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The Significance of the Dimension of the Quantum Function ψ

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Abstract

In the system of units where $\hbar = c = 1$, the dimension of a physical object can be written in the form $[L^n]$, where L denotes length. The innovative features of this work depend on the analysis of the dimension of the quantum function ψ. This analysis yields new arguments concerning the coherence of quantum theories. The dimension of the Dirac and the Schroedinger functions ψ is $[L^{-3/2}]$. This fractional dimension enables the construction of crucial theoretical expressions for the Hilbert space and the expectation value of physical operators. On the other hand, the analysis proves that problems exist with quantum fields of elementary massive particles whose function ϕ has the $[L^{-1}]$ dimension, such as the Klein-Gordon theory, the electroweak theory of the $W^{\pm}$, Z particles, and the Higgs boson theory.

Subject Areas

Quantum Physics

Keywords

Quantum Theories, Dimension of the Quantum Function, Coherence of Quantum Field Theories, Fermions, and Massive Bosons

1. Introduction

The dimension of a physical quantity is an important theoretical property. For example, all terms of a physically coherent expression must have the same dimension. Therefore, expressions that determine a given physical quantity should comprise only terms that have a specific dimension. This work analyzes the dimensional constraints that are imposed on the quantum function ψ. The standard literature apparently neglects this topic. Thus, the consequences of the new results that are derived below, have not yet been adequately discussed. This work shows that in the unit system where $\hbar = c = 1$, the dimension of the Schroedinger and the Dirac functions ψ is [$L^{-3/2}$], where L denotes the unit of length. In contrast, the quantum functions of the Klein-Gordon (KG) field (see e.g., [1], pp. 16, 17), the electroweak fields of the $W^{\pm}$ and the Z particles (see e.g., [1], pp. 700-714), and the field of the Higgs particle (see e.g., [1], p. 715) have the [L^{-1}] dimension. The analysis proves that the [$L^{-3/2}$] dimension of the Dirac and the Schroedinger quantum functions yields coherent theoretical expressions that are needed for a comprehensive quantum theory, whereas problems persist with the theories of a massive elementary particle whose quantum function has the [L^{-1}] dimension.

Formulas and symbols take the standard form. Relativistic expressions use a diagonal metric and its entries are (1, −1, −1, −1). The second section analyses the Schroedinger quantum function ψ. The third section analyses the functions of quantum field theories (QFT). The fourth section discusses the results. The fifth section indicates that the progress of time supports the outcome of this work, and the last section summarizes it.

2. The Function ψ in Quantum Mechanics

Let us examine the Theoretical structure of the non-relativistic quantum mechanics (QM) that was constructed about 100 years ago. A textbook explains the meaning of the quantum function $\psi(x,y,z,t)$, and says that it is a "wave function as 'a measure of probability' of finding the particle at time t at the position $\boldsymbol{r}$" (see [2], p. 36). Obviously, the mathematically complex QM function $\psi(x,y,z,t)$ itself cannot do that, because probability is a mathematically real number. (For the short notation, $\psi(x)$ denotes $\psi(x,y,z,t)$.) Therefore, the product $\psi^*\psi$ is used for this purpose. Thus, QM says that for a very small spatial domain D that includes the point $\boldsymbol{r}$, the probability of finding the particle at the time t and inside D is

$$P = V(D)\psi^*\psi, \tag{1}$$

where $V(D)$ denotes the volume of D.

Conclusion: The probability is a pure number and the dimension of a spatial volume is [L^3]. Hence the definition (1) proves that the dimension of the quantum function ψ is [$L^{-3/2}$].

The following points explain why the dimension of the quantum function ψ is an important property of the theory, although its fractional value looks strange.

Q.1 The dimension of a classical quantity is an integral power of the primary units. Hence, the plain meaning of the fractional power of the dimension of ψ is that its properties may depart from the classical concepts. This attribute enables peculiar quantum features like entanglement to live in peace with logical quantum concepts.

Q.2 For a given quantum state ψ, the expectation value of a variable is required for testing the compatibility of the quantum theory with experimental data. The QM expectation value of a given quantity takes the form

$$\langle O\rangle = \int \psi^* \hat{O}\psi \mathrm{d}^3 x, \tag{2}$$

where $\hat{O}$ is the appropriate operator. The $[L^{-3/2}]$ dimension of ψ indicates that (2) is consistent with the balance of dimension, which is a required property of every coherent physical expression.

Q.3. A quantum theory needs an appropriate Hilbert space (see e.g., [3], pp. 49, 50; [4], pp. 164-166). This space comprises vectors where there is a product for every pair of these vectors that takes the form of a dimensionless complex number. The expression for the product of two QM functions, ψ_i and ψ_j, that belong to an orthonormal basis of a Hilbert space is

$$\left(\psi_j^*, \psi_i\right) = \int \psi_j^* \psi_i \mathrm{d}^3 x = \delta_j^i, \tag{3}$$

where δ_j^i is the Kronecker δ function. This expression shows that the $[L^{-3/2}]$ dimension of the QM functions ψ_i and ψ_j enables their utilization as members of a dimensionally coherent expression for an orthonormal basis of a Hilbert space. (Note that the Gram-Schmit process can be used for the construction of the required orthonormal basis.)

Q.4 Consider a charged quantum particle. The 4-current j^μ of this particle is involved in the inhomogeneous Maxwell equations

$$F^{\mu\nu}{}_{,\nu} = -4\pi j^\mu \tag{4}$$

(see [5], p. 79). The 0-component of the 4-current is the charge density (see [5], p. 75). Therefore, the dimension of the 4-current is $[L^{-3}]$, which is the dimension of density. Hence, the $[L^{-3/2}]$ dimension of the Schroedinger functions yields an expression for the 4-current of the Schroedinger function. Its 4 components are

$$j^\mu_{Sch} = e\left\{\psi^*\psi, -i\left[\psi^*\nabla\psi - \left(\nabla\psi^*\right)\psi\right]/2m\right\}, \tag{5}$$

where m denotes the particle's mass (see [2], p. 37). The 4-current of the Dirac function takes a relativistic covariant form

$$j^\mu_{Dirac} = e\bar{\psi}\gamma^\mu\psi \tag{6}$$

(see [6], p. 85). The $[L^{-3/2}]$ dimension of ψ and the expressions (5) and (6) for the 4-current enables the coherent participation of a charged quantum particle in Maxwellian electrodynamics.

The foregoing points prove that the $[L^{-3/2}]$ dimension of ψ is a vital element of the quantum theory because it enables the coherent construction of crucial expressions.

3. Discussion

The mathematical structure of QM takes the form of a wave equation, which differs from the structure of classical physics. In the early days of QM, people regarded the correspondence between QM and classical physics as a criterion for the correctness of quantum expressions. Here are quotations that indicate this issue:

"Classical mechanics is contained in quantum mechanics as a limiting form ($h \to 0$)." Furthermore: "This requirement, which is a guide in discovering the correct quantum laws, is called the *correspondence principle*." ([2], p. 3).

This goal is reached by the comparison of classical quantities with the appropriate limit of the expectation value of the corresponding QM operators (2). Hence, the $[L^{-3/2}]$ dimension is a crucial element of the compatibility of a quantum theory with the correspondence principle, which is an important criterion of the correctness of quantum theories. In particular, the product $\psi^*\psi$ of (2) removes the mathematically complex form of the QM function ψ and its fractional dimension as well.

3.1. Quantum Field Theories

Quantum mechanics is a nonrelativistic theory of a single particle whose states are elements of a Hilbert space. This theory may be extended to states that comprise several particles. In the case of spin-1/2 electrons, the particles' state is antisymmetric, which is consistent with the Pauli exclusion principle ([2], p. 516). High energy processes enable the fermion pair production where a particle-antiparticle pair is produced. This effect is consistent with the fermion conservation number. Pair production is described in a QFT description of processes (see e.g., [3], p. 29).

It is now recognized that the Lagrangian density is a primary QFT expression. It takes the form

$$\mathcal{L}\left(\psi(x), \partial\psi(x)/\partial x^\mu\right) \tag{7}$$

(see e.g., [3], p. 300). The equations of motion of the fields are the Euler-Lagrange equations (see [6], p. 15)

$$\frac{\partial\mathcal{L}}{\partial\psi_r} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\left(\partial\psi_r/\partial x^\mu\right)} = 0. \tag{8}$$

These equations are derived from a variation of the action

$$S = \int \mathcal{L}\mathrm{d}^4 x \tag{9}$$

This expression proves that the dimension of the Lagrangian density is $[L^{-4}]$. Evidently, the specific form of the QFT Lagrangian density (7) determines the dimension of the quantum function ψ.

The important relationships between QFT and QM are stated in Weinberg's well-known textbook: "First, some good news: quantum field theory is based on the same quantum mechanics that was invented by Schroedinger, Heisenberg, Pauli, Born, and others in 1925-26, and has been used ever since in atomic, molecular, nuclear, and condensed matter physics" (see [3], p. 49). This statement makes sense because the transition between the validity domains of QFT and that of QM is carried out by a *continuous* decrease of the process's energy and in many cases the experimenter can control this process.

3.2. The Dirac Theory

Quantum electrodynamics (QED) is a theory that illustrates these issues. Concerning a charged spin-1/2 elementary particle, this theory abides by the variational principle that utilizes the variation of the action of an appropriate Lagrangian density. Thus, a textbook states: "All field theories used in current theories of elementary particles have Lagrangians of this form" (see [3], p. 300). The QED Lagrangian density is (see e.g., [6], p. 84; [1], p. 78)

$$\mathcal{L}_{QED} = \bar{\psi}\left[\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right) - m\right]\psi - F^{\mu\nu}F_{\mu\nu}/16\pi. \tag{10}$$

This Lagrangian density yields the Dirac equation for an elementary massive spin-1/2 charged particle and Maxwell equations for the electromagnetic fields. Furthermore, it has an amazing experimental confirmation. For example, a textbook says: "That such a simple Lagrangian can account for nearly all observed phenomena from macroscopic scales down to 10^{-13} cm is rather astonishing" ([1], p. 78). Referring to the dimension issue which is examined in this work, one notes that the QED $\bar{\psi}m\psi$ term of (10) and the $[L^{-4}]$ dimension of the Lagrangian density prove that the dimension of the QED Dirac function ψ is $[L^{-3/2}]$. This value agrees with the dimension of the Schroedinger function.

The Dirac expression for the 4-current (6) is a factor of the QED Lagrangian density (10). The derivative-free property of this 4-current is an important attribute. Thus, the Noether theorem provides an expression for a conserved 4-current of the quantum particle:

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)}\psi. \tag{11}$$

(see [7], pp. 314-315). Therefore, if the electromagnetic interaction uses a derivative-dependent 4-current of the quantum function ψ, then the interaction term modifies the 4-current upon which it depends.

Conclusion: A contradiction arises if the 4-current of a charged particle depends on a derivative of its quantum function with respect to x^{μ}.

3.3. Other Quantum Theories

The literature discusses other kinds of QFTs of massive particles, such as the KG field (see e.g., [1], pp. 16, 17), the electroweak theory of the $W^{\pm}$ and the Z particles (see e.g., [1], pp. 700-714), and the field of the Higgs particle (see e.g., [1], p. 715). These are elementary massive particles and their spin is an integral number. They belong to a category called massive bosons. An examination of the definition of these fields shows that the dimension of their quantum function ϕ is $[L^{-1}]$. This property can be deduced from the $m^2\phi^*\phi$ term of the corresponding Lagrangian density because the dimension of the Lagrangian density is $[L^{-4}]$ and the mass dimension is $[L^{-1}]$ (see e.g., [8]).

These fields do not enjoy the benefits of the $[L^{-3/2}]$ dimension of the field functions that are pointed out above. For example, the literature does not show how to construct the Hilbert space of these functions. For this reason, there are serious doubts concerning the physical coherence of these theories.

Another serious problem is the absence of a coherent expression for the 4-current of the electroweak charged particles $W^{\pm}$. In the classical theory and QED, the 4-current is a factor of the electromagnetic interaction term

$$\mathcal{L}_{int} = j^{\mu} A_{\mu}. \tag{12}$$

This term yields the equation of the charged particle *and* the Maxwell equations of the electromagnetic fields. This expression is proportional to the electromagnetic 4-potential *and* the QED 4-current (6) is free of derivatives of the quantum function of the charged particle. In contrast, a CERN paper uses an expression that takes a different form (see Equation (3) of [9]). Its description of the $W^{\pm}$ electromagnetic interaction depends on the electromagnetic fields which violates Maxwellian electrodynamics (12) and on derivatives of the quantum function of $W^{\pm}$ which violates the Noether definition (11) of the 4-current. This CERN article was written by thousands of authors, and this evidence indicates that there is still no coherent 4-current expression for the electroweak theory of the $W^{\pm}$ particles. Thus, Equation (3) of [9]) is certainly a gross theoretical error. It is interesting to mention that these results agree with Dirac's lifelong objection to second-order quantum equations (see [10], pp. 1-4).

4. Clarifying Evidence

The topic of this work is the meaning of the dimension of quantum functions. It is shown above that contemporary quantum theories can be divided into two categories: theories whose quantum function has the $[L^{-3/2}]$ dimension, and theories whose quantum function has the $[L^{-1}]$ dimension.

The primary theory of a charged particle whose quantum function has the $[L^{-3/2}]$ dimension is the Dirac electron theory that is embedded in the QED Lagrangian density (10). It is stated above (see below (10)) that it has an amazing experimental success. This is certainly an impressive positive feature of these quantum functions.

However, here the comparison between the historical progress of the theoretical aspects of these two categories of quantum functions is examined. For this end, let us restate the Dirac expression for the 4-current (6)

$$j^{\mu}_{Dirac} = e\bar{\psi}\gamma^{\mu}\psi. \tag{6'}$$

This 4-current has these important virtues:

V.1 It is a vital variable of Maxwell equation (4).

V.2 It is directly derived from the above-mentioned Noether expression for the 4-current (11) which says

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)}\psi. \tag{11'}$$

V.3 It is free of derivatives of the quantum function $\psi_{,\mu}$. This property of the 4-current is required for a coherent electromagnetic interaction term of a charged particle with electromagnetic fields

$$\mathcal{L}_{int} = j^{\mu} A_{\mu}. \tag{12'}$$

Indeed, if a 4-current of a charged particle depends on the derivative of the quantum function then the Noether theorem (11) proves that the interaction term destroys the original expression for the 4-current.

The $W^{\pm}$ are charged particles whose quantum function ϕ has the dimension $[L^{-1}]$. The comparison between the ways which the progress of time has affected the status of the Dirac electron and the $W^{\pm}$ particles is illuminating.

The coherent publication of the expression for the Dirac electron 4-current took place about one month after the publication of the Dirac electron theory [11]. In contrast, although the electroweak theory of the $W^{\pm}$ particles is more than 50 years old, it still has no valid expression for the required 4-current of these particles. Thus, several authors published in 1987 the paper [12] that uses an *effective* Lagrangian for the $W^{\pm}$ electromagnetic interaction (see Equation (2.1) therein). Several decades later, thousands of authors in the above-mention CERN publication of [9] use an analogous expression (see Equation (3) therein). The expressions used in these articles are certainly theoretically unacceptable because they comprise derivatives of the $W^{\pm}$ quantum functions. Hence, they violate requirement V.3. A second violation of Maxwellian electrodynamics is that unlike (12), *the electromagnetic part is not the 4-potential.*

Conclusion: The 4-current of the Dirac electron, whose quantum function has the $[L^{-3/2}]$ dimension, was found about one month after the publication of the Dirac theory. Consider the $W^{\pm}$ that the electroweak theory assigns it to quantum functions whose dimension is $[L^{-1}]$. The $W^{\pm}$ status is completely different from that of the Dirac electron. Although the electroweak theory is more than 50 years old, it still has no coherent expression for the $W^{\pm}$ 4-current.

These lines restate the previous conclusion:

For a quantum function having the $[L^{-3/2}]$ dimension - success is achieved after one month;

For a quantum function having the $[L^{-1}]$ dimension - a failure persists after more than 50 years.

The authors of [12] and [9] implicitly admit that this unfortunate plight holds for the electroweak $W^{\pm}$ particle, because they state that their expression does not refer to a rigorous Lagrangian but to an *effective* Lagrangian.

This section explains the physical acceptability of the Dirac electron theory, whose quantum function has the $[L^{-3/2}]$ dimension. It also indicates unsettled theoretical problems of the $W^{\pm}$ functions whose dimension is $[L^{-1}]$. This outcome is an example of the usefulness of the concept of the dimension of the quantum functions. In particular, the lack of a coherent 4-current for the electroweak $W^{\pm}$ particles means that it violates Maxwellian electrodynamics (see item V.1).

5. Conclusions

The novelty of this work is the examination of the dimension of the quantum function of several theories. The dimension of a physical quantity is an important element of its theoretical structure because it imposes constraints on its description. For a given QFT, the examination of its Lagrangian density, whose dimension is $[L^{-4}]$, yields a solid mathematical proof of the specific dimension of the theory's quantum function ψ. It turns out that the dimension of the Dirac function is $[L^{-3/2}]$. This is also the dimension of the nonrelativistic Schroedinger function. This fractional dimension enables a straightforward construction of crucial theoretical quantum expressions, such as the Hilbert space and the expectation value of fundamental operators (2).

The fractional dimension of these quantum functions may be regarded as another weird quantum property. Like other issues, this QM property and the Dirac theory should be accepted just because of their coherent mathematical structure and their experimental success. In contrast, some other quantum theories use a quantum function whose dimension is $[L^{-1}]$. The functions of the Klein-Gordon theory, the electroweak theory of the $W^{\pm}$, Z particles, and the theory of the Higgs particle have this dimension. Crucial problems stem from the $[L^{-1}]$ dimension of these quantum theories. For example, despite the very old age of these theories, the literature has not yet published the explicit form of their Hilbert space and of a Maxwellian consistent of the electromagnetic interaction of the electroweak description of the $W^{\pm}$ particles. The penultimate section explains why the $W^{\pm}$ particles have no coherent expression for the 4-current. These long-lasting discrepancies indicate that it is impossible to correct these theories. These results point out the usefulness of the dimension attribute of the quantum function.

Conflicts of Interest

The author declares no conflicts of interest.

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