Open Access Library Journal
2023, Volume 10, e10511
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Problems with the Klein-Gordon Theory

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How to cite this paper: Comay, E. (2023) Problems with the Klein-Gordon Theory. *Open Access Library Journal*, **10**: e10511.
https://doi.org/10.4236/oalib.1110511

Received: July 15, 2023
Accepted: August 12, 2023
Published: August 15, 2023

Abstract

The electromagnetic domain comprises two kinds of physical objects—electromagnetic fields and electrically charged particles. Therefore, the structure of a comprehensive electromagnetic theory is a coherent union of two theories. One theory describes electromagnetic fields, and the second theory describes electrically charged particles. An obvious requirement says that a comprehensive electromagnetic theory must be a coherent union of a theory of electromagnetic fields and a theory of electrically charged particles. The continuity equation is a well-known example showing how Maxwell equations of the electromagnetic fields impose a constraint on a theory that describes the time evolution of a charged particle. The novelty of this work is its proof that the continuity equation is not a unique example. Namely, the Maxwell theory of electromagnetic fields imposes other constraints on a theory of an electric charge. This work shows that the classical theory as well as the Dirac theory of a spin-1/2 charged quantum particle provide a coherent electromagnetic theory. In contrast, new problems arise in the Klein-Gordon theory of a charged spin-0 quantum particle.

Subject Areas

Electromagnetics

Keywords

Classical Electrodynamics, Dirac Theory, Klein-Gordon Theory, The Continuity Equation

1. Introduction

Differential equations are regarded as the cornerstone of a theory of physical objects. This is the original structure of Newtonian mechanics and Maxwellian theory of electromagnetic fields combined with the Lorentz force. Later, the merits of the least action principle have been recognized. This issue is connected to the previous approach that regards the primary role of the differential equations because the Euler-Lagrange equations of the corresponding least action principle that applies to a Lagrangian/Lagrangian density are the required differential equations. (In some texts, the term Lagrangian is used for Lagrangian density.) The present work examines theories of charged particles that are derived from the least action principle.

Not every differential equation can be regarded as an acceptable description of the time evolution of a physical system of given particles. A justification for this assertion is provided by a set of conservation laws, like the conservation of energy, momentum, angular momentum, and electric charge. These conservation laws are imposed as constraints and any specific physical theory of a given particle must abide by them. It means that any proposition of a set of differential equations that aim to describe the time evolution of a given system of particles must undertake the painstaking effort of proving that these equations are consistent with the conservation laws mentioned above. This requirement demonstrates the significance of the Noether Theorem of the least action principle that is applied to a given Lagrangian/Lagrangian density. For example, if these expressions are invariant with respect to a translation or a rotation of the space-time coordinates then the Noether theorem proves that the theory's Euler-Lagrange equations satisfy the conservation of energy-momentum and angular momentum.

The rise of special relativity (SR) imposes another constraint on differential equations, and any theory must be coherent with Lorentz transformations. It turns out that if the Lagrangian density $\mathcal{L}$ comprises terms where each of which is a Lorentz scalar then the theory abides by SR (see [1], p. 300). Another requirement says that the dimension of every term of the Lagrangian density must be $[L^{-4}]$. (Here units where $\hbar = c = 1$ are used and the dimension of any quantity is a power of the dimension of the length $[L]$). Indeed, in these units, the action S is dimensionless and its definition $S = \int \mathcal{L} \mathrm{d}^4 x$ means that the dimension of $\mathcal{L}$ is $[L^{-4}]$.

The foregoing discussion explains the merits of the least action principle and the application of a given Lagrangian density as a basis for a theory of a given particle. This point illustrates the ingenuity of gifted persons like Newton and Maxwell who, without using the least action principle, created theories that are based on differential equations that conserve energy, momentum, and angular momentum. Newtonian mechanics is connected to SR because the low-velocity limit of relativistic mechanics takes the form of Newtonian mechanics (see [2], p. 26), whereas Maxwellian electrodynamics takes a relativistic form (see [2], Chapter 4).

These arguments explain why the present theories of quantum particles are based on the least action principle. For example, Weinberg refers to elementary quantum particles and states (see [1], p. 300) "All field theories used in current theories of elementary particles have Lagrangians of this form."

The second section of this work explains the general properties of theories that describe charged particles. The third section examines the classical theory of charged particles. The fourth section discusses the structure of quantum electrodynamics (QED). This theory is based on the Maxwellian theory of electromagnetic fields and on the Dirac theory of a massive spin-1/2 charged quantum particle that interacts with electromagnetic fields. The fifth section discusses the Klein-Gordon (KG) theory of a massive spin-0 quantum particle. The last section comprises concluding remarks.

Most expressions take the tensorial notation of SR. The metric of the Minkowski space is diagonal and its entries are $(1,-1,-1,-1)$. Boldface variables denote vectors in the 3-dimensional space. The γ^μ matrices are used in the discussion of the Dirac theory of spin-1/2 particles. Electromagnetic symbols take the standard form that is used in the literature. References to textbooks are helpful for readers because they justify many assertions that are stated in this work. Therefore, readers can see the primary objective of this work that describes the coherent mathematical structure of a union of the Maxwellian theory of electromagnetic fields and any specific theory of a charged particle.

2. Elements of a Physical Theory

This section explains the structure of the analysis and the significance of the elements that are used in this work. A physical theory has the form of a mathematical structure that describes the behavior of a given system of physical objects. This objective is realized by means of a measurement device that shows a specific feature of the system. Here time is a vital element of the process because an effect is realized by the transition of the measurement device from its initial state into its final state. Hence, a physical theory takes the form of a time-dependent differential equation. Moreover, SR and its Lorentz transformations indicate that the time is connected to the spatial coordinates. Therefore, a relativistic theory of a given system takes the form of a set of partial differential equations with respect to the four space-time coordinates. In the case of a quantum particle, the equations depend on a quantum function that takes the form $\psi(t,\mathbf{r})$. This function is briefly denoted by $\psi(x)$.

2.1. The Role of Interaction

This work examines the classical theory of a charged particle and two quantum theories of a charged particle, each of which uses a specific form of the function $\psi(x)$. The state of a free particle does not change with time. Hence, every theory uses an interaction term that determines the time evolution of the state. The analysis is restricted to the behavior of an electrically charged particle that interacts with external electromagnetic fields. It means that the Lagrangian density of each of these systems takes this general form

$$\mathcal{L}_{GEN} = G - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - ej^\mu A_\mu . \tag{1}$$

Here j^μ denotes the 4-current of the charged particle. j^μ is a 4-vector and its 0-component is the charge density (see [2], pp. 73-75). A_μ denotes the electromagnetic 4-potential, and this 4-potential yields the electromagnetic fields tensor

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \left(A_{\beta,\alpha} - A_{\alpha,\beta} \right) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2}$$

(see [2], p. 65).

Here the first term G of the general form (1) of the Lagrangian density describes the behavior of either a free classical particle or the state of a given free quantum particle; the second term of (1) is the well-known term of free electromagnetic fields; the last term denotes the interaction between the charged particle and the electromagnetic fields.

As stated above, each term of the Lagrangian density must be a Lorentz scalar whose dimension is $[L^{-4}]$. An obvious requirement of a theory of any specific charged particle is that its mathematical structure must be self-consistent. In this work, special attention is dedicated to the 4-current j^μ of the interaction term of (1). This 4-current is a property of the theory that describes the given charged particle.

Consider the inhomogeneous Maxwell Equation (see [2], p. 79)

$$F^{\mu\nu}_{,\nu} = -4\pi e j^\mu. \tag{3}$$

(Here the definition of the 4-current j^μ represents matter 4-current. Some textbooks define the 4-current so that j^μ represents the charge 4-current. The relation between these definitions is $ej^\mu \to j^\mu$.) The 4-divergence of (3) and the antisymmetry of the electromagnetic field's tensor $F^{\mu\nu}$ (2) proves that

$$F^{\mu\nu}_{,\nu,\mu} = -4\pi e j^\mu_{,\mu} = 0. \tag{4}$$

It means that Maxwellian electrodynamics requires a conserved 4-current of a charged particle

$$j^\mu_{,\mu} = 0. \tag{5}$$

This equation is called *the continuity equation.* Its specific name indicates its significance (see [2], section 29; [3], p. 549).

The Noether theorem yields this expression for a conserved 4-current for a theory that is derived from the Lagrangian density of a quantum particle (see [4], p. 314)

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} \psi. \tag{6}$$

This work examines the mathematical coherence of expressions for the 4-current j^μ of theories of a charged particle. The accomplishment of this objective sheds new light on the theoretical structure of charged particles. It is pointed out above that the continuity equation is an important element of an electromagnetic theory of a charged particle. However, one cannot be sure that the continuity equation is a unique theoretical element. The novelty of this work is the proof that Maxwell equations of the electromagnetic fields impose other constraints on the equations of motion of a charged particle.

2.2. Three Levels of Quantum Theories

The concept of correspondence between physical theories is an important element of theoretical physics. These lines explain it briefly (see also [5], pp. 1-6; [6]). An acceptable physical theory should not explain every experimental result. As a matter of fact, such a physical theory has a domain of validity and the theory provides acceptable explanations for experimental results that belong to that domain. For example, Newtonian mechanics provides good explanations for experiments where the particles' velocity $v \ll c$ (and quantum effects can be ignored).

Excluding quantum effects, the validity domain of relativistic mechanics applies to all physically available velocities. Hence, the validity domain of Newtonian mechanics is a subset of that of relativistic mechanics. This example illustrates correspondence and the Hierarchical relationship between physical theories. Relativistic mechanics has a higher rank with respect to Newtonian mechanics. This issue is the basis of two requirements:

Req1. The higher rank theory should define variables that belong to the lower rank theory, and the limit of the value of the variable of the higher rank theory should agree with the value of the corresponding variable of the lower rank theory.

Req2. If the value of a variable of a higher rank theory does not vanish in the limit that holds for the lower rank theory then the lower rank theory should explain that limit.

Correspondence relationships exist between the three quantum theories of Figure 1. For example, the non-relativistic Schrödinger theory provides good explanations of quantum effects, and many textbooks are dedicated to this topic. The Dirac theory explains relativistic quantum effects where the number of particles takes a definite value. For example, the Schrödinger theory provides quite good explanations for the states of the hydrogen atom but the Dirac theory provides better explanations for these states.

Experiments show high energy states where the number of particles does not take a definite value. For example, the proton comprises three quarks of the *uud* flavor (see e.g., [7], p. xiv). These quarks are called valence quarks. However, it is

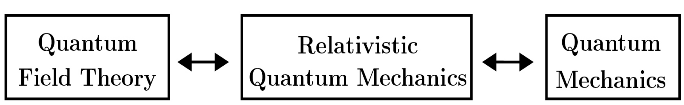

Figure 1. Correspondence between 3 quantum theories.

well known for many decades that the proton comprises antiquarks (see e.g., [8], p. 282). This evidence proves that besides the *uud* quarks, the proton comprises (a probability of) quark-antiquark pairs. Here Quantum Field Theory (QFT) is a better theory. QFT uses the Fock space for a description of a system of several particles (see e.g., [9], pp. 134-137).

Weinberg clearly states in his textbook [1] that correspondence relationships hold between QFT and quantum mechanics. "First, some good news: quantum field theory is based on the same quantum mechanics that was invented by Schrödinger, Heisenberg, Pauli, Born, and others in 1925-26, and has been used ever since in atomic, molecular, nuclear, and condensed matter physics." (see p. 49). Below, these relationships are called the *Weinberg correspondence principle.*

3. The Classical Charged Particle

The well-known textbook of Landau and Lifshitz [2] analyzes the classical theory of charged particles that interact with electromagnetic fields. They use the principle of least action, where the action S takes the form (see [2], p. 75)

$$S = -\sum\int m\mathrm{d}s - \frac{1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}\mathrm{d}^4x - \int ej^{\mu}A_{\mu}\mathrm{d}^4x. \tag{7}$$

Here s is the relativistic interval and the 4-current of a classical charged particle is

$$j^{\mu} = e(\rho, \rho\mathbf{v}). \tag{8}$$

Here e is the particle's charge, ρ denotes its density, and $\mathbf{v}$ is its 3-velocity (see [2], p. 75). The variation of the particle's coordinates and its velocity prove that (7) yields the Lorentz force, which is the law of motion of a classical charge in electromagnetic fields (see [2], section 17). The tensorial form of this force is (see [2], p. 65)

$$m\frac{\mathrm{d}v^{\mu}}{\mathrm{d}s} = eF^{\mu\nu}v_{\nu}, \tag{9}$$

where v^{μ} denotes the particle's velocity 4-vector.

On the other hand, the electromagnetic 4-potential A_{μ} is regarded as the coordinates of the electromagnetic fields. Here the corresponding variation of (7) yields the inhomogeneous Maxwell Equation (3) (see [2], pp. 78, 79). Please note that the mathematical antisymmetry of the electromagnetic field tensor $F^{\mu\nu}$ yields the homogeneous pair of Maxwell equations

$$F^{*\mu\nu}_{,\nu} = 0, \tag{10}$$

where $F^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ (see [2], pp. 70, 71).

Landau and Lifshitz prove that a relativistic elementary particle is pointlike (see [2], pp. 46, 47). In this case, they use the Dirac δ function and prove that the 4-current of a classical particle is coherently defined. Moreover, this 4-current satisfies the continuity Equation (5) (see [2], pp. 74-78).

Relying on the Landau and Lifshitz textbook [2], it is shown here that the action of a classical charged particle (7) takes the general form (1) of Section 2 and the particle's 4-current j^μ is coherently defined.

4. The Dirac Quantum Theory

The Dirac theory describes the state and the time evolution of a spin-1/2 quantum particle. This theory illuminates the theoretical merits of QFT. This topic is discussed in many textbooks (see e.g., [1] [10]). Several features of this issue are described herein. The Lagrangian density of the system of a charged Dirac particle and electromagnetic fields is

$$\mathcal{L}_{QED} = \bar{\psi}\left(\gamma^\mu i\partial_\mu - m\right)\psi - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}e\gamma^\mu A_\mu\psi, \tag{11}$$

where ψ is the Dirac 4-component quantum function, and $\bar{\psi} = \psi^\dagger\gamma^0$. This expression is called the QED Lagrangian density (see e.g., [10], p. 84; [11], p. 78).

The two terms of (11) that are enclosed in the parentheses denote the kinetic and mass terms of a free Dirac particle, respectively; the second term denotes the free electromagnetic fields. This term takes the form that is used in the general structure of the action (1). The last term of (11) denotes the interaction between the charged Dirac particle and the electromagnetic 4-potential A_μ.

An application of the minimal action principle to the Dirac quantum spinor ψ yields the Dirac equation for a spin-1/2 charged particle. This is the Euler-Lagrange equation for the Dirac quantum function ψ of (11) that is obtained from the application of the least action principle to the QED Lagrangian density (11). The standard covariant form of the Dirac equation for the electron is

$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi = 0 \tag{12}$$

(see [10], p. 84; [11], p. 78).

An application of the Noether theorem (6) to the Dirac Lagrangian density (11) yields the Dirac expression for the matter 4-current

$$j^\mu = \bar{\psi}\gamma^\mu\psi \;\rightarrow\; j^0 = \rho_m = \psi^\dagger\psi \tag{13}$$

(see [12], p. 23). This expression depends on the Dirac functions $\bar{\psi},\psi$ and it is charge independent. Therefore, it describes matter properties of the Dirac quantum particle. In (13), the matter density ρ_m is the 0-component of the Dirac 4-current. It proves that $\rho_m > 0$ at a space-time point x where $\psi(x) \neq 0$. For a charge carrying Dirac particle, the electric 4-current is simply a product of the matter 4-current (13) by its charge e (or $-e$, respectively)

$$j_e^\mu = e\bar{\psi}\gamma^\mu\psi \;\rightarrow\; \rho_e = e\psi^\dagger\psi. \tag{14}$$

This expression proves that the last term of the QED Lagrangian density (11) takes the standard form of the electromagnetic interaction

$$\mathcal{L}_{Int} = -ej^\mu A_\mu \tag{15}$$

(see [2], p. 75). The Noether theorem (6) proves that this 4-current satisfies the continuity Equation (5). Furthermore, this 4-current is free of derivatives of the quantum field function $\psi_{,\mu}$. Hence, its usage does not affect the Noether construction of the 4-current (6).

The substitution of the Dirac charged 4-current (14) into the right-hand side of the inhomogeneous Maxwell Equation (3) yields (see [10], p. 85)

$$F^{\mu\nu}_{,\nu} = -4\pi e\bar{\psi}\gamma^{\mu}\psi. \tag{16}$$

This expression shows that the Dirac theory coherently combines Maxwellian electromagnetic fields with charged matter. Here the differential part of the electromagnetic fields term stands on the left-hand side. Furthermore, its right-hand side which depends on the charged particle's 4-current, explicitly defines the solution. In particular, the right-hand side of this equation is *independent* of the electromagnetic field variables A_{μ} and $F^{\mu\nu}$. This is the form of the inhomogeneous Maxwell Equation (3). It takes the form of a linear inhomogeneous partial differential equation: The unknown derivatives of variables (the Maxwellian fields $F^{\mu\nu}$) stand on the left-hand side. It depends on a given 4-current of the charged particle that stands on the right-hand side.

Let us examine the Hilbert space, which is an important element of the theory. The density of the Dirac 4-current (13) proves that $\psi^{\dagger}\psi > 0$ at every space-time point where $\psi \neq 0$. Hence, this relation holds for the Dirac functions

$$\int \psi^{\dagger}\psi \mathrm{d}^3 x > 0. \tag{17}$$

The Hilbert space requires that an element $\left|\psi_i\right\rangle$ of its orthonormal basis satisfies $\left\langle \psi_i^{\dagger} \middle| \psi_i \right\rangle = 1$. This relation together with (17), prove that the Dirac functions ψ_i satisfy a necessary condition for the construction of a Hilbert space. The actual construction of the Hilbert space for a Dirac function is analogous to that of the Schrödinger theory because also the Schrödinger function satisfies (17). The Hilbert space is a crucial element of the Schrödinger theory (see [1], pp. 49, 50). Therefore, the Dirac theory satisfies this element of the Weinberg correspondence principle.

The Hamiltonian is an important quantity of the theory. For a quantum theory that depends on a Lagrangian density $\mathcal{L}$, one finds the Hamiltonian density

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\dot{\psi}}\dot{\psi} - \mathcal{L} \tag{18}$$

(see [1], p. 301; [10], pp. 16, 87). In the case of a charged Dirac particle, one finds the Hamiltonian density

$$\mathcal{H}_{Dirac} = \psi^{\dagger}\left[-\boldsymbol{\alpha}\cdot\left(-i\nabla - e\boldsymbol{A}\right) + \beta m\right]\psi \tag{19}$$

(see [10], p. 87).

The form (19) of the Dirac Hamiltonian density has these important properties:

P.1: The Dirac 4-current (13) proves that $\psi^{\dagger}\psi$ is the density of the Dirac particle. Hence, the quantity that stands inside the square brackets of (19) is the *operator form* of the Dirac Hamiltonian. Furthermore, it is easily factored out and yields the explicit operator form of the Dirac Hamiltonian.

P.2: The quantum function ψ is regarded as the generalized coordinate of the Lagrangian density. It turns out that the Dirac conjugate momentum is $i\psi^\dagger$ (see [10], p. 55). Therefore, formula (19) takes the standard form of the Hamiltonian that is a function of the generalized coordinates and their conjugate momenta.

P.3: Each of the many textbooks of the Schrödinger quantum mechanics discusses the central role of the Hamiltonian operator. Therefore, the explicit form of the Dirac Hamiltonian that is enclosed inside the square brackets of (19) shows this aspect of the Weinberg correspondence principle.

Consider the comparison of the Dirac form (16) of Maxwellian electrodynamics and other specific attributes of the Dirac theory that are discussed above. These issues are compared with the corresponding properties of the KG equation of a charged particle. This comparison, which yields instructive insights, is carried out in the next section.

5. The Klein-Gordon Theory

The literature provides this form of the Lagrangian density of a KG particle that carries a charge e

$$\mathcal{L}_{KG} = -\left[\left(i\partial_\mu + eA_\mu\right)\Phi^*\right]\left[\left(i\partial^\mu - eA^\mu\right)\Phi\right] - m^2\Phi^*\Phi. \tag{20}$$

(see [13], p. 198; [14] p. 73). Like the previous cases, the analysis of the electromagnetic structure of the KG theory requires its 4-current. The original article of Pauli and Weisskopf published an explicit expression for the KG charge-dependent 4-current (see [13], p. 199)

$$j^\nu_{KG} = g^{\mu\nu}\left[ie\phi^*\left(\partial_\mu - \overleftarrow{\partial}_\mu\right)\phi - 2e^2A_\mu\phi^*\phi\right]. \tag{21}$$

(Note that this is *not* the matter-dependent 4-current.) Many textbooks also show this expression (see [12], p. 189; [14], p. 75; [9], p. 63; [15], p. 228). The corresponding Maxwell equation that depends on this KG 4-current is

$$F^{\mu\nu}_{\ \ ,\nu} = -4\pi g^{\mu\nu}\left[ie\phi^*\left(\partial_\mu - \overleftarrow{\partial}_\mu\right)\phi - 2e^2A_\mu\phi^*\phi\right]. \tag{22}$$

Here is a list of problematic points of this KG expression:

Err.1: The right-hand side of the KG expression for the electromagnetic fields (22) comprises a term that is proportional to e^2. In contrast, Maxwell theory proves that the electromagnetic fields of a single source (3) are proportional to the charge e of the source. This Maxwellian attribute has strong experimental support.

Err.2: The KG charge density (21) is not positive definite (see the footnote 11 on p. 197 of the Pauli and Weisskopf paper [13]). Hence, there are problems with the Hilbert space of a KG quantum particle (see Weinberg's textbook [1] 49, 50 and Messiah's textbook, [16], pp. 164-166) because an element of the standard basis of a Hilbert space of quantum functions requires a positive definite normalized scalar product of the quantum functions. For a Dirac and Schrödinger quantum function ψ_j, the scalar product is $\int \psi_j^\dagger \psi_j \mathrm{d}^3 r > 0$, and it is normalized to unity. The same KG problems hold for the Fock space, which is a direct product of Hilbert spaces (see [9], Section 6f). The absence of a KG Hilbert space is inconsistent with the Weinberg correspondence principle of Subsection 2.2.

Err.3: Weinberg's textbook proves that the 4-potential A_μ is *not* a 4-vector. This textbook states: "The fact that A^0 vanishes in all Lorentz frames shows vividly that A^μ cannot be a four-vector." (see [1], p. 251) Therefore, the foregoing KG 4-current is a sum of a derivative ∂_μ which is a 4-vector and A_μ which is not a 4-vector. Hence, the KG 4-current (21) violates relativistic covariance.

Err.4: The inhomogeneous Maxwell Equation (3) is a *partial differential equation* of the fields $F^{\mu\nu}_{,\nu}$, where the right-hand side *is independent* of the electromagnetic variables $A_\mu, F^{\mu\nu}$. On the other hand, the corresponding KG Equation (22) has the ordinary derivative $F^{\mu\nu}_{,\nu}$ of Maxwell theory and another term that depends on A^μ. Therefore, the KG theory of electromagnetic fields does not take the Maxwellian form.

A further examination proves that other inconsistencies exist with the KG theory of a charged particle.

Inc.1: The last term of the KG Lagrangian density is $m^2\Phi^*\Phi$. This term proves that the dimension of the KG function is $[L^{-1}]$. Indeed, the dimension of every term of the Lagrangian density is $[L^{-4}]$ and the mass dimension is $[L^{-1}]$. Hence, the KG theory violates the Weinberg correspondence principle because the dimension of the Schrödinger function is $[L^{-3/2}]$.

Inc.2: The Hamiltonian is the energy operator of the Schrödinger theory. Therefore, the Weinberg correspondence principle says that the KG theory should provide a coherent expression for the Hamiltonian.

The Pauli and Weisskopf KG paper argues that the form of the Hamiltonian is

$$\bar{H}^m = \int\left(\cdots + m^2\phi^*\phi\right)\mathrm{d}^3x \tag{23}$$

(see [13], p. 198). For the simplicity of the argument, other terms of the integrand of (23) are omitted because they are not necessary for the discussion.

Either of these assertions proves that relation (23) is not the required Hamiltonian operator.

- It contains the factor m^2 which is the square of energy, whereas the Hamiltonian is the energy operator.
- The spatial integral $\int \mathrm{d}^3x$ applies to density that has the dimension $[L^{-3}]$ while the dimension of the product of the two KG functions $\phi^*\phi$ is $[L^{-2}]$.
- This problem is uncorrectable because it is already stated above that there is no coherent expression for the matter density of a charged KG particle.

Conclusion: The KG theory has no coherent expression for the Hamiltonian operator.

The KG Hamiltonian problems explain why the KG theory violates the Weinberg correspondence principle because the Schrödinger theory has an explicit form of the Hamiltonian operator.

This section proves several inherent inconsistencies of the KG theory that are not adequately discussed in the general literature.

6. Concluding Remarks

This work explains the significance of the least action principle as the basis of a theory of a given charged particle. Here the Noether theorem provides expressions for quantities that are conserved by the system's Euler-Lagrange equations. The discussion is carried out within the framework of SR. It insists on the obvious requirement that the mathematical structure of a given theory should be error-free. The analysis examines the coherence between several equations of motion of a charged particle and Maxwell equations of the electromagnetic fields. The third section explains why the Landau and Lifshitz textbook [2] proves the coherent structure of the classical theory of a charged particle. The same result is obtained in the fourth section, which discusses the Dirac theory of a spin-1/2 charged quantum particle. In contrast, the fifth section proves inherent errors of the KG theory of a spin-0 charged particle. This outcome points out the novelty of this work. Experiments support these conclusions. In particular, the Dirac theory properly describes the electron, while the KG theory describes no particle. For example, the KG quantum function $\phi(x)$ cannot describe a pion because $\phi(x)$ depends on the 4 space-time independent variables $x \equiv (t, \mathbf{r})$. On the other hand, a pion is a quark-antiquark bound state (see [7], p. xiv). As such, a function of 7 independent variables $\Phi(t, \mathbf{r}_1, \mathbf{r}_2)$ describes a pion. Hence, due to the different number of independent variables, a KG function $\phi(x)$ cannot describe a pion.

The foregoing conclusions extend the known connection between Maxwell's theory of electromagnetic fields and the properties of charged matter. The specific name of the continuity Equation (5) indicates one known example of how the Maxwell theory of the electromagnetic fields requires that any theory of a charged matter must abide with. The novelty of this work is the proof that the continuity equation is not a unique case. For example, the previous KG section proves that the 4-current of the charged matter cannot depend on the square of the charge e^2.

As stated above, the present work proves many discrepancies in the KG theory. A discussion of a few problematic issues of the KG theory has already been published [17]. Besides the KG topic, [17] shows proofs of some general physical issues that are outside the scope of the present work. For example, it proves that the particle-wave duality of quantum mechanics is not contradictory concept. Hence, a quantum theory needs no specific postulate for using both of them. It is interesting to mention that the KG analysis that is published above agrees with Dirac's lifelong objection to second-order quantum equations like that of the KG theory [18].

Conflicts of Interest

The author declares no conflicts of interest.

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