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The Catastrophe of Rapidly Rotating Fluids

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Abstract

In astrophysics, when studying rotating fluid systems (such as stars or gas giant planets), the rotation effect is most often neglected. This is explained by the fact that the various stars that populate our universe revolve around themselves at relatively low speeds. However, when trying to describe the internal structure of an astrophysical body, the speed of rotation of the fluid contained in the core of the star can reach very high values. It, therefore, becomes impossible to neglect centrifugal forces in the equation of fluid motion. In this work, we carry out a simplified but above all general study of rapidly rotating fluid systems. Euler's equation then contains a centrifugal force term. The resolution of this equation leads to a solution that reveals a very particular property of this type of system: "the catastrophe of rapidly rotating fluids".

Keywords

Fluid, Rapidly, Rotating, Catastrophe, Viscosity

1. Introduction

The magnetic field measured in the vicinity of certain astrophysical bodies (a planet or a star) is created by the dynamo effect. In other words, it is the rotational movement of charged fluid particles in the core of the star that produces this magnetic field. Thus, to describe the behavior of the fluid in the core of the star, it becomes essential to consider the centrifugal force in the equation of fluid motion. Some researchers have worked on the behavior of a rotating fluid. We have, for example, Kenyon, K. who studied the behavior of a fluid put into rotation by a cone immersed in the fluid [1]; or even Adachi, T., Takahashi, Y., Akinaga, T. and Okajima, J. who studied the effects of viscosity on the behavior of a Newtonian fluid rotated by a cone immersed in the fluid [2].

In the present work, we will study the behavior of any rapidly rotating fluid, *i.e.* the rotational speed of a fluid particle is very high so that the force of gravity is neglected in front of the centrifugal force in the Euler equation. The solution of this equation is given by a speed which depends not only on the viscosity and the position but also on two real non-zero random variables. It is these two random variables that give the fluid catastrophic properties.

2. Modeling at

1) Hypothesis

For the sake of simplicity, we consider here a cylindrical fluid system in rapid rotation around the axis of the cylinder. Furthermore, the fluid is assumed to be homogeneous, in thermodynamic equilibrium, nonmagnetized and macroscopically neutral.

2) Equation of motion

We place ourselves in a cylindrical frame of reference $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ of coordinates (r, θ, z) where:

- $r \in \mathbb{R}$ is the radial coordinate
- $\theta \in [0, 2\pi]$ is the azimuthal coordinate.
- $z \in \mathbb{R}$ is the axial coordinate.

In Figure 1, the notation $(\boldsymbol{u}_r, \boldsymbol{u}_\theta, \boldsymbol{u}_z)$ was used instead of $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$.

If we assume that the fluid is animated by a pure rotational movement (we therefore neglect any translational movement of the fluid particles), the Euler equation is written:

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{grad})\boldsymbol{u}\right] = -\boldsymbol{grad}P + \rho_c \boldsymbol{E} + \boldsymbol{j} \wedge \boldsymbol{B} + \rho\nu\Delta\boldsymbol{u} + \rho\boldsymbol{g} + \rho\frac{u^2}{r}\boldsymbol{e}_r \tag{1}$$

where:

- ρ is the density
- ρ_c is the total charge density
- ν is the kinematic viscosity
- $\boldsymbol{u}$ is the speed vector (of rotation) of norm u.
- P is the pressure
- $\boldsymbol{E}$ is the electric field
- $\boldsymbol{j}$ is the current density
- $\boldsymbol{B}$ is the magnetic field

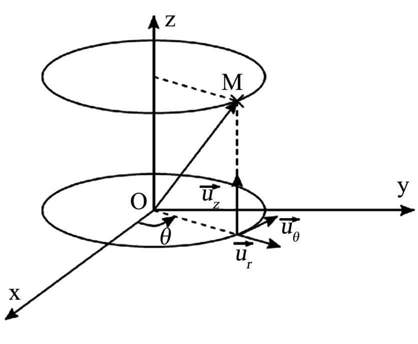

Figure 1. Cylindrical coordinates (r, θ, z).

- $\boldsymbol{g}$ is the acceleration due to gravity.

This is a Navier-Stokes equation with a centrifugal force term.

Let's make the range of forces in Equation (1), we have:

- To the left of the equals sign, we have the force of inertia with the operator $\frac{\partial}{\partial t}+(\boldsymbol{u}\cdot\nabla)$ which is nothing but the derivative convective. In steady state (stable), we have $\frac{\partial}{\partial t}=0$.
- $-\nabla P$ is the pressing force. The fluid being homogeneous and in thermodynamic equilibrium, this force is zero.
- $\rho_c \boldsymbol{E}$ is the electrostatic force. The fluid being macroscopically neutral ($\rho_c = 0$), this force is zero.
- $\boldsymbol{j}\wedge\boldsymbol{B}$ is the Laplace force. The fluid being unmagnetized ($\boldsymbol{B}=\boldsymbol{0}$), this force is zero.
- $\rho v\Delta\boldsymbol{u}$ represents viscous frictional forces (frictional force).
- $\rho\frac{u^2}{r}\boldsymbol{e}_r$ is the centrifugal force. It is a radial force.
- $\rho\boldsymbol{g}$ is the gravitational force. By rapid rotation, we mean rotational speeds $\boldsymbol{u}$ which are such that $u \geq v_l$; where v_l is the escape velocity (or second cosmic velocity) of the star. It is given by: $v_l=\sqrt{\frac{2GM}{R}}$; Where M and R are respectively the mass and the radius of the star, G is the gravitational constant of Newton ($G=6.67\times10^{-11}\ \mathrm{N\cdot kg^{-2}\cdot m^2}$).Thus for fast rotations, one can neglect the effect of gravity compared to the centrifugal force. For the earth, we have $v_l=11.2\ \mathrm{km/s}$ while for the sun, we have $v_l=42.1\ \mathrm{km/s}$.

In short, the equation of motion (1) in steady state simply becomes:

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=v\Delta\boldsymbol{u}+\frac{\|\boldsymbol{u}\|^2}{r}\boldsymbol{e}_r \tag{2}$$

3. Solving the Equation of Motion

We start by recalling the expressions of the differential operators in cylindrical coordinates. Let *f* be a scalar field and $A(A_r, A_\theta, A_z)$ a vector field, then we have:

$$\nabla f=\left(\frac{\partial f}{\partial r},\frac{1}{r}\frac{\partial f}{\partial\theta},\frac{\partial f}{\partial z}\right)$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_z}{\partial z}$$

$$\nabla\times\boldsymbol{A}=\begin{pmatrix}\frac{1}{r}\frac{\partial A_z}{\partial\theta}-\frac{\partial A_\theta}{\partial z}\\ \frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\\ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)\end{pmatrix}$$

$$\Delta f = \frac{1}{r}\frac{\partial\left(r\frac{\partial f}{\partial r}\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta \boldsymbol{A} = \boldsymbol{grad}\left(div\boldsymbol{A}\right) - \boldsymbol{rot}\left(\boldsymbol{rotA}\right).$$

The speed vector $\boldsymbol{u}\left(u_r, u_\theta, u_z\right)$ is given by:

$$\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z$$

As the fluid particle always describes a circular trajectory in a plane orthogonal to the axis of the cylinder, one can always define a cylindrical coordinate system such as the velocity vector $\vec{\boldsymbol{u}}$ either orthogonal to the two axes of the axial and radial coordinates. In such a frame of reference, we will have:

$$u_r = 0 \text{ and } u_z = 0\text{; so } \boldsymbol{u} = u_\theta\boldsymbol{e}_\theta.$$

If being so, let us now solve the equation of motion in such a reference frame. Equation (2) is given by:

$$\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = \nu\Delta\boldsymbol{u} + \frac{u^2}{r}\boldsymbol{e}_r$$

Let's calculate each term of this equation:

$$\boldsymbol{u}\cdot\nabla = u_\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = u_\theta\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\boldsymbol{e}_\theta$$

$$\Delta\boldsymbol{u} = \nabla\left(\nabla\cdot\boldsymbol{u}\right) - \nabla\times\left(\nabla\times\boldsymbol{u}\right)$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$\nabla\left(\nabla\cdot\boldsymbol{u}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\boldsymbol{e}_z$$

In matrix form, we have:

$$\nabla\left(\nabla\cdot\boldsymbol{u}\right) = \begin{pmatrix} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) \\ \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) \\ \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial}{\partial r}r^{-1}\right)\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta} \\ \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} \\ \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial z\partial\theta} \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{-1}{r^2}\right)\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta} \\ \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} \\ \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial z\partial\theta} \end{pmatrix} \quad (3)$$

$$\nabla \times \boldsymbol{u} = \left(-\frac{\partial u_\theta}{\partial z}, 0, \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} \right)$$

$$\nabla \times (\nabla \times \boldsymbol{u}) = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} \right) \\ \frac{\partial}{\partial z}\left(-\frac{\partial u_\theta}{\partial z} \right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} \right) \\ \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_\theta}{\partial z} \right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(u_\theta + r\frac{\partial u_\theta}{\partial r} \right) \\ -\frac{\partial^2 u_\theta}{\partial z^2} - \left[-\frac{1}{r^2}\left(\frac{\partial (r u_\theta)}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial (r u_\theta)}{\partial r} \right) \right] \\ \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial r} \\ -\frac{\partial^2 u_\theta}{\partial z^2} - \left[-\frac{1}{r^2}\left(u_\theta + r\frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial r}\left(u_\theta + r\frac{\partial u_\theta}{\partial r} \right) \right] \\ \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial r} \\ -\frac{\partial^2 u_\theta}{\partial z^2} - \left[-\frac{1}{r^2}u_\theta - \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial r} + r\frac{\partial^2 u_\theta}{\partial r^2} \right) \right] \\ \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial r} \\ -\frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r^2}u_\theta - \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{\partial^2 u_\theta}{\partial r^2} \\ \frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta \partial z} \end{pmatrix} \quad (4)$$

Equation (3) minus Equation (4) gives:

$$\Delta \boldsymbol{u} = \begin{pmatrix} -\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \\ \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{1}{r^2}u_\theta + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2} \\ 0 \end{pmatrix}$$

From all the above, Equation (2) can be written in matrix form as:

$$\begin{pmatrix} \frac{2v}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{u_\theta^2}{r} \\ u_\theta \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}-v\left(\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{1}{r^2}u_\theta+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_\theta}{\partial r^2}\right) \\ 0 \end{pmatrix}(\boldsymbol{e}_r\boldsymbol{e}_\theta\boldsymbol{e}_z)=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad (5)$$

By projecting Equation (5) onto the axis of radial coordinates, we have:

$$\frac{2v}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{u_\theta^2}{r}=0 \Leftrightarrow \frac{2v}{r}\frac{\partial u_\theta}{\partial \theta}-u_\theta^2=0 \quad (6)$$

By projecting the Equation (5) on the axis of the azimuthal coordinates, we have:

$$u_\theta \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}-v\left(\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{1}{r^2}u_\theta+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_\theta}{\partial r^2}\right)=0 \quad (7)$$

u_θ is therefore the solution of the system (6), (7).

This system is equivalent to a partial differential equation problem (6) with Equation (7) as the associated condition.

A characteristic of the physical system allows us to reduce the number of variables on which U_θ depends. Indeed, for fixed *z* (*i.e.* *z* = Constant), the velocity vector $\boldsymbol{u}$ always belongs to a plane orthogonal to the *z* axis as shown in **Figure 2**. Thus, we can study the system at *z* = 0 without losing any generality related to the problem. In this case, $\boldsymbol{u}$ belongs to the plane $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$ and depends only on the variables r and θ. To solve the partial derivative equation problem, let us use the method of separation of variables [3]. We therefore seek solutions of the form $u_\theta(r,\theta)=u_\theta(r)\cdot u_\theta(\theta)$.

By inserting $u_\theta(r,\theta)$ in Equation (6), we find:

$$\frac{2v}{r}\frac{\partial\left(u_\theta(r)\cdot u_\theta(\theta)\right)}{\partial \theta}-u_\theta^2(r)\cdot u_\theta^2(\theta)=0$$

$$\Leftrightarrow \frac{2v}{r}u_\theta(r)\cdot\frac{\partial u_\theta(\theta)}{\partial \theta}-u_\theta^2(r)\cdot u_\theta^2(\theta)=0$$

$$\Leftrightarrow \frac{1}{u_\theta^2(\theta)}\cdot\frac{\partial u_\theta(\theta)}{\partial \theta}=\frac{r u_\theta(r)}{2v} \quad (8)$$

In Equation (8), the left-hand side depends only on θ while the right-hand side depends only on *r*. This equality is only possible if both members are equal to a constant λ.

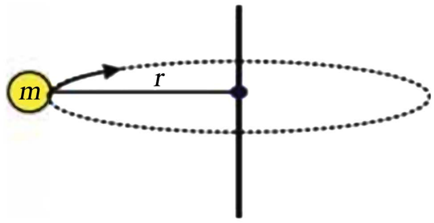

Figure 2. Rotation of a fluid particle in a plane orthogonal to *z* axis.

So we have: $\frac{1}{u_\theta^2(\theta)}\cdot\frac{\partial u_\theta(\theta)}{\partial\theta}=\frac{ru_\theta(r)}{2v}=\lambda$.

We deduce a new system with two equations:

$$\begin{cases}\frac{1}{u_\theta^2(\theta)}\frac{\mathrm{d}u_\theta(\theta)}{\partial\theta}=\lambda & (9)\\ \frac{ru_\theta(r)}{2v}=\lambda & (10)\end{cases}$$

(10) Gives: $ru_\theta=\frac{2v\lambda}{r}$

(9) Gives: $\frac{\mathrm{d}u_\theta(\theta)}{\partial\theta}-\lambda u_\theta^2(\theta)=0$

It can be written as:

$$y'-\lambda y^2=0 \tag{11}$$

with $y=u_\theta(\theta)$ and $y'=\frac{\mathrm{d}u_\theta(\theta)}{\partial\theta}$.

This is Bernoulli's ordinary differential equation [4].

We then put $x=y^{-1}$

$$\Rightarrow x'=(-1)\cdot y^{-2}\cdot y'$$

$$\Leftrightarrow y'=-x'y^2$$

(11) Therefore becomes: $-x'y^2-\lambda y^2=0$

$\Leftrightarrow x'+\lambda=0$ (We look for non-trivial solutions $y\neq 0$).

$$\Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=-\lambda \Leftrightarrow \mathrm{d}x=-\lambda\mathrm{d}\theta$$

$$\Leftrightarrow x=-\lambda\theta+c$$

where c is a constant.

So we have $y^{-1}=-\lambda\theta+c$

$$\Leftrightarrow y=\frac{1}{-\lambda\theta+c}$$

$$\Rightarrow u_\theta(\theta)=\frac{1}{-\lambda\theta+c}$$

The solution of Equation (6) is therefore given by: $u_\theta(r,\theta)=\frac{2v\lambda}{r(-\lambda\theta+c)}$ and this solution must satisfy Equation (7).

So we have:

$$\frac{\partial u_\theta}{\partial\theta}=\frac{2v\lambda}{r}\cdot(-1)(-\lambda\theta+c)^{-2}\cdot(-\lambda)=\frac{2v\lambda^2}{r(-\lambda\theta+c)^2}$$

$$\frac{\partial^2 u_\theta}{\partial\theta}=\frac{2v\lambda^2}{r}\cdot(-2)(-\lambda\theta+c)^{-3}\cdot(-\lambda)=\frac{4v\lambda^3}{r(-\lambda\theta+c)^3}$$

$$\frac{\partial u_\theta}{\partial r}=\frac{2v\lambda^2}{r}\cdot(-1)\cdot r^{-2}=-\frac{2v\lambda}{r^2(-\lambda\theta+c)}$$

$$\frac{\partial^2 u_\theta}{\partial r} = -\frac{2\nu\lambda}{-\lambda\theta + c}\cdot(-2)\cdot r^{-3} = \frac{4\nu\lambda}{r^3(-\lambda\theta + c)}$$

Equation (7) then gives:

$$u_\theta \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \nu\left(\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{1}{r^2}u_\theta + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r}\right)$$
$$= \frac{2\nu\lambda}{r(-\lambda\theta + c)}\frac{1}{r}\frac{2\nu\lambda^2}{r(-\lambda\theta + c)^2} - \nu\frac{1}{r^2}\frac{4\nu\lambda^3}{r(-\lambda\theta + c)^3}$$
$$+ \frac{\nu}{r^2}\frac{2\nu\lambda}{r(-\lambda\theta + c)} + \frac{\nu}{r}\frac{2\nu\lambda}{r^2(-\lambda\theta + c)} - \frac{\nu 4\nu\lambda}{r^3(-\lambda\theta + c)}$$
$$= 0$$

Equation (7) therefore holds for all $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}$.

The solution of the equation of motion (2) is therefore given by:

$$\boldsymbol{u} = \frac{2\nu\lambda}{r(-\lambda\theta + c)}\boldsymbol{e}_\theta \tag{12}$$

with $\nu \in \mathbb{R}_+^*, \lambda \in \mathbb{R}^*, r \in \mathbb{R}^*, c \in \mathbb{R}$ and $\theta \in [0, 2\pi]$.

The fact that λ and c take any non-zero real value in this solution, imparts to the rapidly rotating fluid a very impressive feature as we will see in the following discussion.

4. Discussion

- We begin by pointing out that solution (12) does satisfy the dimensional equation. In fact, in a system of international units, the speed u is measured in m/s, the kinematic viscosity ν in m^2/s and the radius of the circular path r in m. θ, λ and c are numbers.
- Note also that the modeling above is general. In other words, it can be used to describe any rapidly rotating fluid system (fluid in the core, cyclone, vortex, …).
- The first characteristic of this type of system deduced directly from solution (12) is: "the further one moves away from the axis or from the center of rotation, the speed decreases". This property can be verified quite easily in the case of a vortex.
- A second characteristic which follows directly from solution (12) is the fact that for a rotating fluid, the kinematic viscosity depends on the radius of curvature, which is not the case for an irrotational fluid [5]. Indeed, as can be seen in relation (12), the kinematic viscosity ν increases with the radius r. We will illustrate this property later for the case of the solar core.
- The last characteristic of this solution and not at all the least comes from the fact that in the relation (12), the constants λ and c can take any values in $\mathbb{R}^*$. The kinematic viscosity and consequently the frictional forces can thus reach very high values.

This property is very important in hot plasma theory and in astrophysics because it explains the enormous amount of energy produced in the cores of stars.

Indeed, due to its dimensions, the kinematic viscosity is proportional to the collision frequency and to the collision cross-section. The energy in the core of the star is produced by the nuclear fusion reactions between the fluid particles which populate this medium. These reactions only take place when there is a collision between particles in the plasma [6]. So, when the kinematic viscosity is high, the collision frequency and cross-section are also high. There are therefore more and more fusion reactions and as a result, the energy produced by these increases.

5. Numerical Example: Case of the Solar Core [7]

In this example:

- T is the temperature of the fluid;
- ρ is the density of the fluid;
- v_i is the kinematic viscosity of the fluid when not rotating;
- u is the rotational speed of the fluid;
- v_{th} is the thermal velocity of the particles;
- k_B is the Boltzmann constant;
- m is the average atomic mass of the fluid;
- m_H is the atomic mass of hydrogen;
- v_l is the star's escape velocity;
- v_r is the kinematic viscosity of the rapidly rotating fluid.
- For irrotational fluid flow, the viscosity kinematics of a gas of density ρ and of temperature T is in the first approximation given by:

$$v_i = \frac{2.2\times10^{-17}\times T^{\frac{5}{2}}}{\rho};$$

with T in Kelvin et ρ in kg/m^3.

In the case of the solar core, we have:

$$\rho = 15\times10^4\ \text{kg}/\text{m}^3 \quad \text{and} \quad T = 15\times10^6\ \text{K}.$$

$$v_i = \frac{2.2\times10^{-17}\times\left(15\times10^6\right)^{\frac{5}{2}}}{15\times10^4} \simeq 1278\times10^{-7}\ \text{m}^2/\text{s}$$

- For our configuration where the fluid is in rapid rotation, we can take $u \simeq v_{th}$ (v_{th} is the thermal velocity of the particles).

So we have:

$$u \simeq v_{th} = \left(\frac{8K_BT}{\pi m}\right)^{\frac{1}{2}}$$

where k_B is Boltzmann's constant and m is the average atomic mass of the gas.

In the case of the solar core, we have:

$m = 0.5m_H$ with $m_H = 1.67\times10^{-27}$ kg, the atomic mass of hydrogen.

Thereby $u \simeq v_{th} = \left(\frac{8\times1.38\times10^{-23}\times15\times10^6}{3.14\times0.835\times10^{-27}}\right)^{\frac{1}{2}} \simeq 8\times10^5\ \text{m/s} > v_l = 42.1\ \text{km/s}$.

The kinematic viscosity is therefore given by:

$$v_r = \frac{u|r||-\lambda\theta+c|}{2|\lambda|} = 4\times10^5\,\frac{|r||-\lambda\theta+c|}{|\lambda|}$$

Let's make the choice $\lambda=1, c=0$ and $\theta=1$ then determine $|r|$ so that $v_r = v_i$; we have:

$$v_i = 4\times10^5\,|r|$$

$$\Leftrightarrow |r| = \frac{1278\times10^{-7}}{4\times10^5} = 3195\times10^{-13}\ \text{m}$$

We notice that already at distances very close to the center of the sun, v_r is of the order of v_i. However, the radius of the solar core is estimated at about 200,000 km. We therefore realize that far from the center of the star, the viscosity v_r can reach extremely high values. And that's not all. Here we have made the trivial choice $c=0$ but we could also choose $c=10^6$. In the latter case, the values reached by the kinematic viscosity are quite simply "catastrophic".

This is why this configuration where the fluid is in the rapid rotation is the one that best describes the dynamics in the cores of stars.

To this, we would add a heuristic argument drawn from magnetohydrodynamics, again to confirm the previous proposition.

Indeed, for there to be a fusion reaction, the plasma must be hot and very dense [6]. The fluid must therefore reach the effective temperature and density necessary for the fusion reactions.

The movement of charged particles in the core inevitably produces an induced magnetic field.

The plasma is then subjected to a magnetic force given by [8]:

$$\boldsymbol{J}\times\boldsymbol{B} = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B};$$

with $\boldsymbol{J} = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})$ (Maxwell-Ampère equation in which we have neglected the displacement current).

Using vector identity:

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$$

That is $\frac{1}{2}\nabla B^2 = -(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$; we have:

$$\boldsymbol{J}\times\boldsymbol{B} = -\nabla\frac{B^2}{2\mu_0} + \frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$$

Laplace's force $\boldsymbol{J}\times\boldsymbol{B}$ therefore breaks down into two parts. The first can be interpreted as the gradient of a pressure, magnetic pressure, $\frac{B^2}{2\mu_0}$.

The second part $\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$ is non-zero if the field lines are curved or divergent. The curvature of the magnetic field produces a force analogous to that which the field lines would produce if they were elastic strings, so it tends to confine the plasma.

Moreover, in our model of the rapidly rotating fluid, the magnetic field lines are always curved. At the heart of the star, the magnetic force is therefore added to the gravitational force to confine the plasma and the latter can then reach the effective density necessary for the nuclear fusion reaction.

6. Conclusions

In this work, we have used a rather simplified model to demonstrate that the dynamics of slowly rotating fluids differ enormously from the dynamics of rapidly rotating fluids. In a configuration where the fluid is in rapid rotation, the solution (12) of the equation of motion (2) presents a kinematic viscosity which depends not only on the position and the speed of the fluid particles but also on two real non-zero "random" variables. These give the rotating fluid a viscosity that can reach very high values. Since the kinematic viscosity is proportional to the frequency of the fusion reactions as well as to the cross-section of these in hot plasma, the model of the rapidly rotating fluid clearly explains the enormous amount of energy produced in the core of the star.

This particular characteristic of rapidly rotating fluids, which we have named as "catastrophe", can be used by nuclear engineers for the design of a new type of hot plasma reactor. This rapidly rotating hot plasma reactor would then allow us to optimize the yield of energy produced by nuclear fusion reactions, even during a relatively short confinement time of the plasma. With the speed of rotation being proportional to the kinematic viscosity, the energy produced by the fusion reactions will be all the greater as the speed of rotation is high. Moreover, as a good approximation, the speed of rotation to be reached will be given by the thermal speed of the particles at the effective fusion temperature if and only if this is greater than the escape velocity. In the case of our planet, the earth, the speed of rotation will have to exceed 11.2 km/s.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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