_0\right|^2-\frac{\left|P_{x_ix_j}\chi_0\right|^2\left(\bar{B}_2\right)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}\mathrm{d}x\\
&-\int_\Omega\left[m_{x_ix_j}u_t+m_{x_i}u_tx_j+m_{x_j}u_tx_i+(mu\cdot\nabla u)_{x_ix_j}\right]u_{x_ix_j}\chi_0^2\mathrm{d}x\\
&-\int_\Omega\left(\frac{\bar{P}_t+(B_2-\bar{B}_2)\mathrm{div}u+u\cdot\nabla P}{\bar{B}_2}\right)_{x_ix_j}P_{x_ix_j}\chi_0^2\mathrm{d}x-\mu\int_\Omega u_{x_ix_j}\nabla u_{x_ix_j}\nabla\chi_0^2\mathrm{d}x\\
&-(\mu+\lambda)\int_\Omega\mathrm{div}u_{x_ix_j}u_{x_ix_j}\nabla\chi_0^2\mathrm{d}x+\int_\Omega P_{x_ix_j}u_{x_ix_j}\nabla\chi_0^2\mathrm{d}x\\
&\le C\varepsilon\left(\|\nabla u\|_1^2+\|\nabla P\|_1^2+\|\nabla u_t\|_{L^2}^2\right)+C\|\nabla u\|_{L^2}\left(\left\|\nabla^2 u\right\|_{L^2}+\|\nabla P\|_{L^2}\right).
\end{aligned} \tag{3.26}$$

which implies (3.24). So, the Lemma 3.3 can be finished.

Refer to [35], we need a more argument using the trick of estimating the tangential derivatives and the normal derivatives separately to establish the estimates near the boundary. We choose a finite number of bounded open sets $\left\{O_j\right\}_{j=1}^N$ in $\mathbb{R}^3$, such that $\partial\Omega\subset\bigcup_{j=1}^N O_j$. In each open set O_j, we choose the local coordinates $y=\left(y_1,y_2,y_3\right)$ as follows:

1) The surface $O_j\cap\partial\Omega$ is the image of a smooth vector function $z^j\left(y_1,y_2\right)=\left(z_1^j,z_2^j,z_3^j\right)\left(y_1,y_2\right)$ (e.g. take the local geodesic polar coordinate), satifying

$$\left|z_{y_1}^j\right| = 1,\ z_{y_1}^j \cdot z_{y_2}^j = 0 \text{ and } \left|z_{y_2}^j\right| \ge \delta > 0, \tag{3.27}$$

where δ is a positive constant independent of $1 \le j \le N$.

2) For any $x = (x_1, x_2, x_3) \in O_j$ is represented by

$$x_i = \Psi_i(y) = y_3 \eta_i^j \left(z^j(y_1, y_2) + z_i^j(y_1, y_2)\right), \text{ for } i = 1,2,3 \tag{3.28}$$

where $\eta^j(y_1, y_2) = \left(\eta_1^j, \eta_2^j, \eta_3^j\right)\left(z^j(y_1, y_2)\right)$ represents the internal unit normal vector at the point $z^j(y_1, y_2)$ of the surface $\partial\Omega$.

We omit the subscript *j* in what follows for the simplicity of presentation. For $k = 1, 2$, we define the unit vectors

$$e_1 = z_{y_1} \text{ and } e_2 = \frac{z_{y_2}}{\left|z_{y_2}\right|},$$

Then Frenet-Serret's formal gives that there exist smooth functions $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ of (y_1, y_2) satisfying

$$\frac{\partial}{\partial y_1}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i = \begin{pmatrix} 0 & -\gamma_1 & -\alpha_1 \\ \gamma_1 & 0 & -\beta_1 \\ \alpha_1 & \beta_1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i,$$

$$\frac{\partial}{\partial y_1}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i = \begin{pmatrix} 0 & -\gamma_2 & -\alpha_2 \\ \gamma_2 & 0 & -\beta_2 \\ \alpha_2 & \beta_2 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i,$$

where e_m^i denote the *i*-th component of e_m. An elementary calculation shows that the Jacobian *J* of the transform (3.28) is

$$J = \Psi_{y_1} \times \Psi_{y_2} \cdot \eta = \left|z_{y_2}\right| + \left(\alpha_1 \left|z_{y_2}\right| + \beta_2\right) y_3 + \left(\alpha_1 \beta_2 - \alpha_2 \beta_1\right) y_3^2 \tag{3.29}$$

By (3.29), we have the transform (3.28) is regular by choosing y_3 so small that $J \ge \frac{\delta}{2}$ for some positive δ. Therefore, the inverse function of $\Psi(y) := (\Psi_1, \Psi_2, \Psi_3)(y)$ exists, and we use $y = \Psi^{-1}(x)$ denote it. Using a straightforward calculation, $(y_1, y_2, y_3)_{x_i}(x)$ can be expressed by

$$\begin{cases} \partial_{x_i} y_1 = \frac{1}{J}\left(\Psi_{y_2} \times \Psi_{y_3}\right)_i = \frac{1}{J}\left(Ae_i^1 + Be_i^2\right) =: r_{1i}, \\ \partial_{x_i} y_2 = \frac{1}{J}\left(\Psi_{y_3} \times \Psi_{y_1}\right)_i = \frac{1}{J}\left(Ce_i^1 + De_i^2\right) =: r_{2i}, \\ \partial_{x_i} y_3 = \frac{1}{J}\left(\Psi_{y_1} \times \Psi_{y_2}\right)_i = \eta_i =: r_{1i}, \end{cases} \tag{3.30}$$

where $A = \left|z_{y_2}\right| + \beta_2 y_3$, $B = -y_3 \alpha_2$, $C = -\beta_1 y_3$, $D = 1 + \alpha_1 y_3$, and $J = AD - BC \ge \frac{\delta}{2}$. It's easy to find out, (3.30) gives

$$\sum_{i=1}^{3} a_{3i}^2 = |n|^2 = 1,\ r_{1i} r_{3i} = r_{2i} r_{3i} = 0,\ J^2 = (AC + BD)^2 - \left(A^2 + B^2\right)\left(C^2 + D^2\right) \tag{3.31}$$

and

$$\partial_{x_i} = r_{ki} \partial_{yk} \tag{3.32}$$

where we have used the Einstein convention of summing over repeated indices.

Therefore, in each O_j, $(1.10)_3$ - $(1.10)_4$ can be rewritten in the local coordinates (y_1, y_2, y_3) as follows:

$$E^p := \frac{\mathrm{d}P}{\mathrm{d}t} + \frac{\bar{B}_2}{J}\left[\left(Ae_1 + Be_2\right)\cdot u_{y_1} + \left(Ce_1 + De_2\right)\cdot u_{y_2} + J\eta \cdot u_{y_3}\right] = g,$$

$$E^u := mu_t - \frac{\mu}{J^2}\Big[\left(A^2+B^2\right)u_{y_1y_1} + 2\left(AC+BD\right)u_{y_1y_2} + \left(C^2+D^2\right)u_{y_2y_2}$$
$$+ J^2 u_{y_3y_3}\Big] + \text{one order terms of } u + \frac{1}{J}\left(Ae_1+Be_2\right)\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{\mathrm{d}P}{\mathrm{d}t} + P\right]_{y_1}$$
$$+ \frac{1}{J}\left(Ce_1+De_2\right)\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{\mathrm{d}P}{\mathrm{d}t} + P\right]_{y_2} + \eta\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{\mathrm{d}P}{\mathrm{d}t} + P\right]_{y_3} = h,$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + u\cdot\nabla\left(\text{denotes the material derivative}\right),$$
$$g = -\left(B_2 - \bar{B}_2\right)\mathrm{div}u,$$
$$h = mu\cdot\nabla u + \frac{\mu+\lambda}{\bar{B}_2}\nabla g.$$
$$J^2 = \left(AC+BD\right)^2 - \left(A^2+B^2\right)\left(C^2+D^2\right)$$

Let us denote the tangential derivatives by $\partial = \left(\partial_{y_1}, \partial_{y_2}\right)$ and χ_j be arbitrary but fixed-function in $C_0^\infty\left(O_j\right)$. Obviously, $x_j\partial^k u = 0$ on $\partial\Omega_j^{-1}$, where $0 \le k \le 2$ and $\Omega_j^{-1}(y) := \left\{y \mid y = \Psi^{-1}(x), x\in\Omega_j = O_j \cap \Omega\right\}$. Estimating the tangential derivatives in the similar way as the above lemma, we have

Lemma 3.4. *Under the conditions of Theorem* 1.1 *and* (3.1), *there exists a positive constant C such that for any* $t \ge 0, 1 \le j \le N$, *it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}} m\left|\partial u\chi_j\right|^2 + \frac{\left|\partial P\chi_j\right|^2}{\bar{B}_2}\mathrm{d}y + \int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2\mathrm{d}y + \int_{\Omega_j^{-1}}\left|\partial\frac{\mathrm{d}P}{\mathrm{d}t}\chi_j\right|^2\mathrm{d}y$$
$$\le C\varepsilon\left(\|\nabla u\|_1^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla P\|_1^2\right) + C\|\nabla u\|_{L^2}\left(\|\nabla u\|_1 + \|\nabla P\|_{L^2}\right), \tag{3.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}} m\left|\partial^2 u\chi_j\right|^2 + \frac{\left|\partial^2 P\chi_j\right|^2}{\bar{B}_2}\mathrm{d}y + \int_{\Omega_j^{-1}}\left|\partial^2\nabla u\chi_j\right|^2\mathrm{d}y + \int_{\Omega_j^{-1}}\left|\partial^2\frac{\mathrm{d}P}{\mathrm{d}t}\chi_j\right|^2\mathrm{d}y$$
$$\le C\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla P\|_1^2\right) + C\left\|\nabla^2 u\right\|_{L^2}\left(\|\nabla u\|_2 + \left\|\nabla^2 P\right\|_{L^2}\right). \tag{3.34}$$

Next, we begin to deduce the estimates of derivatives in the normal directions.

Lemma 3.5. *Under the conditions of Theorem* 1.1 *and* (*3.1*), *there exists a positive constant C such that for any* $t \ge 0, k+l = 1, k, l \ge 0, 1 \le j \le N$, *it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}}\left|P_{y_3}\chi_j\right|^2\mathrm{d}y + \int_{\Omega_j^{-1}}\left|\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\chi_j\right|^2\mathrm{d}y$$
$$\le C\left[\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \varepsilon\left(\|\nabla P\|_1^2 + \|\nabla u\|_1^2\right) + \int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2\mathrm{d}y\right], \tag{3.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}P\chi_j\right|^2\mathrm{d}y+\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\chi_j\right|^2\mathrm{d}y$$
$$\leq C\left[\|\nabla u\|_1^2+\|u_t\|_1^2+\varepsilon\left(\|\nabla P\|_1^2+\|\nabla^2u\|_1^2\right)+\int_{\Omega_j^{-1}}\left|\partial^{k+1}\partial_{y_3}^{l}\nabla u\chi_j\right|^2\mathrm{d}y\right], \tag{3.36}$$

Proof. First, using $\partial_{y_3}\left(E^P-g\right)=0$ and $\eta\left(E^u-h\right)=0$, that is the following forms:

$$\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}+\frac{\bar{B}_2}{J}\left[\left(Ae_1+Be_2\right)\cdot u_{y_1y_3}+\left(Ce_1+De_2\right)\cdot u_{y_2y_3}+J\eta\cdot u_{y_3y_3}\right]$$
$$+\text{one order terms of } u=g_{y_3}, \tag{3.37}$$

$$\eta mu_t-\frac{\mu}{J^2}\left[\left(A^2+B^2\right)\eta u_{y_1y_1}+2\left(AC+BD\right)\eta u_{y_1y_2}+\left(C^2+D^2\right)\eta u_{y_2y_2}\right.$$
$$\left.+J^2\eta u_{y_3y_3}\right]+\text{one order terms of } u+\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{\mathrm{d}P}{\mathrm{d}t}+P\right]_{y_3}=\eta h, \tag{3.38}$$

In order to eliminate $u_{y_3y_3}$ in equation (3.37), we use (3.37) × $\frac{\mu}{\bar{B}_2}$ + (3.38) yields:

$$\frac{2\mu+\lambda}{\bar{B}_2}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}+P_{y_3}$$
$$=-\frac{\mu}{J^2}\left[\left(A^2+B^2\right)\eta u_{y_1y_1}+2\left(AC+BD\right)\eta u_{y_1y_2}+\left(C^2+D^2\right)\eta u_{y_2y_2}\right]$$
$$-\eta mu_t-\frac{\mu}{J}\left[\left(Ae_1+Be_2\right)\cdot u_{y_1y_3}+\left(Ce_1+De_2\right)\cdot u_{y_2y_3}+J\eta\cdot u_{y_3y_3}\right]$$
$$+\text{one order terms of } u+\eta h+\frac{\mu}{\bar{B}_2}g_{y_3}=\Phi. \tag{3.39}$$

Multiply that by $\chi_j^2\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}$ and integrating on Ω_j^{-1}, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}}\left|P_{y_3}\chi_j\right|^2\mathrm{d}y+\frac{2\mu+\lambda}{\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\chi_j\right|^2\mathrm{d}y$$
$$=\int_{\Omega_j^{-1}}-\left(u\cdot\nabla P\right)_{y_3}P_{y_3}\chi_j^2+\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\Phi\chi_j^2\mathrm{d}y. \tag{3.40}$$
$$=:K_1+K_2.$$

Estimate each term at the right end of the above equation,

$$\left|K_1\right|\leq\left|\int_{\Omega_j^{-1}}u_{y_3}\cdot\nabla PP_{y_3}\chi_j^2\mathrm{d}y\right|+\frac{1}{2}\left|\int_{\Omega_j^{-1}}\left(P_{y_3}\right)^2\mathrm{div}\left(u\chi_j^2\right)\mathrm{d}y\right|$$
$$\leq C\|\nabla u\|_1\|\nabla P\|_1^2\leq C\varepsilon\|\nabla P\|_1^2, \tag{3.41}$$

and

$$\left|K_2\right|\leq\frac{2\mu+\lambda}{2\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\chi_j\right|^2\mathrm{d}y+C\int_{\Omega_j^{-1}}\left|\Phi\chi_j\right|^2\mathrm{d}y$$
$$\leq\frac{2\mu+\lambda}{2\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\chi_j\right|^2\mathrm{d}y+C\int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2\mathrm{d}y$$
$$+C\left(\|\nabla u\|_{L^2}^2+\|u_t\|_{L^2}^2+\varepsilon\|\nabla u\|_1^2\right). \tag{3.42}$$

Substituting (3.41) and (3.42) into (3.40) get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}}\left|P_{y_3}\chi_j\right|^2\mathrm{d}y+\frac{2\mu+\lambda}{\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)\chi_j\right|^2\mathrm{d}y\\&\le C\left[\|\nabla u\|_{L^2}^2+\|u_t\|_{L^2}^2+\varepsilon\left(\|\nabla u\|_1^2+\|\nabla P\|_1^2\right)+\int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2\mathrm{d}y\right],\end{aligned}\tag{3.43}$$

which implies (3.35).

By the same way, using $\partial^k\partial_{y_3}^l$ to (3.38), multiplying the resulting equations by $\chi_j^2\partial^k\partial_{y_3}^{l+1}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)$, then when $k+l=1$ we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}P\chi_j\right|^2\mathrm{d}y+\frac{2\mu+\lambda}{\bar{B}_2}\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{y_3}\chi_j\right|^2\mathrm{d}y\\&\le C\left[\|\nabla u\|_1^2+\|u_t\|_1^2+\varepsilon\left(\|\nabla^2u\|_1^2+\|\nabla P\|_1^2\right)+\int_{\Omega_j^{-1}}\left|\partial^{k+1}\partial_{y_3}^l\nabla u\chi_j\right|^2\mathrm{d}y\right].\end{aligned}\tag{3.44}$$

which implies (3.36). The proof of Lemma3.5 is completed.

Finally, we use Lemma 2.2 to deduce the estimates on the tangential derivatives of (P,u).

Lemma 3.6. *Under the conditions of Theorem* 1.1 *and* (3.1), *there exists a positive constant C such that for any* $t\ge 0$, *it holds*

$$\|\nabla^2u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\le C\left(\left\|\frac{\mathrm{d}P}{\mathrm{d}t}\right\|_1^2+\|u_t\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\|\nabla u\|_1^2\right),\tag{3.45}$$

$$\begin{aligned}&\int_{\Omega_j^{-1}}\left|\partial\nabla^2u\chi_j\right|^2\mathrm{d}y+\int_{\Omega_j^{-1}}\left|\partial\nabla P\chi_j\right|^2\mathrm{d}y\\&\le C\left(\|\nabla u\|_1^2+\|u_t\|_1^2+\|\nabla P\|_{L^2}^2+\|\nabla^3u\|_{L^2}^2\|\nabla u\|_1^2+\|\nabla P\|_{L^2}+\left\|\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\right\|_{L^2}\right)\\&+C\int_{\Omega_j^{-1}}\left|\partial\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\chi_j\right|^2\mathrm{d}y.\end{aligned}\tag{3.46}$$

Proof. We rewrite the perturbed equations as the Stokes problem:

$$\begin{cases}\mathrm{div}u=-\dfrac{1}{B_2}\dfrac{\mathrm{d}P}{\mathrm{d}t},\\-\mu\Delta u+\nabla P=(\lambda+\mu)\nabla\mathrm{div}u-(mu_t+mu\cdot\nabla u),\\u|_{\partial\Omega}=0,\end{cases}\tag{3.47}$$

where applying Lemma 2.2 to (3.47), one can easily get (3.45).

Next we prove (3.46). To do this, by applying $\chi_j\partial$ to equation (3.47)₂, we have

$$\begin{cases}\mathrm{div}(\chi_j\partial u)=-\chi_j\partial\left(\dfrac{1}{B_2}\dfrac{\mathrm{d}P}{\mathrm{d}t}\right)+\nabla\chi_j\partial u,\\-\mu\Delta(\chi_j\partial u)+\nabla(\chi_j\partial P)=-2\mu\nabla\chi_j\nabla(\partial u)-\Delta\chi_j\partial u+\nabla\chi_j\partial P\\-(\lambda+\mu)\chi_j\nabla\partial\left(\dfrac{1}{B_2}\dfrac{\mathrm{d}P}{\mathrm{d}t}\right)-\chi_j\partial(mu_t+mu\cdot\nabla u),\\\chi_j\partial u|_{\partial\Omega_j^{-1}}=0,\end{cases}\tag{3.48}$$

Using the Lemma 2.2 to (3.48) gives (3.46). The proof of Lemma 3.6 is completed.

Now, let's start proving Theorem 1.1. We will do it by four steps.

Step 1: We first estimate the lower order derivatives for (P,u). Suppose D be a fixed but large positive constant. Let $D^2 \times ((3.3) + (3.15)) + D \times ((3.23) + (3.33)) + (3.35)$, there exists a function $H_1(P,u)$ which is equivalent to $\|u\|^2 + \|P-\bar{P}\|^2 + \|u_t\|^2 + \|P_t - \bar{P}_t\|^2 + \|\nabla P\|^2$ and satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{H_1 + \int_\Omega m\left|\nabla u\chi_0\right|^2 \mathrm{d}x + \sum_{j=1}^N \int_{\Omega_j^{-1}} m\left|\partial u\chi_j\right|^2 \mathrm{d}y\right\} + D\left(\|\nabla u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\right)\\&+\left\|\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\right\|_{L^2}^2 + \int_\Omega\left|\nabla^2 u\chi_0\right|^2 \mathrm{d}x + \sum_{j=1}^N \int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2 \mathrm{d}y\\&\le \frac{1}{D^{1/3}}\left(\left\|\nabla^2 u\right\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\right) + C\varepsilon\left\|\nabla^2 P\right\|_{L^2}^2.\end{aligned} \tag{3.49}$$

Substituting equation (3.45) into the above equation and using $\frac{\mathrm{d}P}{\mathrm{d}t} = -B_2 \mathrm{div}u$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{H_1 + \int_\Omega m\left|\nabla u\chi_0\right|^2 \mathrm{d}x + \sum_{j=1}^N \int_{\Omega_j^{-1}} m\left|\partial u\chi_j\right|^2 \mathrm{d}y\right\}\\&+\|\nabla u\|_1^2 + \|\nabla P\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \left\|\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\right\|_{L^2}^2\\&\le C\varepsilon\left\|\nabla^2 P\right\|_{L^2}^2.\end{aligned} \tag{3.50}$$

where D is enough large, ε is arbitrarily small.

Step 2: In this step, we will estimate the higher order derivatives for (P,u). Let $l=0$ in (3.36), by $D\times(2(3.24) + (3.34)) + (3.36)$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{D\int_\Omega m\left|\nabla^2 u\chi_0\right| + \frac{\left|\nabla^2 P\chi_0\right|^2}{\bar{B}_2}\mathrm{d}x + D\sum_{j=1}^N \int_{\Omega_j^{-1}} m\left|\partial^2 u\chi_j\right|^2\right.\\&\left.+\frac{\left|\partial^2 P\chi_j\right|^2}{\bar{B}_2}\mathrm{d}y + \int_{\Omega_j^{-1}}\left|\partial\partial P_{y_3}\chi_j\right|^2 \mathrm{d}y\right\} + \int_\Omega\left|\nabla^3 u\chi_0\right|^2 \mathrm{d}x\\&+\int_\Omega\left|\nabla^2\frac{\mathrm{d}P}{\mathrm{d}t}\chi_0\right|^2 \mathrm{d}x + \sum_{j=1}^N \int_{\Omega_j^{-1}}\left|\partial^2\nabla u\chi_j\right|^2 + \left|\partial\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\chi_j\right|^2 \mathrm{d}y\\&\le CD\left(\|\nabla u\|_1^2 + \|\nabla u_t\|_{L^2}^2\right) + CD\varepsilon\left(\left\|\nabla^2 u\right\|_1^2 + \|\nabla P\|_1^2\right) + CD\left\|\nabla^2 u\right\|_{L^2}\left(\|\nabla u\|_2 + \left\|\nabla^2 P\right\|_{L^2}\right)\end{aligned} \tag{3.51}$$

Then by taking $l=1$ in (3.36) and substituting (3.46) into (3.36), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j^{-1}}\left|\partial_{y_3}^2 P\chi_j\right|^2 \mathrm{d}y + \int_{\Omega_j^{-1}}\left|\partial_{y_3}^2\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)\chi_j\right|^2 \mathrm{d}y\\&\le C\left[\|\nabla u\|_1^2 + \|u_t\|_1^2 + \|\nabla P\|_{L^2}\left(\|\nabla P\|_{L^2} + \left\|\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\right\|_{L^2}\right) + \delta\|\nabla P\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2\right]\end{aligned} \tag{3.52}$$

By the same way, $D\times(3.51) + (3.52)$, there exists $H_2(P)$ which is equivalent to $\left\|\nabla^2 P\right\|_{L^2}^2$, such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{D^2\int_\Omega m\left|\nabla^2 u\chi_0\right|\mathrm{d}x+D^2\sum_{j=1}^N\int_{\Omega_j^{-1}}m\left|\partial u\chi_j\right|^2\mathrm{d}y+H_2\right\}\\&+\int_\Omega\left|\nabla^3 u\chi_0\right|^2\mathrm{d}x+\sum_{j=1}^N\int_{\Omega_j^{-1}}\left|\partial^2\nabla u\chi_j\right|^2+\int_\Omega\left|\nabla^2\frac{\mathrm{d}P}{\mathrm{d}t}\right|^2\mathrm{d}x\\&\le CD^2\left(\|\nabla u\|_1^2+\|\nabla u_t\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)+CD^2\delta\left(\|\nabla P\|_1^2+\left\|\nabla^2 u\right\|_1^2\right)\\&\quad+CD^2\left\|\nabla^2 u\right\|_{L^2}\left(\|\nabla u\|_2+\left\|\nabla^2 P\right\|_{L^2}\right).\end{aligned}\tag{3.53}$$

Applying Lemma 2.2 to (3.47), we obtain

$$\left\|\nabla^3 u\right\|_{L^2}^2+\left\|\nabla^2 P\right\|_{L^2}^2\le C\left(\|\nabla u\|_1^2+\|u_t\|_1^2\|\nabla P\|_{L^2}^2+\left\|\nabla\frac{\mathrm{d}P}{\mathrm{d}t}\right\|_1^2+\|\nabla u\|_1^2\left\|\nabla^3 u\right\|_{L^2}^2\right).\tag{3.54}$$

Step 3: Establish the energy inequality of Gronwall-type. An application of the L^p-estimate of elliptic system to $(1.10)_4$ gives

$$\left\|\nabla^2 u\right\|_{L^2}^2\le C\left(\|u_t\|_{L^2}^2+\|\nabla P\|_{L^2}+\|\nabla u\|_{L^2}^2\right)\tag{3.55}$$

Consider $D^4\times$ (3.50) + $D\times$ (3.53) + (3.54), there exists a function $H_3(P,u)$ which is equivalent to $\|P-\bar{P}\|_2^2+\|u\|_2^2+\|P_t-\bar{P}_t\|_{L^2}^2+\|u_t\|_{L^2}^2$ such that

$$\frac{\mathrm{d}H_3}{\mathrm{d}t}+CH_3+C\left\|\nabla^3 u\right\|_{L^2}^2\le 0,\tag{3.56}$$

where we have used the Poincaré's inequality $\|P_t-\bar{P}_t\|_{L^2}\le\|\nabla P(t)\|$. Integrating the above inequality over [0,t] get (1.15). Using Gronwall's inequality to (3.56), it is clear that there exist two positive constant C_1 and η_1 such that

$$H_3(P,u)\le C_1H_3(P_0,u_0)\mathrm{e}^{-\eta_1 t},\tag{3.57}$$

which together with $(1.10)_3$ yield (1.16).

Step 4: Finally, we prove (1.17), (1.18) and (1.19), and we can conclude the energy estimates on α, c as following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\alpha-\bar{\alpha}\|_2^2\le C\|u\|_2^2\|\alpha-\bar{\alpha}\|_2^2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|c-\bar{c}\|_2^2\le C\|u\|_2\|c-\bar{c}\|_2^2+\|B_1\|_2^2,$$

By Gronwall's inequality, we get

$$\|\alpha-\bar{\alpha}\|_2^2\le C\|\alpha_0-\bar{\alpha}\|_2\exp\left\{C\int_0^t\|u(\tau)\|_2\,\mathrm{d}\tau\right\}$$

$$\|c-\bar{c}\|_2^2\le C\exp\left\{C_1\int_0^t\|u(\tau)\|_2\,\mathrm{d}\tau\right\}\left(\|u_0\|_2+\int_0^t\|B_1\|_2^2\,\mathrm{d}\tau\right),$$

By simple calculation, implies (1.17) and (1.18). To prove (1.19), we first show that $\lim_{t\to\infty}\bar{P}(t)$ exists. In fact, for any arbitrary positive constant ε, there exists a positive constant $T=\max\left\{1,\frac{\ln\frac{\eta_0\varepsilon}{C_0}}{-\eta_0}\right\}$ such that for any $t_2>t_1>T$, it holds that

$$\left|\bar{P}(t_2)-\bar{P}(t_1)\right|=\left|\int_{t_1}^{t_2}\bar{P}_\tau \mathrm{d}\tau\right|\le C_0\int_{t_1}^{t_2}\mathrm{e}^{-\eta_0\tau}\mathrm{d}\tau\le\frac{C_0}{\eta_0}\mathrm{e}^{-\eta_0 t_1}<\varepsilon,$$

which implies that $\lim_{t\to\infty}\bar{P}(t)$ exists. Now, setting $\tilde{P}=\lim_{t\to\infty}\bar{P}(t)$, and combining (3.8), we obtain

$$\left\|\tilde{P}-\bar{P}(t)\right\|=\left|\int_t^\infty \bar{P}_\tau \mathrm{d}\tau\right|\le C\int_t^\infty\left\|u(\tau)\right\|\left(\left\|\nabla P(\tau)\right\|+\left\|\nabla u(\tau)\right\|\right)\mathrm{d}\tau,$$

which together with (1.16) implies (1.19).

Finally, we have finished the proof of Theorem 1.1.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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