Scientific Research Publishing

Open Journal of Applied Sciences, 2021, 11, 789-806
https://www.scirp.org/journal/ojapps
ISSN Online: 2165-3925
ISSN Print: 2165-3917

# The Family of Global Attractors for a Generalized Kirchhoff Equations

**Guoguang Lin, Xiaomei Liu**

Department of Mathematics, Yunnan University, Kunming, China
Email: gglin@ynu.edu.cn, 790612351@qq.com

**How to cite this paper:** Lin, G.G. and Liu, X.M. (2021) The Family of Global Attractors for a Generalized Kirchhoff Equations. *Open Journal of Applied Sciences*, **11**, 789-806.
https://doi.org/10.4236/ojapps.2021.117058

**Received:** June 1, 2021
**Accepted:** July 17, 2021
**Published:** July 20, 2021

Copyright © 2021 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

## Abstract

This paper deals with the existence and uniqueness of solutions of generalized Kirchhoff equations and the family of global attractors for the equation and its dimension estimation. First, the stress term $M\left(\left\|\nabla^m u\right\|_p^p\right)$ of Kirchhoff equation is properly assumed. When certain conditions are met between the order $m$ and the degree $p$ of Banach space $L^p(\Omega)$, the existence and uniqueness of the solution of equation are obtained by a prior estimation and Galerkin's method; Then, the bounded absorption set $B_{0k}$ is obtained by prior estimation, and it is proved that the solution semigroup $S(t)$ generated by the equation has a family of global attractors $A_k$ in phase space $E_k=\left(H^{2m+k}(\Omega)\cap H_0^1(\Omega)\right)\times H_0^k(\Omega)$ by using Rellich-Kondrachov compact embedding theorem. Further, the equation is linearized and rewritten into a first-order variational equation, and it is proved that the solution semigroup $S(t)$ is *Fréchet* differentiable on $E_k$; Finally, the upper bound of Hausdorff dimension and Fractal dimension of $A_k$ is estimated, and the Hausdorff dimension and Fractal dimension are finite.

## Keywords

Kirchhoff Equation, Galerkin's Method, Family of Global Attractor, Dimension Estimation

## 1. Introduction

This paper will study the initial-boundary value problems of the following generalized Kirchhoff equations:

$$u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho(u_t+u)=f(x), \tag{1}$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,2m-1,x\in\partial\Omega,t>0, \tag{2}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_t(x),x\in\Omega\subset R^n. \tag{3}$$

where $m\in N^+,\Omega\subset R^n(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $f(x)$ is an external force term, $M\left(\|\nabla^m u\|_p^p\right)$ is the stress term of Kirchhoff equation, $\beta>0$, $\beta\left(-\Delta^{2m}\right)u_t$ is a strong dissipative term, $|u|^\rho(u_t+u)$ is a nonlinear source term.

Many scholars have studied the existence of global attractor of Kirchhoff equation with strong dissipative term, [1]-[7] can be referred.

In reference [8], scholars considered the following Kirchhoff type wave equation with nonlinear strong damping term

$$u_{tt}-\varepsilon_1\Delta u_t+\alpha|u_t|^{p-1}u_t+\beta|u|^{q-1}u-\varphi\left(\|\nabla u\|^2\right)\Delta u=f(x),(x,t)\in\Omega\times R^+, \tag{4}$$

$$u(x,0)=u_0(x);u_t(x,0)=u_1(x),x\in\Omega, \tag{5}$$

$$u(x,t)\big|_{\partial\Omega}=0,\Delta u(x,t)\big|_{\partial\Omega}=0,x\in\Omega. \tag{6}$$

Here, $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$ in $R^N$, $\varepsilon_1,\alpha,\beta$ are normal numbers.

*TokioMatsuyama* and *RyoIkehata* [9] proved the global solution and attenuation of the solution of Kirchhoff type wave equation with nonlinear damping:

$$u_{tt}-M\left(\|\nabla u(t)\|_2^2\right)\Delta u+\delta|u_t|^{p-1}u_t=\mu|u|^{q-1}u. \tag{7}$$

With compact boundary conditions

$$u(x,t)\big|_{\partial\Omega}=0,t\geq 0. \tag{8}$$

*FucaiLi* discussed the higher-order Kirchhoff type equations with nonlinear terms in reference [10]:

$$u_{tt}+\left(\int_\Omega|\nabla^m u|^2\right)^q(-\Delta)^m u+u_t|u_t|^r=|u|^p u,x\in\partial\Omega,t>0, \tag{9}$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,m-1,x\in\partial\Omega,t>0, \tag{10}$$

$$u(x,0)=u_0,u_t=u_1(x),x\in\Omega. \tag{11}$$

In a bounded domain, where $m>1$ is a positive integer, $p,q,r>0$ is a normal number, if $p\leq r$, the existence of global solution will be obtained, if $p>\max\{r,2q\}$, for any initial value with negative initial energy, the solution explodes in a finite time. For more related research results, please refer to references [11] [12] [13] [14].

In this paper, for the convenience of narration, the following spaces and marks are defined: $H=L^2(\Omega)$, $H_0^{2m}(\Omega)=H^{2m}(\Omega)\cap H_0^1(\Omega)$, $H_0^{2m+k}(\Omega)=H^{2m+k}(\Omega)\cap H_0^1(\Omega)$, $E_k=H_0^{2m+k}(\Omega)\times H_0^k(\Omega)$, $(k=0,1,2,\cdots,2m)$. Define $(\cdot,\cdot)$ and $\|\cdot\|$ to represent the inner product and

norm of $H$ respectively,
namely

$$(u,v)=\int_{\Omega} u(x)v(x)\mathrm{d}x,(u,u)=\|u\|^2.$$

Let $A_k$ be the family of global attractor from $E_0$ to $E_k$, $B_{0k}$ be the bounded absorption set in $E_k$, where $k=1,2,\cdots,2m$. $C_i\left(i=0,1,2,\cdots\right)$ represents a constant.

(H1) assume that Kirchhoff type stress term $M(s)\in C^2\left(\left[0,+\infty\right],R\right)$ satisfies:

$$1<\mu_0\le M(s)\le\mu_1,\mu=\begin{cases}\mu_0,\dfrac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2\ge 0,\\ \mu_1,\dfrac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2<0.\end{cases}$$

where $\mu$ is a constant.

(H2) $\rho\le\dfrac{8m}{n}$

There exist constant $\varepsilon$, $0<\varepsilon<\min\left\{\sqrt{1+\dfrac{\beta\lambda_1^{2m}}{2}}-1,2\mu_0,\dfrac{2\mu_0}{\dfrac{1}{\lambda_1^{2m}}+\beta}\right\}$.

## 2. The Existence and Uniqueness of Global Solution

In this section, under the assumption of Kirchhoff stress term, the existence and uniqueness of global solution are obtained by prior estimation and Galerkin's method.

**Lemma 2.1** *Suppose Kirchhoff stress term* $M(s)$ *satisfies the conditions* (H1), *Assume that* (H2) *holds,* $f\in H$, $(u_0,v_0)\in E_0$, $v=u_t+\varepsilon u$, $\varepsilon>0$, *then the smooth solution of the initial-boundary value problem* (1.1)-(1.3) *satisfies* $(u,v)\in E_0$, $u\in L^\infty\left(0,+\infty;H^{2m}(\Omega)\right)$, $v\in L^\infty\left(0,+\infty;L^2(\Omega)\right)\cap L^2\left(0,T;H^{2m}(\Omega)\right)$, *and satisfy that following inequality*

$$\begin{aligned}\left\|(u,v)\right\|_{E_0}^2&=\left\|\nabla^{2m}u\right\|^2+\|v\|^2\\&\le\left(\|v_0\|^2+\varepsilon^2\|u_0\|^2+\mu\left\|\nabla^{2m}u_0\right\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u_0\|_{L^{\rho+2}(\Omega)}^{\rho+2}\right)\mathrm{e}^{-\alpha_1 t}+\frac{C_0}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right),\end{aligned}\tag{1}$$

$$\frac{\beta}{2}\int_0^T\left\|\nabla^{2m}v\right\|^2\mathrm{d}t\le C_0T+y_1(0).\tag{2}$$

where $\alpha_1=\min\left\{2\varepsilon,\dfrac{a_1}{\mu},a_2,\dfrac{\varepsilon(\rho+2)}{\varepsilon+1}\right\}$,

$$y_1(0)=\|v_0\|^2+\varepsilon^2\|u_0\|^2+\mu\left\|\nabla^{2m}u_0\right\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u_0\|_{L^{\rho+2}(\Omega)}^{\rho+2}.$$

So there is a non-negative real number $C(R_1)$ and $t=t_1(\Omega)>0$, make

$$\left\|(u,v)\right\|_{E_0}^2\le\frac{2C_0}{\alpha_1}=C(R_1),(t>t_1).\tag{3}$$

Proof. Set $v=u_t+\varepsilon u$, take the inner product of both sides of Equation (1.1) with $v$ in $H$, we obtain

$$\left(u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho(u_t+u),v\right)=(f(x),v). \quad (4)$$

$$(u_{tt},v)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\varepsilon^2\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2-\varepsilon\|v\|^2+\varepsilon^3\|u\|^2. \quad (5)$$

According to hypothesis ($H_1$), we can get

$$\begin{aligned}\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u,v\right)&=\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u,u_t+\varepsilon u\right)\\&\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2+\varepsilon\mu_0\left\|\nabla^{2m}u\right\|^2.\end{aligned} \quad (6)$$

By *Young*'s inequality and *Poincaré*'s inequality

$$\begin{aligned}\left(\beta(-\Delta)^{2m}u_t,v\right)&=\beta\left\|\nabla^{2m}v\right\|^2-\varepsilon\beta\left(\nabla^{2m}u,\nabla^{2m}v\right)\\&\geq\beta\left\|\nabla^{2m}v\right\|^2-\beta\left(\varepsilon\left\|\nabla^{2m}u\right\|\left\|\nabla^{2m}v\right\|\right)\\&\geq\frac{\beta}{2}\left\|\nabla^{2m}v\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^{2m}u\right\|^2.\end{aligned} \quad (7)$$

where $\lambda_1$ is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition on $\Omega$.

we can get

$$\begin{aligned}\left(|u|^\rho(u_t+u),v\right)&=\left(|u|^\rho,u_t^2\right)+\varepsilon\left(|u|^\rho u_t,u\right)+\left(|u|^\rho u,u_t\right)+\varepsilon\left(|u|^\rho u,u\right)\\&\geq\varepsilon\int_\Omega|u|^{\rho+2}\mathrm{d}x+(\varepsilon+1)\int_\Omega|u|^\rho u\cdot u_t\mathrm{d}x\\&\geq\varepsilon\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}+\frac{\varepsilon+1}{\rho+2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}.\end{aligned} \quad (8)$$

According to *Young*'s inequality, we get

$$(f(x),v)\leq\|f\|\|v\|\leq\frac{\varepsilon^2}{2}\|v\|^2+\frac{1}{2\varepsilon^2}\|f\|^2. \quad (9)$$

Substitute (2.5)-(2.9) into (2.4),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\left\|\nabla^{2m}u\right\|^2+\|v\|^2+\varepsilon^2\|u\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}\right)+\left(2\varepsilon\mu_0-\varepsilon^2\right)\left\|\nabla^{2m}u\right\|^2\\&+\frac{\beta}{2}\left\|\nabla^{2m}v\right\|^2+\left(\frac{\beta\lambda_1^{2m}}{2}-2\varepsilon-\varepsilon^2\right)\|v\|^2+2\varepsilon^3\|u\|^2+2\varepsilon\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}\leq\frac{1}{\varepsilon^2}\|f\|^2=C_0.\end{aligned} \quad (10)$$

Make $a_1=2\varepsilon\mu_0-\varepsilon^2\geq0, a_2=\frac{\beta\lambda_1^{2m}}{2}-2\varepsilon-\varepsilon^2\geq0$.

Take $\alpha_1=\min\left\{2\varepsilon,\frac{a_1}{\mu},a_2,\frac{(\rho+2)\varepsilon}{\varepsilon+1},\right\}$.

Then (2.10) can be converted into

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\left\|\nabla^{2m}u\right\|^2+\|v\|^2+\varepsilon^2\|u\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}\right)\\&+\alpha_1\left(\mu\left\|\nabla^{2m}u\right\|^2+\|v\|^2+\varepsilon^2\|u\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}\right)+\frac{\beta}{2}\left\|\nabla^{2m}v\right\|^2\leq C_0.\end{aligned} \quad (11)$$

By *Gronwall*'s inequality

$$\begin{aligned}&\left\|\nabla^{2m}u\right\|^2+\|v\|^2\\&\le\left(\|v_0\|^2+\varepsilon^2\|u_0\|^2+\mu\left\|\nabla^{2m}u_0\right\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u_0\|_{L^{\rho+2}(\Omega)}^{\rho+2}\right)\mathrm{e}^{-\alpha_1 t}+\frac{C_0}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right),\end{aligned}\tag{12}$$

$$\frac{\beta}{2}\int_0^T\left\|\nabla^{2m}v\right\|^2\mathrm{d}t\le C_0T+y_1(0).\tag{13}$$

where $y_1=\|v\|^2+\varepsilon^2\|u\|^2+\mu\left\|\nabla^{2m}u\right\|^2+\frac{2(\varepsilon+1)}{\rho+2}\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}$.

So there is a non-negative real number $C(R_1)$ and $t=t_1(\Omega)>0$, such as

$$\|(u,v)\|_{E_0}^2\le\frac{2C_0}{\alpha_1}=C(R_1),(t>t_1).\tag{14}$$

Lemma 2.1 is proved.

**Lemma 2.2** *Assume that* (H1), (H2) *holds, if* $f(x)\in H$, $(u_0,v_0)\in E_k\,(k=1,2,\cdots,2m)$, $v=u_t+\varepsilon u$, $\varepsilon>0$, *Then the smooth solution* $(u,v)\in E_k,(k=1,2,\cdots,2m)$ *of the initial-boundary value problem* (1.1)-(1.3) *satisfies* $u\in L^\infty\left(0,+\infty;H_0^{2m+k}(\Omega)\right)$,
$v\in L^\infty\left(0,+\infty;H^k(\Omega)\right)\cap L^2\left(0,T;H_0^{2m+k}(\Omega)\right)$ *and satisfy that following inequality*

$$\begin{aligned}\|(u,v)\|_{E_k}^2&=\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k v\right\|^2\\&\le\left(\left\|\nabla^k v_0\right\|^2+(\mu-\beta\varepsilon)\left\|\nabla^{2m+k}u_0\right\|^2\right)\mathrm{e}^{-\alpha_2 t}+\frac{C_3}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right),\end{aligned}\tag{15}$$

$$\frac{\beta}{6}\int_0^T\left\|\nabla^{2m+k}v\right\|^2\mathrm{d}t\le y_2(0)+C_3T.\tag{16}$$

So there is a non-negative real number $C(R_2)$ and $t=t_2(\Omega)>0$, make

$$\|(u,v)\|_{E_k}^2\le\frac{2C_3}{\alpha_2}=C(R_2),(t>t_2).\tag{17}$$

Proof. Take the inner product of $(-\Delta)^k v=(-\Delta)^k u_t+(-\Delta)^k\varepsilon u$ and the two sides of Equation (1.1), and get

$$\begin{aligned}&\left(u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho(u_t+u),(-\Delta)^k v\right)\\&=\left(f(x),(-\Delta)^k v\right).\end{aligned}\tag{18}$$

By *Young*'s inequality, *Poincaré*'s inequality

$$\left(u_{tt},(-\Delta)^k v\right)\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^k v\right\|^2-\frac{2\varepsilon+\varepsilon^2}{2}\left\|\nabla^k v\right\|^2-\frac{\varepsilon^2}{2\lambda_1^{2m}}\left\|\nabla^{2m+k}u\right\|^2.\tag{19}$$

According to hypothesis (H1)

$$\left(M\left(\left\|\nabla^m u\right\|_p^p\right),(-\Delta)^k v\right)\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}u\right\|^2+\varepsilon\mu_0\left\|\nabla^{2m+k}u\right\|^2.\tag{20}$$

By *Young*'s inequality, *Poincaré*'s inequality

$$\left(\beta(-\Delta)^{2m}u_t,(-\Delta)^k v\right)=\beta\left\|\nabla^{2m+k}v\right\|^2-\beta\varepsilon\left\|\nabla^{2m+k}u\right\|\left\|\nabla^{2m+k}v\right\| \\ \geq\frac{\beta}{6}\left\|\nabla^{2m+k}v\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|\nabla^{2m+k}u\right\|^2+\frac{\beta\lambda_1^{2m}}{3}\left\|\nabla^k v\right\|^2. \tag{21}$$

where $\lambda_1$ is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition on $\Omega$.

we can get

$$\begin{aligned}&\left(|u|^\rho(u_t+u),(-\Delta)^k v\right)\\&=\left(|u|^\rho v,(-\Delta)^k v\right)-\varepsilon\left(|u|^\rho u,(-\Delta)^k v\right)+\left(|u|^\rho u,(-\Delta)^k v\right)\\&\geq(1-\varepsilon)\left(|u|^\rho u,(-\Delta)^k v\right),\end{aligned}$$

By *Young*'s inequality, *Poincaré*'s inequality

$$\begin{aligned}(1-\varepsilon)\left(|u|^\rho u,(-\Delta)^k v\right)&\leq(1-\varepsilon)\left\|\nabla^{2k}v\right\|\left(\int_\Omega|u|^{2\rho+2}\,\mathrm{d}x\right)^{\frac{1}{2}},\\&\leq C_1\left\|\nabla^{2k}v\right\|\leq\frac{\beta}{12}\left\|\nabla^{2m+k}v\right\|^2+C_2.\end{aligned} \tag{22}$$

By *Schwarz*'s inequality, *Young*'s inequality

$$\left(f(x),(-\Delta)^k v\right)\leq\frac{\varepsilon^2}{2}\left\|\nabla^k v\right\|^2+\frac{1}{2\varepsilon^2}\left\|\nabla^k f(x)\right\|^2. \tag{23}$$

Substitute (2.19)-(2.23) into (2.18) to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2\right)+\left(\frac{2\beta\lambda_1^{2m}}{3}-2\varepsilon-2\varepsilon^2\right)\left\|\nabla^k v\right\|^2\\&+\left(2\varepsilon\mu_0-\frac{\varepsilon^2}{\lambda_1^{2m}}-\beta\varepsilon^2\right)\left\|\nabla^{2m+k}u\right\|^2+\frac{\beta}{6}\left\|\nabla^{2m+k}v\right\|^2\leq C_3.\end{aligned} \tag{24}$$

Take $a_3=\frac{2\beta\lambda_1^{2m}}{3}-2\varepsilon-2\varepsilon^2\geq0, a_4=2\varepsilon\mu_0-\frac{\varepsilon^2}{\lambda_1^{2m}}-\beta\varepsilon^2\geq0$.

Let $\alpha_2=\min\left\{\frac{2\beta\lambda_1^{2m}}{3}-2\varepsilon-2\varepsilon^2,\frac{2\varepsilon\mu_0-\frac{\varepsilon^2}{\lambda_1^{2m}}-\beta\varepsilon^2}{\mu}\right\}$.

Then (2.24) can be converted into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2\right)+\alpha_2\left(\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2\right)+\frac{\beta}{6}\left\|\nabla^{2m+k}v\right\|^2\leq C_3. \tag{25}$$

By *Gronwall*'s inequality

$$\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k v\right\|^2\leq\left(\mu\left\|\nabla^{2m+k}u_0\right\|^2+\left\|\nabla^k v_0\right\|^2\right)\mathrm{e}^{-\alpha_2 t}+\frac{C_3}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right), \tag{26}$$

$$\frac{\beta}{6}\int_0^T\left\|\nabla^{2m+k}v\right\|^2\mathrm{d}t\leq y_2(0)+C_3T. \tag{27}$$

where $y_2=\left\|\nabla^k v\right\|^2+\mu\left\|\nabla^{2m+k}u\right\|^2$.

So there is a non-negative real number $C(R_2)$ and $t = t_2(\Omega) > 0$, make

$$\|(u,v)\|_{E_k}^2 \le \frac{2C_3}{\alpha_2} = C(R_2), (t > t_2). \tag{28}$$

Lemma 2.2 is proved.

**Theorem 2.1** *Under the assumption of lemma* 2.1 *and lemma* 2.2, *and satisfy the hypothesis* (H1), (H2), *Then the initial-boundary value problem* (1.1)-(1.3) *has a unique smooth solution* $(u,v) \in L^\infty(0,+\infty; E_k)$, $v \in L^2(0,T; H_0^{2m+k}(\Omega))$, $(k = 0,1,2,\cdots,2m)$.

Proof. Existence: Galerkin's method is used to prove the existence of global smooth solution.

Step 1: construct an approximate solution.

Let $(-\Delta)^{2m+k} w_j = \lambda_j w_j, k = 0,1,2,\cdots,2m$. where $\lambda_j$ is the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary on $\Omega$, $w_j$ denotes the eigenfunction determined by the corresponding eigenvalue $\lambda_j$, $w_1,\cdots,w_n$ constitute the orthonormal basis of $H$ from the eigenvalue theory.

Let the approximate solution of the problem (1.1)-(1.3) be

$u_s = u_s(t) = \sum_{j=1}^{s} g_{js}(t) w_j$, where $g_{js}(t)$ is determined by the following equations.

$$\left(u_{stt} + M\left(\|\nabla^m u_s\|_p^p\right)(-\Delta)^{2m} u_s + \beta(-\Delta)^{2m} u_{st} + |u_s|^\rho (u_{st} + u_s), w_j\right) = \left(f(x), w_j\right). \tag{29}$$

$$(1 \le j \le s)$$

The formula (2.29) satisfies the initial condition $u_s(0) = u_{0s}, u_{st}(0) = u_{1s}$.

When $s \to +\infty$, $(u_{0s}, u_{1s}) \to (u_0, u_1)$ in $E_k$, according to the basic theory of ordinary differential equations, the approximate solution $u_s(t)$ exists on $(0, t_s)$.

Step 2: Prior estimation.

$v_s(t) = u_{st}(t) + \varepsilon u_s(t)$, multiplying by $g'_{js}(t) + \varepsilon g_{js}(t)$ and summing over *j*, we can get

1) $k = 0$, by lemma 2.1, there is

$$\|(u_s, v_s)\|_{E_0}^2 \le y_1(0) e^{-\alpha_1 t} + \frac{C_0}{\alpha_1}\left(1 - e^{-\alpha_1 t}\right), \tag{30}$$

$$\frac{\beta}{2}\int_0^T \|\nabla^{2m} v\|^2 \, dt \le C_0 T + y_1(0). \tag{31}$$

2) $k = 1,2,\cdots,2m$, by lemma 2.2, there is

$$\|(u_s, v_s)\|_{E_k}^2 \le y_2(0) e^{-\alpha_2 t} + \frac{C_3}{\alpha_2}\left(1 - e^{-\alpha_2 t}\right), \tag{32}$$

$$\frac{\beta}{6}\int_0^T \|\nabla^{2m+k} v\|^2 \, dt \le y_2(0) + C_3 T. \tag{33}$$

From (2.30) and (2.32), $(u_s, v_s)$ is bounded in $L^\infty(0,+\infty; E_k)$, $(u_s, v_s)$ in $L^\infty(0,+\infty; E_0)$ is bounded.

It can be seen that the formula (2.30)-(2.33) holds a priori estimates for lemma 2.1 and lemma 2.2 respectively.

Step 3: Limit process.

In $E_k\left(k=0,1,2,\cdots,2m\right)$ space, select the subsequence $\{u_h\}$ from the sequence $\{u_s\}$,

Make $\left(u_h,v_h\right)\to\left(u,v\right)$ in $L^\infty\left(0,+\infty;E_k\right)$ weak * convergence.

According to Rellich-Kondrachov compact embedding theorem, $E_k$ compactly embeds $E_0$, Then $\left(u_h,v_h\right)\to\left(u,v\right)$ converges strongly almost everywhere in $E_0$.

Let $s=h$ in (2.29), and take the limit, for fixed *j*, $h\geq j$,

Then from (2.29), make $u_s\to u$ in $L^\infty\left(0,+\infty;H_0^{2m+k}\left(\Omega\right)\right)$ weak * convergence.

Thus $\left(u_h\left(t\right),\left(-\Delta\right)^k w_j\right)\to\left(u\left(t\right),\lambda_j^k w_j\right)$ in $L^\infty\left(0,+\infty\right)$ weak * convergence.

$\left(u_{ht}\left(t\right),\left(-\Delta\right)^k w_j\right)\to\left(u_t\left(t\right),\lambda_j^k w_j\right)$ in $L^\infty\left(0,+\infty\right)$ weak * convergence.

So $\left(u_{htt},\left(-\Delta\right)^k w_j\right)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(u_{ht},\left(-\Delta\right)^k w_j\right)\to\left(u_{ht},\lambda_j^k w_j\right)$ in $D'\left[0,+\infty\right)$ convergence.

$D'\left[0,+\infty\right)$ is a conjugate space of $D\left[0,+\infty\right)$ infinite differentiable space.

$\left(M\left(\left\|\nabla^m u_h\right\|_p^p\right)\left(-\Delta\right)^k u_h,\left(-\Delta\right)^k w_j\right)\to\left(M\left(\left\|\nabla^m u_h\right\|_p^p\right)\left(-\Delta\right)^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}w_j\right)$ in $L^\infty\left(0,+\infty\right)$ weak * convergence.

$\left(\beta\left(-\Delta\right)^{2m}u_{ht},\left(-\Delta\right)^k w_j\right)\to\beta\left(\left(-\Delta\right)^{\frac{k}{2}}v,\lambda_j^{\frac{2m+k}{2}}w_j\right)-\varepsilon\beta\left(\left(-\Delta\right)^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}w_j\right)$

in $L^\infty\left(0,+\infty\right)$ weak * convergence.

$u_{ht}\to u_t$ in $E_0$ strong convergence, almost everywhere convergence.

$$\left|u_h\right|^\rho\left(u_{ht}+u_h\right)\to w. \tag{34}$$

In $L^\infty\left(0,+\infty;L^{\frac{\rho+2}{\rho+1}}\left(\Omega\right)\right)$ weak * convergence.

Take $D=\Omega, g_h=\left|u_h\right|^\rho\left(u_{ht}+u_h\right)$. (2.34) almost everywhere

$g_h\to\left|u\right|^\rho\left(u_t+u\right)$, $g_h\to w$ in $L^{\frac{\rho+2}{\rho+1}}\left(\Omega\right)$ weak convergence.

Therefore $w=g=\left|u\right|^\rho\left(u_t+u\right)$, so

$\left(\left|u_h\right|^\rho\left(u_{ht}+u_h\right),\left(-\Delta\right)^k w_j\right)\to\left(\left|u\right|^\rho\left(u_t+u\right),\left(-\Delta\right)^k w_j\right)$ in $L^\infty\left(0,+\infty\right)$ weak * convergence. In particular, $u_{0h}\to u_0$ weak convergence in the $E_k$, $u_{1h}\to u_1$ weak convergence in $E_k$, From all *j* and $h\to+\infty$,it can be introduced

$$\begin{aligned}&\left(u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)\left(-\Delta\right)^{2m}u+\beta\left(-\Delta\right)^{2m}u_t+\left|u\right|^\rho\left(u_t+u\right),\left(-\Delta\right)^k w_j\right)\\&=\left(f\left(x\right),\left(-\Delta\right)^k w_j\right).\end{aligned}$$

Because of the density of $w_1,w_2,\cdots,w_k,\cdots$.

$$\left(u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho(u_t+u),v\right)$$
$$=(f(x),v),\forall v\in H_0^{2m+k}(\Omega).$$

Therefore, the existence is proved.

The uniqueness of the solution.

Set $u^*,v^*$ is equations of two solutions, make $w=u^*-v^*$, $w$ satisfies

$$\begin{gathered}w_{tt}+M\left(\left\|\nabla^m u^*\right\|_p^p\right)(-\Delta)^{2m}u^*-M\left(\left\|\nabla^m v^*\right\|_p^p\right)(-\Delta)^{2m}v^*+\beta(-\Delta)^{2m}w_t\\=\left|v^*\right|^\rho\left(v_t^*+v^*\right)-\left|u^*\right|^\rho\left(u_t^*+u^*\right),\\w(0)=0,w'(0)=0,x\in\Omega\subset R^n.\end{gathered}\tag{35}$$

Take inner product of (2.35) and $w_t$ in $H$ is as follows

$$\begin{gathered}\left(w_{tt}+M\left(\left\|\nabla^m u^*\right\|_p^p\right)(-\Delta)^{2m}u^*-M\left(\left\|\nabla^m v^*\right\|_p^p\right)(-\Delta)^{2m}v^*+\beta(-\Delta)^{2m}w_t,w_t\right)\\=\left(\left|v^*\right|^\rho\left(v_t^*+v^*\right)-\left|u^*\right|^\rho\left(u_t^*+u^*\right),w_t\right).\end{gathered}\tag{36}$$

Therefore

$$\left(w_{tt},w_t\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|w_t\right\|^2,\tag{37}$$

$$\left(\beta(-\Delta)^{2m}w_t,w_t\right)=\beta\left\|\nabla^{2m}w_t\right\|^2.\tag{38}$$

By differential mean value theorem, *Young*'s inequality

$$\begin{aligned}&\left(M\left(\left\|\nabla^m u^*\right\|_p^p\right)(-\Delta)^{2m}u^*-M\left(\left\|\nabla^m v^*\right\|_p^p\right)(-\Delta)^{2m}v^*,w_t\right)\\&\geq\frac{1}{2}M\left(\left\|\nabla^m u^*\right\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}w\right\|^2-\left(M'(\xi_1)\left(\left\|\nabla^m u^*\right\|_p^p-\left\|\nabla^m v^*\right\|_p^p\right)(-\Delta)^{2m}v^*,w_t\right)\\&\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}w\right\|^2-\frac{\beta}{2}\left\|\nabla^{2m}w_t\right\|^2-\frac{C_4^2}{2\beta}\left\|\nabla^{2m}w\right\|^2.\end{aligned}\tag{39}$$

where $\xi_1=\delta u^*+(1-\delta)v^*,0<\delta<1$.

By *Young*'s inequality

$$\begin{aligned}&\left(\left|v^*\right|^\rho\left(v_t^*+v^*\right)-\left|u^*\right|^\rho\left(u_t^*+u^*\right),w_t\right)\\&\leq C_5\left\|w_t\right\|^2\int_\Omega\left(\left|v^*\right|^\rho+\left|u^*\right|^\rho\right)\mathrm{d}x+C_6\|w\|\left\|w_t\right\|\int_\Omega\left(\left|v^*\right|^\rho+\left|u^*\right|^\rho\right)\mathrm{d}x\\&\leq C_5\left(\left\|\left|v^*\right|^\rho\right\|_{L^\infty(\Omega)}+\left\|\left|u^*\right|^\rho\right\|_{L^\infty(\Omega)}\right)\left\|w_t\right\|^2\\&\quad+\frac{C_6\left(\left\|\left|v^*\right|^\rho\right\|_{L^\infty(\Omega)}+\left\|\left|u^*\right|^\rho\right\|_{L^\infty(\Omega)}\right)}{2}\left\|w_t\right\|^2+\frac{C_6\left(\left\|\left|v^*\right|^\rho\right\|_{L^\infty(\Omega)}+\left\|\left|u^*\right|^\rho\right\|_{L^\infty(\Omega)}\right)}{2}\|w\|^2\end{aligned}$$

by the interpolation inequality

$$\left\|v^*\right\|_\infty^\rho\leq C_7\left\|\nabla^{2m}v^*\right\|^{\frac{\rho n}{4m}},$$

in the same way with

$$\left\|u^*\right\|_\infty^\rho \le C_8 \left\|\nabla^{2m} u^*\right\|^{\frac{\rho n}{4m}},$$

where $\rho \le \frac{8m}{n}$.

By *Poincaré*'s inequality

$$\begin{aligned}
&\left(\left|v^*\right|^\rho \left(v_t^* + v^*\right) - \left|u^*\right|^\rho \left(u_t^* + u^*\right), w_t\right) \\
&\le C_9 \left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)\left\|w_t\right\|^2 + C_{10}\left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)\left\|w\right\|^2 \qquad (40) \\
&\le C_9 \left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)\left\|w_t\right\|^2 + \frac{C_{10}}{\lambda_1^{2m}}\left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)\left\|\nabla^{2m} w\right\|^2.
\end{aligned}$$

By (2.37)-(2.40)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|w_t\right\|^2 + \mu\left\|\nabla^{2m} w\right\|^2\right) &\le 2C_9 \left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)\left\|w_t\right\|^2 \\
&\quad + \left(\frac{C_4^2}{\beta} + \frac{2C_{10}\left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)}{\lambda_1^{2m}}\right)\left\|\nabla^{2m} w\right\|^2. \qquad (41)
\end{aligned}$$

Take the

$$\alpha_3 = 2C_9 \left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right) + \frac{\frac{C_4^2}{\beta} + \frac{2C_{10}\left(\left\|\nabla^{2m} v_t^*\right\|^{\frac{\rho n}{4m}} + \left\|\nabla^{2m} u_t^*\right\|^{\frac{\rho n}{4m}}\right)}{\lambda_1^{2m}}}{\mu},$$

there are

$$\frac{\mathrm{d}}{\mathrm{d}t} y_3(t) \le \alpha_3(t) y_3(t). \qquad (42)$$

where $y_3(t) = \left\|w_t\right\|^2 + \mu\left\|\nabla^{2m} w\right\|^2$.

By *Gronwall*'s inequality

$$y_3(t) \le y_3(0) \mathrm{e}^{\int_0^T \alpha_3(t)\mathrm{d}t} = 0. \qquad (43)$$

Thus $y_3(t) = 0$, or $u^* = v^*$, therefore, the uniqueness is proved.

**Theorem 2.2** [11] *Let E be a Banach space and* $S(t): E \to E$ *semigroups satisfy the following conditions.*

1) semigroup $S(t)$ is uniformly bounded in *E*, and $\forall R > 0$, there is a constant $C(R)$, so that when $\|u\|_E \le R$, there is

$$\left\|S(t)u\right\|_E \le C(R).\left(\forall t \in [0,\infty)\right).$$

2) There is a bounded absorption set $B_0$ in *E*.

3) $S(t)(t > 0)$ is a fully continuous operator

Then a semigroup $S(t)$ is said to have a compact global attractor $A_0$.

Theorem 2.2 in Banach space *E* change to the Hilbert space $E_k$, has the following the existence theorem of the family of global attractor.

**Theorem 2.3** *If the global smooth solution of the problem* (1.1)-(1.3) *satisfies the assumptions and conditions of lemma* 2.1 *and lemma* 2.2, *then the problem* (1.1)-(1.3) *have a family of global attractor* $A_k,(k=1,2,\cdots,2m)$. *That is, there is a compact set* $A_k$, *which makes*:

1) $S(t)A_k = A_k,(t>0)$.

2) $\lim_{t\to\infty} dist(S(t)B_{0k},A_k)=0,(\forall B_{0k}\subset E_k)$.

where $dist(S(t)B_{0k},A_k)=\sup_{x\in B_{0k}}\inf_{y\in A_k}\|S(t)x-y\|_{E_k}$, $S(t)$ is the solution semigroup of (1.1)-(1.3).

Proof. It is necessary to verify the hypothesis (1), (2), (3) of theorem 2.2. It is easy to know that the Equation (1.1) has a solution semigroup $S(t):E_k\to E_k$ under the hypothesis of theorem 2.3.

1) by lemma 2.1, lemma 2.2, bounded set for $\forall B_{0k}\subset E_k$ and contained in $\left\{\|(u,v)\|_{E_k}\le R_k\right\}$.

$$\|S(t)(u_0,v_0)\|_{E_k}^2=\|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2\le\|u_0\|_{H_0^{2m+k}(\Omega)}^2+\|v_0\|_{H_0^k(\Omega)}^2\le R_k^2.$$

where $t\ge0,(u_0,v_0)\in B_{0k}$, this suggests that the $\{S(t)\}(t\ge0)$ in $E_k$ uniformly bounded.

2) by lemma 2.1, lemma 2.2, there are further

$$\|S(t)(u_0,v_0)\|_{E_k}^2=\|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2\le R_k^2,t\ge t_0=t_0(R_k).$$

So $B_{0k}=\left\{(u,v)\in E_k:\|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2\le R_k^2\right\}$ is the bounded absorbing set of semigroup $S(t)$.

3) since $E_k\to E_0$ is embedded, then $E_k$ bounded set of compact set of $E_0$, so the family of semigroup operators $S(t)$ is continuous, so the equation exists a family of global attractor $A_k=w(B_{0k})=\bigcap_{\tau\ge0}\overline{\bigcup_{t\ge\tau}S(t)B_{0k}}$.

## 3. Estimation of the Dimension of the Family of Global Attractor

Firstly, we linearize the equation into a first-order variational equation and prove that the solution semigroup $S(t)$ is *Fréchet* differentiable on $E_k$, and further prove the attenuation of the volume element of the linearization problem. Finally, the upper bound of *Hausdorff* dimension and *Fractal* dimension of $A_k$ is estimated.

The Equations (1.1)-(1.3) is linearized

$$U_{tt}+M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}U+M'\left(\|\nabla^m u\|_p^p\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u$$
$$+\beta(-\Delta)^{2m}U_t+\rho|u|^{\rho-1}u_t\cdot U+|u|^\rho U_t+(\rho+1)|u|^\rho U=0, \quad (1)$$

$$U(x,t)=0,x\in\partial\Omega,t>0, \quad (2)$$

$$U(x,0)=\xi, U_t(x,0)=\eta. \tag{3}$$

where $(\xi,\eta)\in E_k,(u,u_t)=S(t)(u_0,u_1)$ is the solution of the problem (1.1)-(1.3) with $(u_0,u_1)\in A_k$. Given $(u_0,u_1)\in A_k, S(t):E_k\to E_k$, it can be proved that for any $(\xi,\eta)\in E_k$, there is a unique solution $(U(t),U_t(t))\in L^\infty(0,+\infty;E_k)$ to the linearized initial-boundary value problem.

**Lemma 3.1** *if* $S(t):E_k\to E_k$, *Fréchet differential on* $\eta_0=(u_0,u_1)$ *is a linear operator* $F:(\xi,\eta)\to(U(t),U_t(t))$, *let* $t>0,R>0$, *and the mapping* $S(t):E_k\to E_k$ *is Fréchet differentiable on* $E_k$, *where* $(U(t),U_t(t))$ *is the solution of linearized initial-boundary value problem.*

Proof.set $\eta_0=(u_0,u_1)\in E_k$, $\bar{\eta}_0=(u_0+\xi,u_1+\eta)\in E_k$, and $\|\eta_0\|_{E_k}\le R$, $\|\bar{\eta}_0\|_{E_k}\le R$, make $\eta_1=S(t)\eta_0=(u,v)$, $\bar{\eta}_1=S(t)\bar{\eta}_0=(\bar{u},\bar{v})$, In which the semigroup $S(t)$ is *Lipschitz* continuous on the bounded set of $E_k$, that is, $\|S(t)\eta_0-S(t)\bar{\eta}_0\|_{E_k}^2\le e^{C_{25}t}\|(\xi,\eta)\|_{E_k}^2$, Make $\theta=U_t+\varepsilon U$. so you can get it.

$$\begin{aligned}&\theta_t-\varepsilon\theta+\varepsilon^2U+M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}U+M'\left(\|\nabla^m u\|_p^p\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u\\&+\beta(-\Delta)^{2m}(\theta-\varepsilon U)+\rho|u|^{\rho-1}u_t\cdot U+|u|^\rho(\theta-\varepsilon U)+(\rho+1)|u|^\rho U=0,\end{aligned} \tag{4}$$

$$\theta(0)=0,\theta_t(0)=0. \tag{5}$$

Make $(\phi,\varphi)=(\bar{u}-u-U,\bar{v}-v-\theta)$.

$$\begin{cases}\bar{u}_{tt}+M\left(\|\nabla^m\bar{u}\|_p^p\right)(-\Delta)^{2m}\bar{u}+\beta(-\Delta)^{2m}\bar{u}_t+|\bar{u}|^\rho(\bar{u}_t+\bar{u})=f(x),\\ u_{tt}+M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+|u|^\rho(u_t+u)=f(x),\\ U_{tt}+M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}U+M'\left(\|\nabla^m u\|_p^p\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u\\ +\beta(-\Delta)^{2m}U_t+\rho|u|^{\rho-1}u_t\cdot U+|u|^\rho U_t+(\rho+1)|u|^\rho U=0.\end{cases} \tag{6}$$

Subtract these three equations to get:

$$\begin{aligned}&\phi_{tt}+M\left(\|\nabla^m\bar{u}\|_p^p\right)(-\Delta)^{2m}\bar{u}-M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u-M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}U\\&-M'\left(\|\nabla^m u\|_p^p\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u+\beta(-\Delta)^{2m}\phi_t\\&+|\bar{u}|^\rho(\bar{u}_t+\bar{u})-|u|^\rho(u_t+u)-\rho|u|^{\rho-1}u_t\cdot U-|u|^\rho U_t-(\rho+1)|u|^\rho U=0.\end{aligned} \tag{7}$$

where

$$\begin{aligned}H=&M\left(\|\nabla^m\bar{u}\|_p^p\right)(-\Delta)^{2m}\bar{u}-M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}u-M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m}U\\&-M'\left(\|\nabla^m u\|_p^p\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u+|\bar{u}|^\rho(\bar{u}_t+\bar{u})\\&-|u|^\rho(u_t+u)-\rho|u|^{\rho-1}u_t\cdot U-|u|^\rho U_t-(\rho+1)|u|^\rho U.\end{aligned} \tag{8}$$

Make $H=h_1+h_2$.

$$\begin{aligned} h_1 &= M\left(\left\|\nabla^m \bar{u}\right\|_p^p\right)(-\Delta)^{2m}\bar{u} - M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u - M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}U \\ &\quad - M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u \\ &= M\left(\left\|\nabla^m \bar{u}\right\|_p^p\right)(-\Delta)^{2m}\bar{u} - M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}\bar{u} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}\phi \\ &\quad - M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m(\bar{u}-u-\phi)(-\Delta)^{2m}u. \end{aligned} \tag{9}$$

By the differential mean value theorem

$$\begin{aligned} h_1 &= M'\left(\left\|\nabla^m \bar{\zeta}\right\|_p^p\right)\left(\left\|\nabla^m \bar{\zeta}\right\|_p^p\right)'\nabla^m(\bar{u}-u)(-\Delta)^{2m}\bar{u} \\ &\quad - M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m(\bar{u}-u)(-\Delta)^{2m}u \\ &\quad + M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m\phi(-\Delta)^{2m}u \\ &\quad + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}\phi \\ &= M''\left(\left\|\nabla^m \varsigma\right\|_p^p\right)\left(\left\|\nabla^m \varsigma\right\|_p^p\right)'(1-s)\left(\nabla^m(\bar{u}-u)\right)^2(-\Delta)^{2m}\bar{u} \\ &\quad + M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m(\bar{u}-u)(-\Delta)^{2m}(\bar{u}-u) \\ &\quad + M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m\phi(-\Delta)^{2m}u + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}\phi. \end{aligned} \tag{10}$$

where $\bar{\zeta} = s\nabla^m\bar{u} + (1-s)\nabla^m u, s\in(0,1)$, $\varsigma = s\nabla^m\bar{\zeta} + (1-s)\nabla^m u, s\in(0,1)$.

Take inner product of $h_1$ and $(-\Delta)^k\phi_t$, there is

$$\begin{aligned} &\left|\left(M''\left(\left\|\nabla^m \varsigma\right\|_p^p\right)\left(\left\|\nabla^m \varsigma\right\|_p^p\right)'(1-s)\left(\nabla^m(\bar{u}-u)\right)^2(-\Delta)^{2m}\bar{u}, (-\Delta)^k\phi_t\right)\right| \\ &\le C_{11}\left|\int_\Omega\left(\nabla^m(\bar{u}-u)\right)^2\nabla^{2m+k}\bar{u}\cdot\nabla^{2m+k}\phi_t \mathrm{d}t\right| \\ &\le \frac{C_{12}^2\lambda_1^{-2m-2k}}{2\mu_0}\left\|\nabla^{2m+k}(\bar{u}-u)\right\|^4 + \frac{\mu_0}{2}\left\|\nabla^{2m+k}\phi_t\right\|^2. \end{aligned} \tag{11}$$

where $C_{11} = \left\|M''\left(\left\|\nabla^m \varsigma\right\|_p^p\right)\left(\left\|\nabla^m \varsigma\right\|_p^p\right)'(1-s)\right\|_\infty$, $C_{12} = C_{11}\left\|\nabla^{2m+k}\bar{u}\right\|$.

$$\begin{aligned} &\left|\left(M'\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m(\bar{u}-u)(-\Delta)^{2m}(\bar{u}-u), (-\Delta)^k\phi_t\right)\right| \\ &\le C_{13}\left|\int_\Omega\left(\nabla^m(\bar{u}-u)\right)\cdot\nabla^{2m+k}(\bar{u}-u)\cdot\nabla^{2m+k}\phi_t \mathrm{d}x\right| \\ &\le C_{13}\lambda_1^{-\frac{m+k}{2}}\left\|\nabla^{2m+k}(\bar{u}-u)\right\|^2\left\|\nabla^{2m+k}\phi_t\right\| \\ &\le \frac{\mu_0}{2}\left\|\nabla^{2m+k}\phi_t\right\|^2 + \frac{C_{13}^2\lambda_1^{-m-k}}{2\mu_0}\left\|\nabla^{2m+k}(\bar{u}-u)\right\|^4. \end{aligned} \tag{12}$$

$$\left|\left(M''\left(\left\|\nabla^m u\right\|_p^p\right)\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m\phi\cdot(-\Delta)^{2m}u,(-\Delta)^k\phi_t\right)\right|$$
$$\le C_{14}\left|\int_\Omega \nabla^m\phi\cdot\nabla^{2m+k}u\cdot\nabla^{2m+k}\phi_t\mathrm{d}x\right|$$
$$\le C_{14}\left\|\nabla^m\phi\right\|\left\|\nabla^{2m+k}\phi_t\right\| \tag{13}$$
$$\le\frac{\mu_0}{2}\left\|\nabla^{2m+k}\phi_t\right\|^2+\frac{C_{15}^2\lambda_1^{-m-k}}{\mu_0}\left\|\nabla^{2m+k}\phi\right\|^2.$$

$$\left|\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}\phi,(-\Delta)^k\phi_t\right)\right|\ge\frac{\mu_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}\phi\right\|^2. \tag{14}$$

The $h_2=|\bar{u}|^\rho(\bar{u}_t+\bar{u})-|u|^\rho(u_t+u)-\rho|u|^{\rho-1}u_t\cdot U-|u|^\rho U_t-(\rho+1)|u|^\rho U$.

Take inner product of $h_2$ and $(-\Delta)^k\phi_t$.

$$\left|\left(|\bar{u}|^\rho(\bar{u}_t+\bar{u})-|u|^\rho(u_t+u)-\rho|u|^{\rho-1}u_t\cdot U-|u|^\rho U_t-(\rho+1)|u|^\rho U,(-\Delta)^k\phi_t\right)\right|$$
$$\le C_{16}\left|\int_\Omega\phi_t\cdot(-\Delta)^k\phi_t\mathrm{d}x\right|\le C_{16}\lambda_1^{-k}\left\|\nabla^{2k}\phi_t\right\|^2. \tag{15}$$

where

$$|h_2|=\left|\left(|\bar{u}|^\rho\bar{u}_t-|u|^\rho u_t-|u|^\rho U_t\right)+\left(|\bar{u}|^\rho\bar{u}-|u|^\rho u-\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)U\right)\right|$$
$$\le C_{17}\left(|\bar{u}|^\rho+|u|^\rho\right)w_t-|u|^\rho U_t+C_{18}\left(|\bar{u}|^\rho+|u|^\rho\right)w_t-\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)U$$
$$\le C_{19}\left(C_{17}\left(|\bar{u}|^\rho+|u|^\rho\right)+|u|^\rho\right)\phi_t$$
$$+C_{20}\left(C_{18}\left(|\bar{u}|^\rho+|u|^\rho\right)+\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)\right)\phi_t$$
$$\le C_{21}\phi_t$$

Combined with (3.11)-(3.15), there are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^{2k}\phi_t\right\|^2+\mu_0\left\|\nabla^{2m+k}\phi\right\|^2\right)$$
$$\le\frac{C_{12}^2\lambda_1^{-2m-2k}+C_{13}^2\lambda_1^{-m-k}}{\mu_0}\left\|\bar{u}-u\right\|_{E_k}^4+C_{22}\left(\left\|\nabla^{2k}\phi_t\right\|^2+\mu_1\left\|\nabla^{2m+k}\phi\right\|^2\right). \tag{16}$$

Through the *Gronwall's* inequality, there is

$$\left\|\nabla^{2k}\phi_t\right\|^2+\mu_0\left\|\nabla^{2m+k}\phi\right\|^2\le C_{22}\mathrm{e}^{C_{23}t}\left\|(\xi,\eta)'\right\|_{E_k}^4. \tag{17}$$

When $\left\|(\xi,\eta)'\right\|_{E_k}^4\to 0$

$$\frac{\left\|s(t)\bar{\eta}-s(t)\eta-Fs(t)(\xi,\eta)\right\|}{\left\|(\xi,\eta)\right\|_{E_k}^2}\le C_{24}\mathrm{e}^{C_{25}t}\left\|(\xi,\eta)\right\|_{E_k}^2\to 0. \tag{18}$$

So the lemma 3.1 is proved.

**Lemma 3.2** *Under the assumption and condition of lemma* 3.1, *the family of global attractor* $A_k$ *of initial-boundary value problem* (1.1)-(1.3) *has Hausdoff* *dimension and Fractal dimension, and* $d_H(A_k)\le\frac{1}{7}n$, $d_F(A_k)\le\frac{8}{7}n$.

Proof. Make $\phi=R_\varepsilon\varphi=(u,v)'$, $\varphi=(u,u_t)'$, $v=u_t+\varepsilon u$, $R_\varepsilon:\{u,u_t\}\to\{u,u_t+\varepsilon u\}$ is an isomorphic mapping, let $A_i(i=1,2,\cdots,2m)$ be the global attractor of $\{S(t)\}$, and $A_{\varepsilon i}$ is the global attractor of $\{S_\varepsilon(t)\}$, and they have the same dimension, from lemma 3.1, we can know that $S(t):E_k\to E_k$ is *Fréchet* differentiable. The linearized first-order variational Equation (3.1) can be rewritten as

$$P_t+\Lambda(\Psi)P=0. \tag{19}$$

$$P_t=F_t(\Psi). \tag{20}$$

where

$$\Lambda(\Psi)=\begin{pmatrix}\varepsilon I & -I\\ M\left(\left\|A^{\frac{m}{2}}u\right\|_p^p\right)A^{2m}+\varepsilon^2I-\varepsilon\beta A^{2m}-\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)\varepsilon+\varepsilon^2|u|^\rho+\Phi & \beta A^{2m}-\varepsilon I+|u|^\rho\varepsilon\end{pmatrix},$$

$I$ is an identity operator. $-\Delta=A$, $P=(U,\theta)'\in E_k$. Make $\Psi=R_\varepsilon\varphi=(U,V)$, $\varphi=(U,U_t)$, $\theta=U_t+\varepsilon U$ $R_\varepsilon:\{u,u_t\}\to\{u,u_t+\varepsilon u\}$ is an isomorphic mapping. For a fixed $(u_0,v_0)\in E_k$, let $r_1,r_2,\cdots,r_n$ be $n$ element of $E_k$, let $U_1(t),U_2(t),\cdots,U_n(t)$ be $n$ solutions of the linear Equation (3.19), its initial value is $U_1(0)=r_1,U_2(0)=r_2,\cdots,U_n(0)=r_n$.

so

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots\Lambda U_n(t)\right\|^2_{\Lambda E_k}\\&-2trF_t\left(\Psi(\tau)\cdot Q_n(\tau)\right)\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots\Lambda U_n(t)\right\|^2_{\Lambda E_k}=0.\end{aligned}\tag{21}$$

Further, by the same *Gronwall*'s inequality, available:

$$\begin{aligned}&\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots\Lambda U_n(t)\right\|^2_{\Lambda E_k}\\&=\left\|r_1\Lambda r_2\Lambda\cdots\Lambda r_n\right\|_{\Lambda E_k}\exp\left(\int_0^t trF_t\left(\Psi(\tau)\cdot Q_n(\tau)\right)\mathrm{d}\tau\right).\end{aligned}\tag{22}$$

where $\Lambda$ stands for outer product and $tr$ stands for trace. $Q_n(\tau)$ is an orthogonal projection from space $E_k$ to $span\{U_1(t),U_2(t),\cdots,U_n(t)\}$.

Given a certain moment $\tau$,

set $w_j(\tau)=(\xi_j(\tau),\eta_j(\tau))'$, $j=1,2,\cdots,n$ is $span\{U_1(t),U_2(t),\cdots U_n(t)\}$ orthonormal basis.

We define the inner product in $E_k$ is

$$\left((\xi,\eta),(\bar\xi,\bar\eta)\right)=\left(\left(\nabla^{2m+k}\xi,\nabla^{2m+k}\bar\xi\right)+\left(\nabla^k\eta,\nabla^k\bar\eta\right)\right). \tag{23}$$

To sum up, it is available

$$\begin{aligned}trF'\left(\Psi(\tau)\cdot Q_n(\tau)\right)&=\sum_{j=1}^n\left(F'\left(\Psi(\tau)\right)\cdot Q_n(\tau)w_j(\tau),w_j(\tau)\right)_{E_k}\\&=\sum_{j=1}^n\left(F'\left(\Psi(\tau)\right)w_j(\tau)\cdot w_j(\tau)\right)_{E_k}.\end{aligned}\tag{24}$$

where $\left(F'\left(\Psi(\tau)\right)w_j(\tau)\cdot w_j(\tau)\right)_{E_k}=-\left(\Lambda(\Psi)w_j,w_j\right)$.

$$=-\varepsilon\left\|\nabla^{2m+k}\xi_j\right\|^2+\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\eta_j\right)-M\left(\left\|A^{\frac{m}{2}}u\right\|_p^p\right)\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\eta_j\right)$$
$$-\varepsilon^2\left(\nabla^k\xi_j,\nabla^k\eta_j\right)+\beta\varepsilon\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\eta_j\right)$$
$$+\varepsilon\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)|u_t|^{\rho-1}\left(\nabla^k\xi_j,\nabla^k\eta_j\right)-\Phi\left(\nabla^k\xi_j,\nabla^k\eta_j\right)$$
$$-\beta\left\|\nabla^{2m+k}\eta_j\right\|^2+\varepsilon\left\|\nabla^k\eta_j\right\|^2-\varepsilon^2|u|^\rho\left(\nabla^k\xi_j,\nabla^k\eta_j\right)-\varepsilon|u|^\rho\left\|\nabla^k\eta_j\right\|^2$$
$$\le-\varepsilon\left\|\nabla^{2m+k}\xi_j\right\|^2+\frac{\mu_0-\beta\varepsilon-1}{2}\left\|\nabla^{2m+k}\xi_j\right\|^2+\frac{\mu_0-\beta\varepsilon-1}{2}\left\|\nabla^{2m+k}\eta_j\right\|^2+\frac{\varepsilon}{2}\left\|\nabla^k\xi_j\right\|^2$$
$$+\frac{\Phi+\varepsilon^2|u|^\rho-\varepsilon\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)}{2\lambda_1^{2m}}\left\|\nabla^{2m+k}\xi_j\right\|^2$$
$$+\frac{\Phi+\varepsilon^2|u|^\rho-\varepsilon\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)+3\varepsilon+2\varepsilon|u|^\rho-2\beta\lambda_1^{2m}}{2}\left\|\nabla^k\eta_j\right\|^2 \tag{25}$$
$$\le-C_{26}\left(\left\|\nabla^{2m+k}\xi_j\right\|^2+\left\|\nabla^k\eta_j\right\|^2\right)+\frac{\varepsilon}{2}\left\|\nabla^k\xi_j\right\|^2.$$

where

$$C_{26}=\min\left\{-\left(\frac{\mu_0-\beta\varepsilon-1}{2}-\varepsilon+\frac{\Phi+\varepsilon^2|u|^\rho-\varepsilon\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)}{2\lambda_1^{2m}}\right),\right.$$
$$\left.-\left(\frac{\mu_0-\beta\varepsilon-1}{2}\lambda_1^{2m}+\frac{\Phi+\varepsilon^2|u|^\rho-\varepsilon\left(\rho|u|^{\rho-1}u_t+(\rho+1)|u|^\rho\right)+3\varepsilon+2\varepsilon|u|^\rho-2\beta\lambda_1^{2m}}{2}\right)\right\}.$$

Above all there is

$$\left(F_t\left(\Psi(\tau)\right)w_j(\tau),w_j(\tau)\right)\le-C_{26}\left(\left\|\nabla^{2m+k}\xi_j\right\|^2+\left\|\nabla^k\eta_j\right\|^2\right)+\frac{\varepsilon}{2}\left\|\nabla^k\xi_j\right\|^2. \tag{26}$$

Because $w_j(\tau)=\left(\xi_j(\tau),\eta_j(\tau)\right)^{\mathrm{T}}, j=1,2,\cdots,n$ is $span\left\{U_1(t),U_2(t),\cdots,U_n(t)\right\}$ orthonormal basis. Therefore

$$\left\|\nabla^{2m+k}\xi_j\right\|^2+\left\|\nabla^k\eta_j\right\|^2=1. \tag{27}$$

$$\sum_{j=1}^n\left(F'\left(\Psi(\tau)\right)w_j(\tau),w_j(\tau)\right)_{E_k}\le-nC_{26}+\frac{\varepsilon}{2}\sum_{j=1}^n\left\|\nabla^k\xi_j\right\|^2. \tag{28}$$

Almost all the *t*.

$$\sum_{j=1}^n\left\|\nabla^k\xi_j\right\|^2\le\sum_{j=1}^n\lambda_j^{s-1}. \tag{29}$$

The $s-1=-2m$ and $s\in[0,1]$, $\lambda_j$ is $A^{2m}$ characteristic value, and $\lambda_1<\lambda_2<\cdots<\lambda_n$, so there is

$$TrF'\left(\Psi(\tau)\cdot Q_n(\tau)\right)\le-nC_{26}+\varepsilon\sum_{j=1}^n\lambda_j^{s-1}. \tag{30}$$

Set

$$q_n(t) = \sup_{\Psi_0 \in B_{0k}} \sup_{\eta_j \in E_k} \left( \frac{1}{t} \int_0^t TrF'\left(s(t)\Psi_0\right) Q_n(\tau) \mathrm{d}\tau \right). \quad (31)$$

and

$$q_n = \lim_{t \to \infty} q_n(t). \quad (32)$$

Therefore,

$$q_n \le -nC_{26} + \varepsilon \sum_{j=1}^{n} \lambda_j^{s-1}. \quad (33)$$

Therefore, $B_{0k}$ *Lyapunov* index $\kappa_1, \kappa_2, \cdots, \kappa_n\ (n > 1)$ is uniformly bounded, and

$$\kappa_1 + \kappa_2 + \cdots + \kappa_n \le -nC_{26} + \varepsilon \sum_{j=1}^{n} \lambda_j^{s-1}. \quad (34)$$

make

$$(q_j)_+ \le -nC_{26} + \varepsilon \sum_{j=1}^{n} \lambda_j^{s-1} \le \varepsilon \sum_{j=1}^{n} \lambda_j^{s-1} \le \frac{nC_{26}}{8}. \quad (35)$$

$$q_n \le -nC_{26} \left( 1 - \frac{\varepsilon}{nC_{26}} \sum_{j=1}^{n} \lambda_j^{s-1} \right) \le -\frac{7}{8} nC_{26}. \quad (36)$$

Further,

$$\max_{1 \le j \le n} \frac{(q_j)_+}{|q_n|} \le \frac{1}{7}. \quad (37)$$

From this we can get $d_H(A_k) \le \frac{1}{7} n, d_F(A_k) \le \frac{8}{7} n$, Then the *Hausdorff* dimension and *Fractal* dimension of the family of global attractor $A_k$ are finite.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

## References


[1] Cavalcanti, M.M., Cavalcanti, V.N.D., Filho, J.S.P. and Soriano, J.A. (1998) Existence and Exponential Decay for a Kirchhoff-Carrier Model with Viscosity. *Journal of Mathematical Analysis and Applications*, **226**, 40-60. https://doi.org/10.1006/jmaa.1998.6057

[2] Ono, K. (1997) Global Existence, Decay, and Blow up of Solutions for Some Mildly Degenerate Nonlinear Kirchhoff Strings. *Journal of Differential Equations*, **137**, 273-301. https://doi.org/10.1006/jdeq.1997.3263

[3] Ono, K. (1997) On Global Existence, Asymptotic Stability and Blowing up of Solutions for Some Degenerate Non-Linear Wave Equtions of Kirchhoff Type with a Strong Dissipation. *Mathematical Methods in the Applied Sciences*, **20**, 151-177. https://doi.org/10.1002/(SICI)1099-1476(19970125)20:2<151::AID-MMA851>3.0.CO;2-0

[4] Yang, Z.J. (2007) Longtime Behavior of the Kirchhoff Type Equation with Strong

Damping on $R^N$. *Journal of Differential Equations*, **242**, 269-286.

[5] Yang, Z.J., Ding, P.Y. and Liu, Z.M. (2014) Global Attractor for the Kirchhoff Type Equations with Strong Nonlinear Damping and Supercritical Nonlinearity. *Applied Mathematics Letters*, **33**, 12-17. https://doi.org/10.1016/j.aml.2014.02.014

[6] Yang, Z.J., Wang, Y.Q. (2010) Global Attractor for the Kirchhoff Type Equation with a Strong Dissipation. *Journal of Differential Equations*, **249**, 3258-3278. https://doi.org/10.1016/j.jde.2010.09.024

[7] Yang, Z. and Li, X. (2011) Finite Dimensional Attractors for the Kirchhoff Equation with a Strong Dissipation. *Journal of Mathematical Analysis and Applications*, **375**, 579-593. https://doi.org/10.1016/j.jmaa.2010.09.051

[8] Lin, G.G. (1973) Several Types of the Kirchhoff Equation of Dynamics Characteristic. *Theoretical Population Biology*, **4**, 331-356.

[9] Tokio, M. and Ryo, I. (1996) On Global Solutions and Energy Decay for the Wave Equations of Kirchhoff Type with Nonlinear Damping Terms. *Journal of Mathematical Analysis and Applications*, **204**, 729-753. https://doi.org/10.1006/jmaa.1996.0464

[10] Li, F.C. (2004) Global Existence and Blow-up of Solutions for a Higher-Order Kirchhoff-Type Equations with Nonlinear Dissipation. *Applied Mathematics letters*, **17**, 1409-1414. https://doi.org/10.1016/j.am1.2003.07.014

[11] Lin, G.G. (2011) Nonlinear Evolution Equations. Kunming: Yunnan University Press.

[12] Gao, Y.L., Sun, Y.T. and Lin, G.G. (2016) The Global Attractors and Their Hausdorff and Fractal Dimensions Estimation for the Higer-Order Nonlinear Kirchhoff-Type Equation with Strong Lineae Damping. *International Journal of Modern Nonlinear Theory and Application*, **5**, 185-202. https://doi.org/10.4236/ijmnta.2016.54018

[13] Chen, L., Wang, W. and Lin, G.G. (2016) The Global Attractors and the Hausdorff and Fractal Demensions Estimation for the Higher-Order Nonlinear Kirchhoff-Type Equation. *Journal of Advances in Mathematics*, **12**, 6608-6621. https://doi.org/10.24297/jam.v12i9.133

[14] Ma, Q.Z., Sun, C.Y. and Zhong, C.K. (2007) The Existence of Strong Global Attractors for Nonlinear Beam Equations. *Journal of Mathematical Physics*, **27A**, 941-948.